SPECTRAL AND SCATTERING THEORY FOR SCHRODINGER OPERATOR WITH A CLASS OF MOMENTUM DEPENDENT LONG RANGE POTENTIALS

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C O N T E N T S

§ 1. INTRODUCTION

Schrodinger Operator : In Non-Relativistic Quantum Mechanics the Schrodinger operator given by the Hamiltonian

$$H = \tfrac{1}{2}P^2+W(Q,P) = -\tfrac{1}{2}\Delta+W(Q,P) = H_0+W(Q,P) \tag{1.1}$$

where Q,P are the position and momentum operators on $\mathcal{H} = L^2(R^n)$ is of great interest.

(i) When W = 0, H is the energy operator of a freely moving particle.

(ii) When W(Q,P) = W(Q) with W(x) real valued i.e. when W is independent of P, H is the energy operator of a particle moving under the influence of the potential W(x). Of special interest will be the case when W is the Coulomb potential : $W(x) = k/|x|$ where κ is a real constant.

(iii) When $W(Q,P) = W_0(Q) + W_1(Q)P_1 + \ldots + W_n(Q)P_n$, then H is closely related to the energy operator of a particle moving in a static magnetic and electric fields. Note that when $W_1,W_2,\ldots,W_n$ are all zero we get (ii). Thus (ii) is a particular case of (iii). In (iii) we assume that the intensity of the potentials decreases with the distance from the origin i.e., $\lim_{|x|\to\infty} \sum_{j=0}^{n} |W_j(x)| = 0$. The main interest of this thesis is the operator H of (iii).

(iv) When $W(Q,P) = W(Q) = \tfrac{1}{2}Q^2$ or $\tfrac{1}{2}(Q^2+Q^4)$ then H is called the Hamiltonian of the harmonic or an-harmonic oscillator.

Self adjointness : Since Non-Relativistic Quantum Mechanics requires that the operator H should be self adjoint, the first task in Mathematics is to prove rigorously that H is, indeed, self adjoint when suitable conditions are imposed on W(Q,P). Even when W = 0, $H = -\tfrac{1}{2}\Delta = H_0$ is not defined on the whole space. Thus to determine the self adjointness of an operator is not an obvious one. There are various techniques and we cite only two, that of Kato-Rellich and Friedrichs [1,2,3,5]. The first technique is useful for (iii) while the second is useful for (iv).

Let A be densely defined (linear) operator on $\mathcal{H}$, i.e., A is defined on D(A) with values in $\mathcal{H}$ and D(A) is a dense subspace of $\mathcal{H}$. A is said to be closable if there exists a closed operator A_1 extending A. In such a case there is an operator A_0 which is closed, extension of A and every closed extension of A is also an extension of A_0. A_0 is called the closure of A and D(A) is called a core for A_0. If A_0 is self adjoint then A is said to be essentially self adjoint.

Let $C_0^\infty(R^n)$, $S(R^n)$ denote the space of all infinitely differentiable functions with compact support and the Schwartz space of rapidly decreasing functions. Then it is known [1,2,3,5] that H_0 with its maximal domain is self adjoint and that both $C_0^\infty(R^n)$, $S(R^n)$ are cores for H_0.

For self adjointness a necessary condition is symmetry. The operators H in (i), (ii) and (iv) are obviously symmetric. For (iii) we assume that W_j are real valued. Then for symmetry we need $\sum_{j=1}^{n} \partial W_j(x)/\partial x_j = 0$ in the sense of distributions which we always assume. At a later stage we shall split W_j into $W_j = W_j^L + W_j^S$ where W_j^L is smooth. In such a case we shall further assume that $0 = \sum_{j=1}^{n} \partial W_j^L(x)/\partial x_j = \sum_{j=1}^{n} \partial W_j^S(x)/\partial x_j$.

The energy operator with a static electro-magnetic field is given by $H = \frac{1}{2} \Sigma(P_j+W_j(Q))^2 + W_0(Q)$. Then $H = -\frac{1}{2}\Delta+\sum_j W_j(Q)P_j+W_0(Q)-i \sum_j \partial W_j(Q)/\partial Q_j$ which reduces to (iii) if the last summand is zero.

In (iii) H can be thought of as a perturbation of H_0 and Kato-Rellich Theorem [1,2,3,5] answers when is the perturbation of a self adjoint operator self adjoint.

Theorem 1.1 (Kato-Rellich) [1,2,3,5] : Let A be a self adjoint operator, B symmetric with $D(B) \supset D(A)$. Define

$$\rho(B,A) = \inf\{a > 0 : \text{for some } b > 0 \text{ and all } f \in D(A), ||Bf|| \leq a||Af|| + b||f||.\}$$

If $\rho(B,A) < \infty$ then B is said to be bounded relative to A or simply B is A bounded. $\rho(B,A)$ is called the relative bound of B w.r.t A or simply A bound. If $\rho(B,A) < 1$ then

(i) A+B with $D(A+B) = D(A)$ is self adjoint,

(ii) any core for A is a core for A+B,

(iii) A+B is bounded below if A is.

Remark 1.2 : Let A be self adjoint, $D(B) \supset D(A)$. If $B(A+i)^{-1}$ is compact then $\rho(B,A) = 0$.

As a consequence we have the following

Theorem 1.3 : Define for $\lambda < n$

$$M_\lambda = \{q : R^n \to R : \sup_x \int_{|y-x|\leq 1} dy\, |q(y)|^2 |x-y|^{\lambda-n} < \infty\}$$

and for $\lambda \geq n$

$$M_\lambda = \{q : R^n \to R : \sup_x \int_{|x-y|\leq 1} |q(y)|^2 dy < \infty\}.$$

If $W_j \in M_\lambda$ for some $\lambda < 2$ for $j = 0,1,\ldots,n$ then

(i) $\rho(\Sigma W_j(Q)P_j + W_0(Q), H_0) = 0$

(ii) $H = H_0 + \Sigma W_j(Q)P_j + W_0(Q)$ is self adjoint with $D(H) = D(H_0)$.

Proof : Refer Theorem 10.18 [3].

Remark 1.4 : M_λ is a linear space for each λ. Further if $q(x) = |x-a|^{-\delta}$ for some $a \in R^n$ and $\delta \in (0,\frac{1}{2}n)$ then $q \in M_\lambda$. This shows that there are also unbounded functions in M_λ. Certainly all the bounded functions are in M_λ.

Theorem 1.5 : (Friedrichs) [1,2,3,5] : Let A be a densely defined symmetric nonnegative operator in a Hilbert space $\mathcal{H}$. Then A has an extension to

$$D_1 = \{f \in \mathcal{H} : \text{there exists a sequence } f_n \text{ in } \mathcal{D}(A) \text{ such that } f_n \to f \text{ strongly and } \lim_{n,m\to\infty} \langle A(f_n-f_m), f_n-f_m\rangle = 0\}$$

and the extension is self adjoint, nonnegative called Friedrichs extension of A.

Now P^2, Q^2 are both defined on $S(R^n)$ and so is P^2+Q^2 on $S(R^n)$ on which it is nonnegative. By Friedrichs' theorem P^2+Q^2 has a self adjoint extension. Similarly $P^2+Q^2+Q^4$ has a self adjoint extension. By the same argument $P^2+(PQ+QP)^2$ and $P^6+(PQ+QP)^4$ defined on $S(R^n)$ have self adjoint extensions, infact $S(R^n)$ is a core.

Some other methods for proving self adjointness are (i) perturbation of quadratic forms [2,5] (ii) Nelson's theory of analytic vectors [3,5].

Spectrum (essential, point) Having decided the self adjointness, the next important thing is to decide the spectrum and the spectral properties of the operator. If A is self adjoint then the only general result known is that $\sigma(A)$ = spec (A) [1,2,3,4,5,7] is a closed subset of R. Infact given any closed set C of R we can find a self adjoint A with $\sigma(A) = C$. (Let $\{x_1, x_2, \dots\}$ be any countable dense subset of C. On

$$\ell^2 = \{(a_1, a_2, \dots) : a_n \text{ are complex numbers}, \Sigma|a_n|^2 < \infty\}$$

define $A(a_1, a_2, \dots) = (x_1a_1, x_2a_2, \dots)$ so that C = spec A.) The operator P^2 has no eigenvalue at all where as P^2+Q^2 has only eigenvalues in the spectrum; $\sigma(P^2) = [0,\infty)$ where as $\sigma(P^2+Q^2) = \{(n+\frac{1}{2})\pi : n = 0,1,2,\dots\}$

Let $\sigma_d(A)$ = all the discrete eigenvalues of A of finite multiplicity and $\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A)$ [1,2,3,4]. The $\sigma_{ess}(A)$ is stable under compact perturbations. More precisely

Theorem 1.6 (Kato-Weyl) [1,2,3,7] Let A be self adjoint. If B is symmetric with $D(B) \supset D(A)$ and $B(A+i)^{-1}$ is compact then A+B is self adjoint with $D(A+B) = D(A)$ and $\sigma_{ess}(A+B) = \sigma_{ess}(A)$.

To decide the compactness of $f(Q)g(P)$ we have

Theorem 1.7 Let $\int dx(|f(x)|^2 + |g(x)|^2)(1+|x|)^{\mu} < \infty$ for some $\mu > n$. Then the operator $f(Q)g(P)$ on $L^2(R^n)$ is a trace class operator.

The above theorem is Theorem X1.21 of [6].

Corollary 1.8 Let f,g be bounded (real valued) functions on R^n vanishing at ∞. Then $f(Q)g(P)$ is a compact operator.

Proof : Let $f_m(x) = F(|x| \leq m)f(x)$, $g_m(k) = F(|k| \leq m)g(k)$ where F stands for the indicator function. Then by Theorem 1.7 $f_m(Q)g_m(P)$ is compact. Since $\lim_{m \to \infty} ||f_m(Q)g_m(P)-f(Q)g(P)|| = 0$, $f(Q)g(P)$ is compact.

Q.E.D.

Note that, however, by Weyl-von Neumann Theorem [1,2] that even when A does not have any eigenvalue at all, one can choose a Hilbert-Schmidt operator B such that A+B has only eigenvalues in its spectrum.

The operator $H_o = -\frac{1}{2}\Delta$ has $\sigma_{ess}(H_o) = [0,\infty)$. If W_j are bounded functions vanishing at ∞ for $j = 0,1,\ldots n$, then $\{\Sigma W_j(Q)P_j + W_o(Q)\}(H_o+1)^{-1}$ is compact by Corollary 1.8. By Theorem 1.6 $H = -\frac{1}{2}\Delta + \Sigma W_j(Q)P_j + W_o(Q)$ has $\sigma_{ess}(H) = [0,\infty)$; consequently all the negative eigenvalues of H, if any, are of finite multiplicity and they can accumulate, possibly, only at zero. For positive eigenvalues the same result is true when $W_j(x)$ are bounded and behave like $|x|^{-1-\varepsilon}$ at ∞ for some $\varepsilon > 0$ and all $j = 0,1,2,\ldots n$ [8].

Regarding the finiteness or infiniteness, positivity or negativity, degenerate or nondegenerate properties of $-\frac{1}{2}\Delta + W(Q)$ and about properties of eigen vectors we refer to [7].

Let $\mathcal{H}_p(A), \mathcal{H}_c(A), \mathcal{H}_{ac}(A), \mathcal{H}_{sc}(A)$; $E_p(A)$, $E_c(A)$, $E_{ac}(A)$, $E_{sc}(A)$; A_p, A_c, A_{ac}, A_{sc} be the point, continuous, absolutely continuous, singularly continuous space; the respective orthogonal projections on these spaces; and $AE_p(A)$, $AE_c(A)$, $AE_{ac}(A)$, $AE_{sc}(A)$ respectively for self adjoint A. Let $\sigma_p(A)$ be the eigenvalues of A, $\sigma_c(A)$, $\sigma_{ac}(A)$, $\sigma_{sc}(A)$ the spectrum of A_c, A_{ac}, A_{sc} respectively. For all these refer [1,2,3,4]. Note the inclusion $\sigma_c(A) \subset \sigma_{ess}(A)$ so that $\sigma_c(H) \subset [0,\infty)$ where $H = -\frac{1}{2}\Delta + \Sigma W_j(Q)P_j + W_o(Q)$ where W_j are bounded and vanish at ∞ for every $j = 0,1,\ldots,n$.

The operator $H_o = -\frac{1}{2}\Delta$ in the momentum representation is given by the polynomial $\frac{1}{2}k^2$ and so has absolutely continuous spectrum; in other words $E_{ac}(H_o) = 1$.

Using the idea of dilation analyticity [9] $-\Delta + W_o(Q)$ is shown to have no singular continuous spectrum where W_o is the physically important Coulomb potential i.e., $W_o(x) = k/|x|$. The same result is proved for N-body case in [10].

Scattering Theory (short range) One way, viz, the Physicist's way of proving that two self adjoint operators are unitarily equivalent is to use the notion of wave operators. For any two self adjoint operators A and B define the wave operators

$$\Omega_{\pm}(B,A) = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itB} e^{-itA} E_{ac}(A)$$

if it exists. If $\Omega_{\pm}$ exists, it is clear that $\Omega_{\pm}$ is an isometry on $\mathcal{H}_{ac}(A)$ into $\mathcal{H}_{ac}(B)$ and satisfies the intertwining property:

$$e^{itB}\,\Omega_{\pm} = \Omega_{\pm}\, e^{itA}$$

or $\Omega_{\pm} A_{ac} \subset B_{ac} \Omega_{\pm}$. If further Range $\Omega_{+} = \mathcal{H}_{ac}(B)$ (or Range $\Omega_{-} = \mathcal{H}_{ac}(B)$) then Ω_{+} (or Ω_{-}) is unitary between $\mathcal{H}_{ac}(A)$ and $\mathcal{H}_{ac}(B)$ and $B_{ac}\Omega_{+} = \Omega_{+}A_{ac}$ (or $B_{ac}\Omega_{-} = \Omega_{-}A_{ac}$). In other words B_{ac} and A_{ac} are unitarily equivalent.

For an example where $\Omega_{\pm}$ (B,A) exists but Range Ω_{+}, Range Ω_{-}, $\mathcal{H}_{ac}(B)$ are all distinct refer [11].

We say that asymptotic completeness holds for the pair (B,A) if Range $\Omega_{+}(B,A) = \mathcal{H}_{ac}(B)$ = Range $\Omega_{-}(B,A)$.

If $\Omega_{\pm}(A,B)$ exists then asymptotic completeness holds. Thus showing B_{ac} is unitarily equivalent to A_{ac} seems to be an easy work. But in practice only one of the operators A or B is simple about which lot of details are available.

If all the W_j j=0,1,...,n are bounded with $O(|x|^{-\mu}), \mu > 1$ at ∞ then using Cook's method [12,13,1,2,3,6] it is easy to show that $\Omega_{\pm} = \Omega_{\pm}(H,H_o)$ exist where $H_o = -\frac{1}{2}\Delta$, $H = H_o + \Sigma W_j(Q)P_j + W_o(Q)$. Using eigen function expansions Agmon [14], Kuroda [15] have proved (i) $\sigma_{sc}(H)$ is empty (ii) $\mathcal{H}_{ac}(H)$ = Range $\Omega_{\pm}$ (iii) the positive eigenvalues of H are of finite multiplicity and can accumulate only at zero. When $W_j = 0$ for j = 1,2,...,n questions (i) and (ii) above have been proved to be in the affirmative by using stationary methods or eigenfunction expansions or smooth perturbations by various authors. The literature is too vast and a good reference is [2,6,7].

Trace method (time-dependent, short range) A typical theorem of this method will be "If $\varphi(A,B)$ [which is antisymmetric in A and B] is of trace class for two self adjoint operators A,B then the wave operators $\Omega_{\pm}(B,A)$ exist and asymptotic completeness holds". While Kato takes $\varphi(A,B)$ = A-B ,Kato-Kuroda take $\varphi(A,B) = (A+i)^{-1} - (B+i)^{-1}$. Birman's Theorem is stated in § 7.

The advantage of trace method is that it is abstract. The disadvantages are: (i) no information is given about the singular continuous spectrum, (ii) even for the simplest model where $H_0 = -\frac{1}{2}\Delta$, $H = H_0 + W_0(Q)$ with W_0 bounded one requires $W_0(x) = O(|x|^{-\mu})$ where $\mu > n$ (Theorem X1.30[6]). However, if W_0 is assumed to be spherically symmetric, i.e., a function of $|x|$ only, $\mu > 1$ would be enough (Theorem X1.31[6]).

Time dependent "Geometric" method : Let $H = H_0+W_0(Q)$, $H_0 = -\frac{1}{2}\Delta$, W_0 bounded with $W_0(x) = O(|x|^{-\mu})$, $\mu > 1$.

Recently Enss [16] has proved (i) $\sigma_{sc}(H)$ = empty and (ii) asymptotic completeness for H using a geometric method. Let $F(A \in E)$ or $F(E|A)$ denote, for any Borel subset E of R and any self adjoint A the spectral projection of A on E. (This notation $F(A \in E)$ or $F(E|A)$ we shall follow throught this thesis). Choose $f \in \mathcal{H}_c(H)$ such that H spectral support of f is compact in $(0,\infty)$. Then one finds a $\varphi \in C_0^\infty(0,\infty)$ such that $\varphi(H)f = f$. Next one chooses a sequence f_n increasing to ∞ so that $[\varphi(H)-\varphi(H_0)]V_{t_n} f$ and $F(|Q| \leq n)V_{t_n} f$ both converge to zero strongly where $V_t = \exp(-itH)$. Put $f_n = V_{t_n} f$. Then the space $S(n) = \{(Q,P) : |Q| \geq n, \varphi(\frac{1}{2}P^2) \neq 0\}$ is split into $S^+(n)$ and $S^-(n)$; on $S^+(n)$, "Q.P" is ≥ 0 and on $S^-(n)$ "Q.P" ≤ 0. Corresponding to this splitting Enss is able to construct f_n (out) and f_n (in) so that (i) $(\Omega_+ -1) f_n$ (out), $(\Omega_- -1)f_n$ (in) both converge to zero strongly and (ii) $f_n = f_n$ (in) + f_n (out) + f_n (waste) where f_n (waste) = $\{1-\varphi(H_0)\}f_n + F(|Q| \leq n)f_n$. So one concludes that $\mathcal{H}_c(H) \subset$ Range $\Omega_\pm$ which at one stoke proves absence of singular continuous spectrum and asymptotic completeness for H.

Simon [8] extended the techniques of Enss to a large class of operators; $H = -\frac{1}{2}\Delta + \Sigma W_j(Q)P_j + W_0(Q)$ where W_j, W_0 are bounded and of $O(|x|^{-\mu})$, $\mu > 1$ is only a particular case of his work.

Mourre [17] soon got the same results as in [16] by putting the ideas of Enss in the more elegant form $(\Omega_\pm -1)\ \varphi(H_0)\ F(PQ+QP \gtrless 0)$ is compact for every $\varphi \in C_0^\infty(0,\infty)$; here H and H_0 are as in the case of Enss. To get the necessary estimates Mourre uses certain differential inequalities.

Perry [18] also gets, for the same H, H_o as in Mourre, $(\Omega_{\pm} -1)\varphi(H_o)\, F(PQ+QP \gtrless 0)$ is compact; he uses the idea of Mellin's transform to get the necessary estimates.

Sinha [19] gets the same result as of Perry by noting that $\log H_o$ and $(PQ + QP)/4$ can be considered to be canonical conjugate variables and using integration by parts.

Since $\text{s-}\lim_{n \to \infty} F(|Q| \geq n) = 0$, by results of [17,18,19] it is clear that one can expect $\lim_{n \to \infty} ||(\Omega_{\pm} -1)F(QP+PQ \gtrless 0,\ |Q| \geq n,\ a < |P| < b)|| = 0$ for every $a,b \in (0,\infty)$. That it is so was proved by Davies [20] in the form, where T is as in § 3,

$$\lim_{n \to \infty} ||(\Omega_{\pm} -1)\ T\{(x,k) : a < |k| < b, \quad kx \gtrless 0, |x| \geq n\}|| = 0$$

using the notion of generalised coherent states from [21, 22] and by the method of stationary phase.

By using the techniques of [20] it is not hard to show (Lemma 4.3, Theorem 4.6) that for every $\varphi \in C_o^{\infty}(0,\infty)$ there exists an $a > 0$ such that

$$\lim_{t \to \pm\infty} ||(\Omega_{\pm}-1)U_t\varphi(H_o)F(|Q| \leq a|t|)|| = 0 \quad .$$

This together with the "folk theorem" $V_t^* U_t\, Q\, U_t^* V_t/|t|$ converges to 0 strongly on $\mathcal{H}_c(H)$ as $|t| \to \infty$ (proved as Theorem 2.11 in [23]) gives $\mathcal{H}_c(H) \subset \text{Range } \Omega_{\pm}$.

<u>Long range (existence)</u> For the Hamiltonians H_o and H let $U_t = \exp(-itH_o)$, $V_t = \exp(-itH)$. When $H_o = -\frac{1}{2}\Delta$, $H = H_o + k|Q|^{-1}$ it was known that $V_t^* U_t$ does not have a strong limit as $t \to +\infty$ or as $t \to -\infty$. Dollard [24] proved that $\Omega_{\pm} = \text{s-}\lim_{t \to \pm\infty} V_t^* Z_t$ exists where

$$Z_t = \exp(-iX(t,P)),\ X(t,P) = \tfrac{1}{2}tP^2 + k(\text{sign } t)|P|^{-1} \log(-2|t||P|^2).$$

Also he proved, by the method of eigenfunction expansions, asymptotic completeness i.e. $\mathcal{H}_{ac}(H) = \text{Range } \Omega_{\pm}$.

Let W_o be a C^∞ potential on R^n such that $|D^\alpha W_o(x)| \leq k_\alpha(1+|x|)^{-\delta-|\alpha|}$ for every multi-index α and for some δ in $(0,1)$; $H_o = -\frac{1}{2}\Delta$, $H = H_o+W_o(Q)$.

Define $I(0,t,P) = 0$, $I(1,t,P) = \int_0^t d\tau W_o(\tau P),\ldots,I(m,t,P) = \int_0^t d\tau W_o(\tau P+\partial I(m-1,\tau,P)/\partial P)$ inductively; $X(m,t,P) = \frac{1}{2}tP^2 + I(m,t,P)$.

Buslaev-Matveev [25], Alsholm [26] have shown that given $\delta > 0$ there exists a m depending on δ such that if $Z_t = \exp(-iX(m,t,P))$ then $\Omega_\pm = \text{s-lim}_{t\to\pm\infty} V_t^* Z_t$ exist. While Buslaev-Matveev use stationary phase methods, Alsholm uses mean value theorem for operator valued functions.

Bertheir-Collet [27] prove, using stationary phase, existence of wave operators for $H = -\frac{1}{2}\Delta + T$ where T is a suitable pseudo differential operator which includes as a particular case $T = W_o(Q)$ with $\delta > \frac{1}{2}$.

Finally Hormander [28] has proved the existence of wave operators for a large class of pseudo differential operators. For a statement refer § 2.

Long range (completeness-eigenfunction methods). Spectral properties and completeness for $H = -\frac{1}{2}\Delta + W_o(Q)$ have been studied by Lavine [29], Kitada [30], Saito [31]. Agmon [32] has proved asymptotic completeness for a large class of operators of the form $H = h_o(P) + \sum_\alpha W_\alpha(Q)P^\alpha$ where $h_o(P)$ is an elliptic operator and the W_α's are long range smooth potentials.

Long range (algebraic theory) Amrein-Martin-Misra [33] have proved asymptotic completeness for $H = -\frac{1}{2}\Delta + k|Q|^{-\delta}$ for δ in $(\frac{3}{4},1]$, k constant. Thomas [34] has proved asymptotic completeness for $H = -\frac{1}{2}\Delta + W_o(Q)$ where W_o is spherically symmetric and is of $O(|x|^{-\delta})$ at ∞ for some $\delta > 0$.

Long range (completeness-time dependent theory): Enss [35,36] extended his ideas of [16] to prove absence of singular continuous spectrum and asymptotic completeness for $H = -\frac{1}{2}\Delta + W_o(Q)$ where $D^\alpha W_o(x) = O(|x|^{-|\alpha|-\delta})$ where $\delta > (2n+2)/(2n+3)$. He also proved that the positive eigenvalues of H, if any are of finite multiplicity and accumulate, possibly, only at zero.

Perry [37] proved the same results as of Enss [35] for $\delta > \frac{1}{2}$ when W_0 is assumed to be dilation analytic. His method consists in showing (i) given $\varphi \in C_0^\infty(0,\infty)$ there exist a,b (depending on φ) such that for every $\beta < \delta$

$$0 = \lim_{s\to\infty} \sup_{t \geq s} \| (V_{t-s} - Z_t Z_s^*) F(|Q-sP| < s^\beta) \varphi(H_0) F(PQ+QP \in (sa,sb)) \|$$

and (ii) $V_t^* U_t Q U_t^* V_t |t|^{-\gamma}$ converges to 0 strongly on $\mathcal{H}_c(H)$ for every $\gamma > 1-\delta$.

Muthuramalingam and Sinha have proved similar results in [38] without assuming dilation analyticity. They also have estimates similar to (i) of Perry.

All of [35, 37,38] use Alsohlm's construction of Z_t.

Recently Kitada-Yajima [39] have proved the above results for $\delta > 0$. Their method extends to time dependent Hamiltonians. While they achieve the maximum, their estimates are complicated. The method employed in this thesis are reasonably simple. However we are not able to get to $\delta > 0$ because the theory of asymptotic evolution of observables is in its infancy.

<u>Arrangement of the aricle</u>. In § 2, we take as in Hormander [28] $H = h_0(P) + \sum_\alpha W_\alpha(Q)P^\alpha$, $H_0 = h_0(P)$ where each W_α is a sum of a short range potential W_α^S and a smooth long range potential W_α^L with $D^\theta W_\alpha^L(x) = O(|x|^{-\delta-|\theta|})$ for all multiindices θ with $0 < \delta < 1$; $h_0 : R^n \to R$ is a polynomial such that $G = \{\xi : |\nabla h_0(\xi)| + |\det(\partial h_0/\partial\xi_i\, \partial\xi_j)| \neq 0\}$ is an open set whose complement has Lebesgue measure zero. Following [28], we solve the Hamilton Jacobi equation $\partial X(t,\xi)/\partial t = h_0(\xi) + W^L(\partial X(t,\xi)/\partial\xi, \xi)$ where $W^L(x,\xi) = \sum_\alpha W_\alpha^L(x)\xi^\alpha$. Because of our stronger assumptions on the decay of the W_α^L's we get stronger decay rates for $X(t,\xi)$ and all its derivatives, following [28]. Also the wave operators $\Omega_\pm = \text{s-}\lim_{t\to\pm\infty} V_t^* Z_t$, $V_t = \exp(-itH)$, $Z_t = \exp(-iX(t,P))$, $U_t = \exp(-itH_0)$ are introduced and its various properties stated.

In § 3, using the ideas of [20,21,22] we introduce the positive operator valued measure T on $R^n \times R^n$. It will help us to "sharply" localise in the momentum space and localise in the position space. Also we get some elementary but useful results about T.

In § 4, with H, H_o etc. as in § 2, we prove that $\Omega_\pm$ is "near" 1 in the distance future or past respectively. More precisely for $\varphi \in C_o^\infty(G)$, $\beta < (n + 4\delta)/(2n+4)$, a > 0

$$0 = \lim_{t \to \pm\infty} \| (\Omega_\pm Z^*_t U_t - 1) Z_t \varphi(P) F(|Q| \leq a|t|^\beta) \| .$$

Proof of this result depends on T of § 3, stationary phase method and the existence proof of wave operators in [28].

In § 5, we state a theorem of Kalf [40] on the absence of positive eigenvalues and apply it to our model.

In § 6, we take $H_o = -\frac{1}{2}\Delta$, $H = H_o + \Sigma W_j^L(Q)P_j + W_o^L(Q)$ and prove, following [23], that on $\mathcal{H}_c(H)$, V_t and U_t are indistinguishable; more precisely, for $\beta > 1 - \delta$, a > 0 we get $\text{s-}\lim_{t \to \pm\infty} F(|Q| \geq a|t|^\beta) U_t^* V_t = 0$ on $\mathcal{H}_c(H)$. This together with § 4 gives (i) absence of singular continuous spectrum and (ii) asymptotic completeness for H where $\delta > \frac{1}{2}$.

In § 7, we show that we can allow non-smooth W_j's (i.e. short range).

In the appendix we collect lot of results, mostly from [28], which were used in § 2 and § 4.

§ 2. SOLUTION OF A HAMILTON-JACOBI EQUATION AND THE EXISTENCE OF WAVE OPERATORS

Let $P = (P_1,P_2,\ldots,P_n)$, $P_j = -iD_j = -i\partial/\partial x_j$, $Q = (Q_1,\ldots,Q_n)$ be the self adjoint operators denoting the momentum and position on the Hilbert space $\mathcal{H} = L^2(R^n)$. Let $h_o : R^n \to R$ be a polynomial such that

$$G = \{\xi \in R^n : |\nabla h_o(\xi)| + |\det (\partial h_o(\xi)/\partial\xi_i\partial\xi_j)| \neq 0\} \tag{2.1}$$

is an open set whose complement in R^n has Lebesque measure zero. Define the free Hamiltonian H_o by

$$H_o = h_o(P) \tag{2.2}$$

Clearly H_o has only absolutely continuous spectrum.

Let $\alpha = (\alpha_1,\ldots,\alpha_n)$ be a multi-index with all the entries nonnegative integers; and length $|\alpha|$ given by $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. For a fixed $m \geq 0$, let W^S_α, W^L_α for $|\alpha| \leq m$, be real valued measurable functions on R^n satisfying the following properties.

There exists an $\varepsilon > 0$ and a positive integer N such that for all $|\alpha| \leq m$ the operator

$$W^S_\alpha(Q)(1+|Q|)^{1+\varepsilon}\,(1+|P|)^{-N} \text{ is compact} \tag{2.3}$$

For each $|\alpha| \leq m$, W^L_α is a C^∞ function and there exists a δ in $(0,1)$ such that for all multi-indices θ

$$|D^\theta W^L_\alpha(x)| \leq K_\theta\,(1+|x|)^{-|\theta|-\delta} \tag{2.4}$$

for suitable constants K_θ.

Throughout this thesis K will stand for a generic constant.

Let $H : S(R^n) \to L^2(R^n)$ be given by

$$H = H_o + \sum_{|\alpha|\leq m} W^L_\alpha(Q)P^\alpha + \sum_{|\alpha|\leq m} W^S_\alpha(Q)P^\alpha . \tag{2.5}$$

We make the following

Hypothesis 1 H has a self adjoint extension. We shall denote the extension also by the same letter H. Denote W^S, W^L by

and

$$\left.\begin{aligned} W^S(x,\xi) &= \sum_\alpha W^S_\alpha(x)\xi^\alpha \\ W^L(x,\xi) &= \sum_\alpha W^L_\alpha(x)\xi^\alpha \\ W^S(Q,P) &= \sum_\alpha W^S_\alpha(Q)P^\alpha \\ W^L(Q,P) &= \sum_\alpha W^L_\alpha(Q)P^\alpha \end{aligned}\right\} \quad (2.6)$$

Lemma 2.1 : Let C be a compact subset of G. If t_0^{-1} and ε_0 are sufficiently small, then the Hamilton's equations

$$\left.\begin{aligned} d\xi/dt &= -\,\partial W^L(x,\xi)/\partial x \,, \\ dx/dt &= (\partial h_0/\partial\xi) + \partial W^L(x,\xi)/\partial\xi \end{aligned}\right\} \quad (2.7)$$

have solutions for all $t \geq t_0$ with arbitrary Cauchy-data

where

$$\left.\begin{aligned} &\xi(t_0) = \eta\,, \quad x(t_0) = t_0\, y\,, \\ &\eta \in C \quad \text{and} \quad |y - h_0'(\eta)| < \varepsilon_0 \end{aligned}\right\} \quad (2.8)$$

Denote the solution by $\xi(t,y,\eta)$, $x(t,y,\eta)$. Then there exist constants K_θ, K such that

$$|D^\theta_{y,\eta}\{\xi(t,y,\eta)-\eta\}| \leq K_\theta\, t_0^{-\delta}, \quad |\theta| \leq 1\,, \quad (2.9)$$

$$|x(t,y,\eta)t^{-1} - h_0'(\xi(t,y,\eta))| \leq K\, t^{-\delta} \quad . \quad (2.10)$$

Moreover

$$|D^\theta_{y,\eta}\,(\xi(t,y,\eta),\, x(t,y,\eta)t^{-1})| \leq K_\theta\, t^{\mu(|\theta|)} \text{ for all } \theta, \quad (2.11)$$

where $\mu(0) = 0$, $\mu(k) = (k-1)\,\mu(2)$ for $k \geq 1$, $\mu(2)$ is arbitrary subject to $0 < \mu(2) < \min\{\tfrac{1}{2},\delta,1-\delta\}$.

<u>Proof</u> : We follow [28]. Refer [28] for the proof of (2.7), (2.8), (2.9), (2.10) and (2.11) for $|\theta| \leq 1$.

Let us now consider a derivative of order k of the Hamilton's equations with respect to the variables (y,η), assuming that $k > 1$ and that (2.11) has already been proved for derivatives of order $< k$. Let

$$\varphi(t,y,\xi) = (\nabla_x W^L)(ty,\xi)$$

so that for suitable constants K_θ

$$|D^\theta_{y,\xi}\, \varphi| \leq K_\theta\, t^{-1-\delta} \quad \text{for all } \theta.$$

Note that $\{\mu(k)\}$ is a convex sequence and therefore, by Lemma A.1, in the derivatives of order k of $\nabla_x W^L(x,\xi) = \varphi(t,x/t,\xi)$ w.r.t. (y,η) the terms where φ is differentiated at least twice can be bounded by t raised to the power

$$\max\{-1-\delta+\mu(k-1),-1-\delta\} = -1-\delta+\mu(k-1) \;. \tag{2.12}$$

So, for $|\theta| = k$

$$\frac{d}{dt} D^\theta_{y,\eta}\, \xi = D^\theta_{y,\eta} \frac{d\xi}{dt} = - D^\theta_{y,\eta} (\nabla_x W^L)(x,\xi) \quad \text{by (2.7)}$$

$$= - \varphi'_z(t,z,\xi) D^\theta_{y,\eta}(x/t) - \varphi'_\xi(t,z,\xi) D^\theta_{y,\eta}\xi + O(t^{-1-\delta+\mu(k-1)}). \tag{2.13}$$

Let

$$\psi(t,y,\xi) = (\nabla_\xi W^L)(ty,\xi)$$

so that

$$|D^\theta_{y,\xi}\, \psi| \leq K_\theta\, t^{-\delta} \quad \text{for all } \theta. \tag{2.14}$$

Now Lemma A.1 shows that in the derivatives of order k of $(\nabla_\xi W^L)(x,\xi) = \psi(t,x/t,\xi)$ w.r.t. (y,η), the terms where ψ is differentiated at least twice can be bounded by t raised to the power

$$\max\{-\delta+\mu(k-1),-\delta\} = -\delta+\mu(k-1) \;. \tag{2.15}$$

Thus, for $|\theta| = k$, as in the derivation of (2.13), using (2.15), we get

$$\frac{d}{dt} D^{\theta}_{y,\eta}\, x = D^{\theta}_{y,\eta} h_o'(\xi) + \psi_z'(t,z,\xi) D^{\theta}_{y,\eta}(x/t) + \psi_z'(t,z,\xi) D^{\theta}_{y,\eta}\xi + O(t^{-\delta+\mu(k-1)})$$

$$= h_o''(\xi) D^{\theta}_{y,\eta}\xi + O(t^{\mu(k-1)}) + \psi_z'(t,z,\xi) D^{\theta}_{y,\eta}(x/t) + \psi_\xi'(t,z,\xi) D^{\theta}_{y,\eta}\xi + O(t^{-\delta+\mu(k-1)})$$

$$= \{h_o''(\xi) + \psi_\xi'(t,z,\xi)\} D^{\theta}_{y,\eta}\xi + \psi_z'(t,z,\xi) D^{\theta}_{y,\eta}(x/t) + O(t^{\mu(k-1)}) \, . \qquad \ldots (2.16)$$

From (2.13) and (2.16) the k-th order derivatives of ξ, x w.r.t. (y,η) are bounded by solutions of differential inequalities:

$$d\Xi/dt \geq K(t^{-2-\delta} X + t^{-1-\delta}\,\Xi) + t^{-1-\delta+\mu(k-1)}$$

$$dX/dt \geq K(\Xi + t^{-1-\delta}\, X) + t^{\mu(k-1)} \, ,$$

which are bounds initially. Taking $\Xi = t^a K_1$, $X = t^b K_2$ with $a,b > 0$, we see that the above inequalities are satisfied iff

$$a - 1 \geq \max\{b-2-\delta, -1-\delta+a, \; -1-\delta+\mu(k-1)\}$$

and

$$b - 1 \geq \max\{a \, , \; -1-\delta+b, \; \mu(k-1)\} \, .$$

It is easily seen that for $a = \mu(k)$, $b = a+1$ the above inequalities are satisfied.

Q.E.D.

Using the above lemma we solve the Hamilton Jacobi equations in a natural way in

<u>Theorem 2.2</u> : There exists a C^ω function $X = X(t,\xi)$, $X: R \times G \to R$ such that for ξ in any compact subset of G, we have for large $|t|$

$$\partial X/\partial t = h_o(\xi) + W^L(\partial X/\partial\xi, \xi) \, , \qquad (2.17)$$

$$|D_\xi^\theta X(t,\xi)| \leq K_\theta \, t^{1+\mu(|\theta|-1)}, \quad |\theta| \geq 1 , \tag{2.18}$$

$$|D_\xi^\theta \{t^{-1} X(t,\xi) - h_o(\xi)\}| \leq K_\theta \, t^{\nu(|\theta|)} , \tag{2.19}$$

$$|D_\xi^\theta \{W^L(\partial X/\partial\xi, \xi)\}| \leq K_\theta \, t^{\nu(|\theta|)} , \tag{2.20}$$

where

$$\nu(k) = \mu(k) - \delta .$$

Proof : Similar to Theorem 3.8 of [28]. Q.E.D.

Let V_t, U_t, Z_t be the total, free, modified evolutions given by

$$V_t = \exp(-itH) ,$$

$$U_t = \exp(-itH_o) ,$$

$$Z_t = \exp(-iX(t,P)) .$$

Then we have the following

Theorem 2.3 The modified wave operators

$$\Omega_\pm = \underset{t \to \pm\infty}{\text{s-lim}} \; V_t^* Z_t \tag{2.21}$$

exist on $L^2(R^n)$. They are isometries and satisfy the intertwining relations

$$V_t \, \Omega_\pm = \Omega_\pm \, U_t . \tag{2.22}$$

Also,

$$\text{Range } \Omega_\pm \subset \mathcal{H}_{ac}(H) , \tag{2.23}$$

Further Z_t is feebly oscillating; i.e. for every real s

$$\underset{t \to \pm\infty}{\text{s-lim}} \; Z_{t+s}^* Z_t U_s = 1 \tag{2.24}$$

Proof : Refer [28]. Q.E.D.

Definition 2.4 : The system (H,H_0) is said to have

(i) weak asymptotic completeness if Range Ω_+ = Range Ω_- ,

(ii) asymptotic completeness if Range $\Omega_+ = \mathcal{H}_{ac}(H)$ = Range Ω_- ,

(iii) strong asymptotic completeness if Range $\Omega_+ = \mathcal{H}_c(H)$ = Range Ω_- .

Certainly strong asymptotic completeness is equivalent to asymptotic completeness together with $\mathcal{H}_{sc}(H) = \{0\}$.

The following Lemma on Range $\Omega_\pm$ follows from (2.21).

Lemma 2.5 (i) Range $\Omega_\pm$ is closed,

(ii) Range $\Omega_\pm = \{f \in \mathcal{H} : \underset{t \to \pm\infty}{\text{s-lim}} (\Omega_\pm Z_t^* U_t - 1)V_t f = 0\}$,

(iii) If $f \in$ Range $\Omega_\pm$, then for every $a > 0$ and every $\beta > 0$

$$\lim_{t \to \pm\infty} ||F(|Q| \geq a|t|^\beta) Z_t^* V_t f|| = 0 \qquad . \qquad (2.25)$$

A partial converse of (2.25) shall be proved in Corollary 4.7. To achieve this we require some technicalities presented in § 3.

The next Lemma shows that $f \in$ Range $\Omega_\pm$ iff "every H-part of f" is in Range $\Omega_\pm$.

Lemma 2.6 : Let φ_k be any sequence in $C_0^\infty(G)$ such that $0 \leq \varphi_k \leq 1$ and $\varphi_k^{-1}(1)$ increases to G. Let $f \in \mathcal{H}$. Then $f \in$ Range $\Omega_\pm$ iff there exists a sequence f_k in Range $\Omega_\pm$ such that $\underset{k \to \infty}{\text{s-lim}} f_k = f$ and for every k, $\underset{t \to \pm\infty}{\text{s-lim}} \{1-\varphi_k(P)\}V_t f_k = 0$.

Proof : Since Range $\Omega_\pm$ is closed, sufficiency is clear. Necessity part will be proved only for the positive sign. If $f \in$ Range Ω_+, define $f_k = \Omega_+ F(\varphi_k^{-1}(1)|P)g$ where $f = \Omega_+ g$, so that $\underset{k \to \infty}{\text{s-lim}} f_k = f$. Further

$$||\{1-\varphi_k(P)\}V_t\ f_k||$$

$$\leq ||\{1-\varphi_k(P)\}\{V_t\ f_k - Z_t\ F(\varphi_k^{-1}(1)|P)g\}|| + ||\{1-\varphi_k(P)\}F(\varphi_k^{-1}(1)|P)Z_t g||$$

$$\leq 2||V_t\ f_k - Z_t\ F(\varphi_k^{-1}(1)|P)g|| + 0\ .$$

Thus $\operatorname{s-lim}_{t \to \infty} (1-\varphi_k(P))\ V_t\ f_k = 0.$

Q.E.D.

§ 3. A POSITIVE OPERATOR VALUED MEASURE

Choose and fix c in (0, 1/3), $\eta \in S(R^n)$, such that $\hat{\eta}$-the Fourier transform of η defined by

$$\hat{\eta}(k) = (2\pi)^{-n/2} \int dx\, e^{-ixk}\, \eta(x)$$

has

$$\text{supp}\ \hat{\eta} \subset \{k \in R^n : |k| \leq c/8)\} \ . \tag{3.1}$$

We further normalize η as

$$||\eta||^2 = \int_{R^n} dx\, |\eta(x)|^2 = 1 \ . \tag{3.2}$$

Define, for $(x,k) \in R^n \times R^n$, η_{xk} by

$$\widehat{(\eta_{xk})}\ (p) = (\exp(+i\ x.p))\hat{\eta}(p-k) \tag{3.3}$$

so that

$$\eta_{xk}(y) = (\exp\ ix.(y+x))\eta(y-x) \ . \tag{3.4}$$

η_{xk} is called a generalized coherent state.

For any Borel subset M of $R^n \times R^n$, define an operator R(M) on $L^2(R^n)$ by the weak integral

$$T(M) = (2\pi)^{-n} \int_M dx\, dk < \eta_{xk}, > \eta_{xk} \tag{3.5}$$

where <,> denotes the inner product in $L^2(R^n)$, linear in the second variable. Then T is a positive operator valued measure defined on the Borel subsets of $R^n \times R^n$ so that for Borel subsets M_1, M_2 of $R^n \times R^n$

$$0 \leq T(M_1 \cup M_2) \leq T(M_1) + T(M_2) \ , \tag{3.6}$$

$$T(M_1 \cup M_2) = T(M_1) + T(M_2) \text{ if } M_1, M_2 \text{ are disjoint} \ , \tag{3.7}$$

$$0 \leq T(M_1) \leq T(M_2) \text{ if } M_1 \subset M_2 \ . \tag{3.8}$$

Furthermore, for any M,

$$0 \leq T^2(M) \leq T(M) \leq T(R^n \times R^n) = 1 \quad (3.9)$$

Of special interest is when $M = B_1 \times R^n$ or $M = R^n \times B_2$, B_1, B_2 Borel in R^n. In such a case, $T(B_1 \times R^n)$ $[T(R^n \times B_2)]$ is a multiplication operator in the position [momentum] space and is given by

$$T(B_1 \times R^n) = (\chi_{B_1} * |\eta|^2)(Q) , \quad (3.10)$$

$$T(R^n \times B_2) = (\chi_{B_2} * |\hat{\eta}|^2)(P) \quad (3.11)$$

All the above can be found in [20,21,22].

<u>Lemma 3.1</u> (i) For M_1, M_2 Borel subsets of $R^n \times R^n$ and $f \in \mathcal{H}$

$$||T(M_1 \cup M_2)f||^2 \leq (||T(M_1)f|| + ||T(M_2)f||)||f|| ,$$

(ii) If Y is any bounded operator on $\mathcal{H}$ then for every M_1, M_2

$$||T(M_1 \cup M_2)Y||^2 \leq (||T(M_1)Y|| + ||T(M_2)Y||)||Y||,$$

(iii) If $M_1 \subset M_2$ and $f \in \mathcal{H}$ then

$$||T(M_1)f||^2 \leq ||T(M_2)f|| \, ||f|| ,$$

(iv) For any bounded operator Y on $\mathcal{H}$ and $M_1 \subset M_2$

$$||T(M_1)Y||^2 \leq ||T(M_2)Y|| \, ||Y|| ,$$

(v) If φ is any bounded continuous real valued function on R^n such that dist (supp φ, B_2) > c/8 then for every B_1 in R^n

$$\varphi(P)\, T(B_1 \times B_2) = 0 = T(B_1 \times B_2)\varphi(P),$$

(vi) There exist constants K_o, $t_o \geq 1$ such that for $t \geq t_o$

$$F(|Q| \leq t) \leq (K_o/t) + T\{(x,k) : |x| \leq 2t\}$$

Proof (i) Follows from (3.6), (3.9) and Cauchy-Schwartz inequality.

(ii) Follows from (i).

(iii) Follows from (3.8), (3.9) and Cauchy-Schwartz inequality.

(iv) Follows from (iii)

(v) Case 1. Let $B_1 = R^n$. Then by (3.11) $T(R^n \times B_2)$ is a multiplication operator in the momentum space with support in a c/8 neighbourhood of B_2 and so the result is clear.

Case 2. Let B_1 be any Borel subset of R^n. Then for $f \in \mathcal{H}$, by (3.9), (3.8)

$$||T(B_1 \times B_2)\varphi(P)f||^2 \leq \langle T(R^n \times B_2)\varphi(P)f, \varphi(P)f\rangle = 0 \text{ by case (i)}.$$

So $T(B_1 \times B_2)\varphi(P) = 0$. Taking adjoints one gets $\varphi(P)T(B_1 \times B_2) = 0$.

(vi) By (3.10) there exist constants K_0, $t_0 \geq 1$ such that for all $t \geq t_0$

$$0 \leq T\{(x,k) : |x| \geq t\} \leq F(|Q| \geq t/2) + (K_0/t) \qquad (3.12)$$

$$\begin{aligned} F(|Q| &\leq t/2) \\ &= 1-F(|Q| \geq t/2) \\ &\leq 1-T\{(x,k) : |x| \geq t\} + (K_0/t) && \text{by (3.12)} \\ &= T\{(x,k) : |x| \leq t\} + (K_0/t) && \text{by (3.7) and (3.9).} \end{aligned}$$

Q.E.D.

Corollary 3.2 : Let Y_k be a norm bounded sequence of operators; $M_j(k)$ Borel subset of $R^n \times R^n$ for $k = 1,2,\ldots$, $j = 1,2,\ldots,r$. If $\lim_{k \to \infty} ||Y_k T(M_j(k))|| = 0$ for $j = 1,2,\ldots,r$ then for any sequence $M(k)$ such that $M(k) \subseteq \bigcup_{j=1}^{r} M_j(k)$ we have $\lim_{k \to \infty} ||Y_k T(M(k))|| = 0$.

Proof Follows by noting $||A|| = ||A^*||$ for any operator A and using Lemma 3.1 (ii), (iv).

Q.E.D.

§ 4. REDUCTION TO THE EVOLUTION OF ASYMPTOTIC OBSERVABLES.

<u>Lemma 4.1</u> (i) (Interpolation) Let $\mathcal{H}$ be any Hilbert space, $J:\mathcal{H} \to \mathcal{H}$ a bounded operator; A,B positive self adjoint operators (not necessarily bounded) on $\mathcal{H}$. Let for two numbers a,b with $0 \leq a \leq b$, $B^a JA^{-a}$ and $B^b JA^{-b}$ be bounded. Then for every c in [a,b] the operator $B^c JA^{-c}$ is bounded and $||B^c JA^{-c}|| \leq ||B^a JA^{-a}||^{(b-c)/(b-a)} ||B^b JA^{-b}||^{(c-a)/(b-a)}$

(ii) Let $\varphi \in S(R^n)$ and M an integer ≥ 0. Then for every $\gamma \in [0,1]$, $(1+P^2)^M (1+Q^2)^{-\gamma} \varphi(P)(1+Q^2)^{\gamma}$ is a bounded operator.

<u>Proof</u> (i) Similar to Proposition 9, page 44 [5].

(ii) <u>Step (i)</u> M = 0. By interpolation, it is enough to prove the result for $\gamma = 0$ and $\gamma = 1$. For $\gamma = 0$ it is trivial; for $\gamma = 1$ write $(1+Q^2)^{-1} \varphi(P)(1+Q^2) = \varphi(P) + (1+Q^2)^{-1} [\varphi(P),Q^2]$ and use the commutation relations between P and Q.

<u>Step (ii)</u> For arbitrary M expand $(1+P^2)^M$ by Binomial Theorem, use commutation rules, the result for M = 0 and the fact that if $\varphi \in S(R^n)$ then for every polynomial f, $f\varphi \in S(R^n)$.

Q.E.D.

<u>Lemma 4.2</u> (Stationary phase) Let G, h_o be as in § 2. Let $\varphi \in C_o^\infty(G)$ have support in the compact set C and G_1 any open set containing $\{\nabla h_o(\xi) : \xi \in C\}$. Then given any integer $M \geq 0$ there exists a constant K_M (independent of t,φ) such that

$$|\int d\xi\, \varphi(\xi) \exp(i[x.\xi - th_o(\xi)])|$$

$$\leq K_M(1+|x| + |t|)^{-M} \sum_{|\theta|\leq M} ||D^\theta \varphi||_\infty \text{ whenever } x/t \notin G_1 .$$

Proof Similar to Lemma 2, page 336 of [6]. Q.E.D.

Let E be any subset of R^n and $b > 0$. Then $\{x:\text{dist}(x,E) < b\}$ will be called the b neighbourhood of E and $\{x:\text{dist}(x,E) \leq b\}$ will be called the closed b neighbourhood of E.

Lemma 4.3 Let Z_t, X, G, h_o be as in § 2; T as in § 3. Let $B(t) = \{x \in R^n: |x| \leq t\}$ be the closed ball of radius t with origin as center; let E be a bounded subset of G such that for some c in (0,1/3) the c neighbourhood of E is in G and in that neighbourhood $\xi \to \nabla h_o(\xi)$ is a diffeomorphism. Let $\varphi \in C^\infty(G)$. Then there exists an $a > 0$ such that

(i) $$\lim_{s\to\infty} \int_0^\infty dt\,||(1+|Q|)^{-1-\varepsilon}\varphi(P)Z_{t+s}\,T(B(as)\times E)|| = 0$$

(ii) $$\lim_{s\to\infty} \int_0^\infty dt\,||W^s(Q,P)\varphi(P)Z_{t+s}\,T(B(as)\times E)|| = 0$$

Proof Let $\psi \in C_o^\infty(G)$ be such that $\psi = 1$ in the 3c/(16) neighbourhood of E and 0 out side the 7c/(32) neighbourhood of E. Then by Lemma 3.1(v), $\{1-\psi(P)\}T(B(as)\times E) = 0$ for all $a > 0$ so that $\varphi(P)T(B(as)\times E) = \varphi(P)\psi(P)T(B(as)\times E)$. So replacing φ by $\varphi\psi$ if necessary, we can, and shall, assume that φ vanishes out side the 7c/(32) neighbourhood of E.

(i) The result follows if, for some $b > 0$,

$$\lim_{s\to\infty} \int_0^\infty dt\,||F(|Q| \leq b(t+s))\varphi(P)Z_{t+s}\,T(B(as)\times E)|| = 0$$

Choose $4a = 4b = \inf\{|\nabla h_o(\xi)|:\text{dist}(\xi,E) \leq c/4\}$. By (2.19) there exists $s_o \geq 0$ such that for all $s \geq s_o$, $t \geq 0$ and all ξ in the closed c/4 neighbourhood of E

$$|D_\xi^\theta\{X(t+s,\xi)-(t+s)h_o(\xi)\}| \leq K_\theta(t+s)^{1+\nu(|\theta|)} \text{ for all } \theta. \quad (4.1)$$

Take $C = \{\xi : \mathrm{dist}(\xi, \mathcal{E}) \le c/8\}$

$G_1 = \{y : |y| > 7b/2\}$

and apply stationary phase Lemma 4.2 so that whenever $|Q+x| \le 2b(t+s)$ with $t \ge 0$, $s \ge s_0$ and $k \notin E$

$$|\int d\xi\ \varphi(\xi)\hat{\eta}(\xi-k)\exp(i[(Q+x)\cdot\xi - X(t+s,\xi)])|$$

$$\le K_M(1+|Q+x|+|t+s|)^{-M} \sum_{|\theta|\le M} ||D_\xi^\theta\{\varphi(\xi)\hat{\eta}(\xi-k)\exp(i[(t+s)h_0(\xi)-X(t+s,\xi)])\}||_\infty$$

$$\le K_M(t+s)^{-M\delta} \qquad \text{by (4.1) and Lemma A.2 .} \qquad (4.2)$$

Now as in [20], denoting by $|Y|$ the Lebesgue measure of Y,

$$||F(|Q| \le b(t+s))\varphi(P)Z_{t+s}\ T(B(as) \times E)||$$

$$\le |B(b(t+s))|^{\frac{1}{2}} \sup_{|Q| \le b(t+s)} \int\int_{B(as)\times E} dx\ dk\ |I(Q,x,k,t+s)|$$

where

$$I(Q,x,k,t+s) = \int d\xi\ \varphi(\xi)\hat{\eta}(\xi-k)\exp(i[(Q+x)\cdot\xi - X(t+s,\xi)]).$$

Using (4.2)

$$||F(|Q| \le b(t+s))\varphi(P)Z_{t+s}\ T(B(as) \times E)||$$

$$\le K_M(t+s)^{\frac{1}{2}n} s^n (t+s)^{-M\delta} \quad \text{for every } M.$$

The result follows by choosing M large enough.

(ii) We deduce (ii) from (i) using Lemma 4.1 (ii). Let $|\alpha| \le m$, J any bounded operator and $\psi \in C_0^\infty(G)$ be such that $\psi = 1$ on supp φ. Then

$$||W_\alpha^S(Q)P^\alpha\varphi(P)J||$$

$$\le K||W_\alpha^S(Q)(1+|Q|)^{1+\varepsilon}(1+|P|)^{-N}||\cdot||(1+|P|)^N(1+Q^2)^{-\frac{1}{2}(1+\varepsilon)}P^\alpha\psi(P)(1+|Q|)^{1+\varepsilon}||.$$

$$||(1+|Q|)^{-1-\varepsilon}\varphi(P)J||$$

$$\le K||(1+|Q|)^{-1-\varepsilon}\varphi(P)J|| \quad \text{by (2.3) and Lemma 4.1 (ii).}$$

The above inequality yields

$$||W^S(Q,P)\varphi(P)Z_{t+s}\ T(B(as) \times E)||$$

$$\leq K||(1+|Q|)^{-1-\varepsilon}\varphi(P)Z_{t+s}\ T(B(as) \times E)||$$

and the result follows from (i).

Q.E.D.

Lemma 4.4 Let $\varphi, E, B(t), T$ be as in Lemma 4.3. Then for every $a > 0$ and for every β in $(0,(n+4\delta)/(2n+4))$

$$\lim_{s \to \infty} \int_0^\infty dt||\{W^L(Q,P)-W^L(X_\xi^!(t+s,P),P)\}\varphi(P)Z_{t+s}\ T(B(as^\beta) \times E)|| = 0$$

Proof The proof closely follows the existence proof of wave operators in [28].

As in Lemma 4.3 we can assume that φ vanishes outside the $7c/(32)$ neighbourhood of E. Choose, by Theorem 2.2, $s_o \geq 1$ such that for all $s \geq s_o$, $t \geq 0$ and ξ in the closed $3c/4$ neighbourhood of E

$$\partial X(t+s,\xi)/\partial t = h_o(\xi) + W^L(X_\xi^!(t+s,\xi),\xi) \quad , \tag{4.3}$$

$$|D_\xi^\theta X(t+s,\xi)| \leq K_\theta(t+s)^{1+\mu(|\theta|-1)} \quad \text{for} \quad |\theta| \geq 1 \ , \tag{4.4}$$

$$|D_\xi^\theta\{(t+s)^{-1} X(t+s,\xi)-h_o(\xi)\}| \leq K_\theta(t+s)^{\nu(|\theta|)} \quad , \tag{4.5}$$

and

$$|D_\xi^\theta W^L(X_\xi^!(t+s,\xi),\xi)| \leq K_\theta (t+s)^{\nu(|\theta|)} \ . \tag{4.6}$$

Unless otherwise specified, in what follows, we always assume $t \geq 0$, $s \geq s_o$, $k \in E$ and $|x| \leq as^\beta$.

Let $f \in L^2(R^n)$ with $||f|| = 1$. Then

$$[\{W^L(Q,P)-W^L(X_\xi^!(t+s,P),P)\}\varphi(P)Z_{t+s}\ T(B(as^\beta) \times E)f](Q)$$

$$= \int\int_{B(as^\beta)\times E} dx\, dk\ \langle\eta_{xk}|f\rangle\ I(Q,x,k,t+s) \tag{4.7}$$

where

$I(Q,x,k,t+s)$

$$= \int d\xi \{W^L(Q,\xi)-W^L(X_\xi'(t+s,\xi),\xi)\}\varphi(\xi)\hat{\eta}(\xi-k)\exp(i[(Q+x).\xi-X(t+s,\xi)]). \quad (4.8)$$

Further for all $0 < b_1 < b_2 < \infty$

$$1 = F(|Q+x|/(t+s)\notin [b_1,b_2]) + F(|Q+x|/(t+s)\in [b_1,b_2]) \quad (4.9)$$

where F stands for the indicator function.

We shall estimate the norm of L.H.S. of (4.7) by using the integrals on R.H.S. For this purpose following [28], we shall split the space R^n into various regions and make computations in each region. (4.9) is only the beginning of such splitting. Since the proof of this Lemma is rather lengthy, we present it in nine steps.

Step (i) ((Q+x)/(t+s) is out side an annulus). Choose

$$8b_1 = \inf\{|h_0'(\xi)|:\mathrm{dist}(\xi,E) \le c/4\}$$

$$b_2 = 8\sup\{|h_0'(\xi)|:\mathrm{dist}(\xi,E) \le c/4\}$$

$$C = \{\xi : \mathrm{dist}(\xi,E) \le c/4\}$$

$$G_1 = \{y : 2b_1 < |y| < \tfrac{1}{2} b_2\}$$

and apply stationary phase Lemma 4.2 so that whenever $k \in E$ and $|Q+x|/(t+s) \notin [b_1,b_2]$, one gets

$|I(Q,x,k,t+s)|$

$$\le K_M\{1+|Q+x|+(t+s)\}^{-M} \sum_{|\theta|\le M} ||D_\xi^\theta\{W^L(Q,\xi)\varphi(\xi)\hat{\eta}(\xi-k)\exp(i[(t+s)h_o(\xi)-X(t+s,\xi)])\}||_\infty$$

$$K_M\{1+|Q+x|+(t+s)\}^{-M} \sum_{|\theta|\le M} ||D_\xi^\theta\{W^L(X_\xi'(t+s,\xi),\xi)\varphi(\xi)\hat{\eta}(\xi-k)\cdot \exp(i[(t+s)h_o(\xi)-X(t+s,\xi)])\}||_\infty$$

$\le K_M\{1+|Q+x|+(t+s)\}^{-M}\{(t+s)^{M(1-\delta)} + (t+s)^a\}$ where by Leibnitz rule, (4.6) and Lemma A.2

$$a = \max\{\nu(j)+k(1-\delta) : 0 \le j + k \le M\}$$

$$= -\delta + M(1-\delta).$$

Or in other words,

$$F(|Q+x|/(t+s) \notin [b_1,b_2])F(k \in E)|I(Q,x,k,t+s)|$$

$$\le K_M\{1+|Q+x|+(t+s)\}^{-M} (t+s)^{M(1-\delta)} F(|Q+x|/(t+s) \notin [b_1,b_2])F(k \in E) \cdot \quad (4.10)$$

<u>Step (ii)</u> (A partition of unity) Let λ be a lattice point in R^n, i.e., all the n coordinates of λ are integers and let Λ be the set of all lattice points of R^n. Let $\chi_0 \in C_0^\infty(R^n)$, $\chi_0 \ge 0$, $\chi_0(0) > 0$, supp $\chi_0 \subset \{y : |y| \le n\}$ be such that if χ_λ is defined by $\chi_\lambda(x) = \chi_0(x-\lambda)$ then $\Sigma\{\chi_\lambda(x) : \lambda \in \Lambda\} = 1$ for each x, i.e., $\{\chi_\lambda : \lambda \in \Lambda\}$ is a partition of unity for the space R^n. Choose σ in $(0,\frac{1}{2})$, $\lambda \in \Lambda$, $L > 0$, $t > 0$ and define

$$J(\sigma,\lambda,t,L) = \{x \in R^n : |x|/t \in [b_1,b_2], |x - X_\xi^!(t,\lambda t^{-\sigma})| > Lt^{1-\sigma}\} ,$$

$$J^*(\sigma,\lambda,t,L) = \{x \in R^n : |x|/t \in [b_1,b_2]\} \sim J(\sigma,\lambda,t,L),$$

$$A(\sigma,t,L) = \bigcap \{J(\sigma,\lambda,t,L) : \lambda \in \Lambda\} ,$$

$$A^*(\sigma,t,L) = \bigcup \{J^*(\sigma,\lambda,t,L) : \lambda \in \Lambda\}.$$

Since for every fixed σ,λ,t,L the sets J, J^* form a partition of the annulus $[tb_1,tb_2]$ (in R^n) the same is true for sets A and A^*. Now

$$F(|Q+x|/(t+s) \in [b_1,b_2])\, I(Q,x,k,t+s) = \text{first term} + \text{second term} \quad (4.11)$$

where first term

$$= \sum_{\lambda \in \Lambda} \int d\xi\, F(Q+x \in A(\sigma,t+s,L))(t+s)^{-n\sigma} \chi_\lambda(\xi)\varphi(\xi(t+s)^{-\sigma})\hat{\eta}(-k+\xi(t+s)^{-\sigma}) \cdot$$
$$\{W^L(Q,\xi(t+s)^{-\sigma}) - W^L(X_\xi^!(t+s,\xi(t+s)^{-\sigma}),\xi(t+s)^{-\sigma})\} \cdot$$
$$\exp(i(t+s)^{1-2\sigma}[(Q+x)\xi t^{\sigma-1} - t^{2\sigma-1} X(t+s,\xi(t+s)^{-\sigma})])$$

$$= \sum_{\lambda \in \Lambda} \int d\xi\, F(Q+x \in A(\sigma,t+s,L))(t+s)^{-n\sigma} \mathcal{J}(\lambda,\xi,\sigma,k,Q,x,t+s)$$

and second term

$$= F(Q+x \in A^*(\sigma,t+s,L))\, I(Q,x,k,t+s) .$$

Step 3 ((Q+x)/(t+s) is inside the annulus $[b_1,b_2]$ but away from critical points). We analyse the first summand of R.H.S. of (4.11). Now $\chi_\lambda(\xi)\varphi(\xi(t+s)^{-\sigma}) \not\equiv 0$ only if $\lambda(t+s)^{-\sigma}$ is in a fixed neighbourhood of support φ. Thus the sum is over only $K(t+s)^{n\sigma}$ terms, where K depends only on the volume of supp φ.

We can choose L large so that whenever $\chi_\lambda(\xi)\varphi(\xi(t+s)^{-\sigma}) \not\equiv 0$ and $Q+x \notin J(\sigma,\lambda,t+s)$ then by (4.4), $|Q+x-X_\xi^!(t+s,\xi(t+s)^{-\sigma})| \geq \frac{1}{2} L(t+s)^{1-\sigma}$. Now we can apply Lemma A.3, to get, for all $M \geq 1$

$$|\int d\xi\, F(Q+x \notin J(\sigma,\lambda,t+s))\, \mathcal{J}(\lambda,\xi,\sigma,k,Q,x,t+s)|$$

$$\leq \quad K_M(t+s)^{-M(1-2\sigma)}\, F(|Q+x|/(t+s) \in [b_1,b_2]).$$

So modulus of the first term of R.H.S. of (4.11)

$$\leq \quad K_M(t+s)^{-M(1-2\sigma)}\, F(|Q+x|/(t+s) \in [b_1,b_2]) \,. \tag{4.12}$$

Step 4 (A finiteness condition for the second term of (4.11) and an expansion) By choosing s_0 large, if necessary, we can assume by (4.5) that $\xi \to (t+s)^{-1} X_\xi^!(t+s,\xi)$ is a diffeomorphism on $\{\xi:\text{dist}(\xi,E) < 15c/(16)\}$; and on this set $|(t+s)^{-1}\{X_\xi^!(t+s,\xi_1)-X_\xi^!(t+s,\xi_2)\}|$ behaves like $K|\xi_1-\xi_2|$ with K independent of t+s; and the range contains $\{h_0^!(\xi):\text{dist}(\xi,E) < 7c/8\}$ for all $t \geq 0$, $s \geq s_0$. If $Q+x \in A^*(\sigma,t+s,L)$ then, for some λ, $|Q+x-X_\xi^!(t+s,\lambda(t+s)^{-\sigma})| \leq L(t+s)^{1-\sigma}$. Because of the presence of $\hat{\eta}(\cdot-k)$ in the integral, we can always assume that $\text{dist}(\lambda(t+s)^{-\sigma},E) \leq 3c/(16)$. So, we can find a unique ξ_0 with $\text{dist}(\xi_0,E) \leq 7c/(32)$ such that $Q+x = X_\xi^!(t+s,\xi_0)$. Since $\xi \to (t+s)^{-1} X_\xi^!(t+s,\xi)$ is a diffeomorphorm it follows that $|\lambda(t+s)^{-\sigma}-\xi_0| \leq KL(t+s)^{-\sigma}$. Thus λ has at the most Γ choices where Γ is independent of Q+x, t+s. So, second term of R.H.S. of (4.11)

$$\leq \Gamma \sup_{\lambda} \Big| \int d\xi \;\; F(Q+x \in J^{*}(\sigma,\lambda,t+s,L))\varphi(\xi)\hat{\eta}(\xi-k) \cdot$$

$$\{W^{L}(Q,\xi)-W^{L}(X_{\xi}'(t+s,\xi),\xi)\}\exp(i[(Q+x)\cdot\xi-X(t+s,\xi)])\Big| \cdot \quad (4.12a)$$

Now for $Q+x \in A^{*}(\sigma,t+s,L)$ choose ξ_o,λ as above with $Q+x = X_{\xi}'(t+s,\xi_o)$. Now apply Lemma A.6 to the function $\xi \to \{(Q+x)\xi-X(t+s,\xi)\}/(t+s)$ to get a diffeomorphism ψ in a neighbourhood of 0 such that

$$\left.\begin{aligned}
&\psi(0) = \xi_o \\
&\{(Q+x)\psi(y)-X(t+s,\psi(y))\}/(t+s) \\
&= [\{(Q+x)\xi_o - X(t+s,\xi_o)\}/(t+s)] + \langle \mathcal{A}y,y\rangle/2 \\
&\text{where} \\
&\mathcal{A} = X_{\xi}''(t+s,\xi_o)/(t+s) \cdot
\end{aligned}\right\} \quad (4.13)$$

$\mathcal{A}$ is a metrix depending on $t+s,\xi_o$ but under the assumptions on t,s and ξ_o, $\mathcal{A}$ varies only over a compact subset of $GL(n,R)$ - the set of all $n \times n$ invertible matrices with real entries.

Further by Lemma A.6 we get

$$|(D^{\theta}\psi)(0)| \leq K_{\theta}(t+s)^{\mu(|\theta|)} \quad . \qquad (4.14)$$

If we define

$$g(y) = |\det \psi'(y)|, \qquad (4.15)$$

then

$$|(D^{\theta}g)(0)| \leq K_{\theta}(t+s)^{\mu(|\theta|+1)} \quad . \qquad (4.16)$$

Define

$$v(k,y) = \varphi(\psi(y))\hat{\eta}(\psi(y)-k) \quad , \qquad (4.17)$$

and

$$w(t+s,k,y) = W^{L}(X_{\xi}'(t+s,\psi(y)),\psi(y))v(k,y) \; . \qquad (4.18)$$

Now applying Lemma A.5 and then Lemma A.4.

$$|\int d\xi\, F(Q+x \in J^*(\sigma,\lambda,t+s,L))\varphi(\xi)\hat{\eta}(\xi-k)\cdot$$

$$\{W^L(Q,\xi)-W^L(X_\xi^!(t+s,\xi),\xi)\}\exp(i[(Q+x)\xi-X(t+s,\xi)])|$$

$$\leq K_{M,p}(t+s)^{-M-\frac{1}{2}n} \sum_{|\alpha|\leq m} ||\xi^\alpha\, \varphi(\xi)\hat{\eta}(\xi-k)||_{p,\infty}|W_\alpha^L(Q)| +$$

$$K_{M,p}(t+s)^{-M-\frac{1}{2}n}\, ||W^L(X_\xi^!(t+s,\xi),\xi)\varphi(\xi)\hat{\eta}(\xi-k)||_{p,\infty} +$$

$$K_0 \sum_{|\alpha|\leq m} |\sum_{j=1}^{M-1} (t+s)^{-j-\frac{1}{2}n}\, \Gamma_j\{<\mathbb{A}^{-1} D,D>_*^j\, \psi^\alpha\, v\, g\}(0)|\,|W_\alpha^L(Q)|F(Q+x \in J^*(\ldots)) +$$

$$K_0\, |\sum_{j=1}^{M-1} (t+s)^{-j-\frac{1}{2}n}\, \Gamma_j\{<\mathbb{A}^{-1}D,D>_*^j\, w\, g\}(0)|F(Q+x \in J^*(\ldots)) +$$

$$K_0\, |\sum_{j=0}^{M-1} (t+s)^{-j-\frac{1}{2}n}\, \Gamma_j\{<\mathbb{A}^{-1}D,D>^j\, g\}(0)|\;\; |W^L(Q,\xi_0)-W^L(Q+x,\xi_0)|F(Q+x \in J^*(\ldots)) \cdot \quad \ldots\ldots (4.19)$$

where (i) for any function b which also depends on ξ, $||b||_{p,\infty}= \sum_{|\theta|\leq p} \sup\{|D_\xi^\theta b|:\xi\}$,

(ii) $p > 2M + \frac{1}{2}n$ (iii) $K_{M,p}$, Γ_j are all bounded functions of $t+s,\xi_0$,

(iv) $\psi^\alpha = \psi_1^{\alpha_1} \ldots \psi_n^{\alpha_n}$ where $\psi = (\psi_1,\ldots,\psi_n)$ and $\alpha = (\alpha_1,\ldots\alpha_n)$ (v) $<,>_*$ denotes that ψ^α v of $\psi^\alpha vg$(or w of w g) is differentiated at least once.

<u>Step 5</u> (first and second terms of R.H.S. of (4.19)). By (4.6), taking $p = 2M+n$ with $M \geq 1$

sum of the first two terms of R.H.S. of (4.19)

$$\leq K_M(t+s)^{-M-\frac{1}{2}n+(2M+n)\mu(2)} \quad (4.20)$$

<u>Step 6</u> (Third term of R.H.S. of (4.19)) Since by our assumptions $|x| \leq as^\beta$ with $\beta < 1$ and presently $|Q+x| \geq b_1(t+s)$ one gets $|Q| \geq \frac{1}{2} b_1(t+s)$. So

$$|W_\alpha^L(Q)|F(Q+x \in J^*(\ldots)) \leq K(t+s)^{-\delta} \quad (4.21)$$

Further for all α with $|\alpha| \leq m$ and all $\theta \neq 0$ by Lemma A.1 and (4.14),

$$|(D^{\theta}\psi^{\alpha}v)(0)| \leq K_{\theta}(t+s)^{\mu(|\theta|)}, \tag{4.22}$$

Using Leibnitz rule for differentiation, (4.16) and (4.22)

$$|(<\mathcal{A}^{-1}D,D>^{j}_{*}\psi^{\alpha} v g)(0)|$$

$$\leq K_j(t+s)^{\max\{\mu(a)+\mu(b+1)\ :\ a \geq 1,\ a+b = 2j\}}$$

$$= K_j(t+s)^{\mu(2j)} \quad \text{for} \quad j \geq 1 . \tag{4.23}$$

Third term of R.H.S. of (4.19)

$$\leq K_M(t+s)^{\max\{-j-\delta-\frac{1}{2}n+\mu(2j):\ 1\leq j\leq M-1\}}$$

$$\leq K_M(t+s)^{-1-\delta+\mu(2)-\frac{1}{2}n} F(Q+x \in J^*(\cdot,\cdot)) \text{ if } M \geq 2 . \tag{4.24}$$

Step 7 (Fourth term of R.H.S. of (4.19)) As in step 6, fourth term of R.H.S. of (4.19)

$$\leq K_M(t+s)^{-1-\delta+\mu(2)-\frac{1}{2}n} F(Q+x \in J^*(\cdot,\cdot)) \text{ if } M \geq 2. \tag{4.25}$$

Step 8 (fifth term of R.H.S. of (4.19)) As in step 6 we get for all r in [0,1], $|Q+rx| \geq \frac{1}{2} b_1(t+s)$. So,

$$|W^L(Q+x,\xi_0) - W^L(Q,\xi_0)|$$

$$\leq \sum_{|\alpha|\leq m} |\xi_0|^{\alpha} |W^L_{\alpha}(Q+x) - W^L_{\alpha}(Q)|$$

$$\leq \sum_{|\alpha|\leq m} |\xi_0|^{\alpha} |x| \sup\{|(\nabla W^L_{\alpha})(Q+rx)| : 0 \leq r \leq 1\}$$

$$\leq K |x| (t+s)^{-1-\delta} . \tag{4.26}$$

By (4.26) and (4.16) we see that

last term of R.H.S. of (4.19)

$$\leq K_M|x|(t+s)^{-1-\delta-\frac{1}{2}n} F(Q+x \in J^*(\cdot,\cdot,\cdot)) . \tag{4.27}$$

Step 9 (Estimation of the norm in the Lemma) : Collecting (4.10), (4.12), (4,12a) (4.20), (4.24), (4.25) and (4.27) we have for all $|x| \leq as^{\beta}$, $k \in E$ and $M \geq 2$

$$
\begin{aligned}
&|I(Q,x,k,t+s)| \\
&\leq K_M\{1+|Q+x|+(t+s)\}^{-M}(t+s)^{M(1-\delta)} F(|Q+x|/(t+s) \notin [b_1,b_2]) + \\
&\quad K_M(t+s)^{-M(1-2\sigma)} F(|Q+x|/(t+s) \in [b_1,b_2]) + \\
&\quad K_M(t+s)^{-M-\frac{1}{2}n+(2M+n)\mu(2)} F(|Q+x|/(t+s) \in [b_1,b_2]) + \\
&\quad K_M(t+s)^{-1-\delta-\frac{1}{2}n}\{(t+s)^{\mu(2)}+|x|\} F(|Q+x-z_o| \leq L(t+s)^{1-\sigma}) \qquad (4.28)
\end{aligned}
$$

where $z_o = X_{\xi}^{\prime}(t+s, \lambda(t+s)^{-\sigma})$ for some λ.

Applying Cauchy's inequality and then a change of variable, one gets

$$
\begin{aligned}
&\int dQ \{ \int_{B(as^{\beta})\times E} dx\, dk |\langle\eta_{xk}|f\rangle| F(|Q+x-z_o| \leq L(t+s)^{1-\sigma}\}^2 \\
&\leq \int dQ \int_{B(as^{\beta})\times E} dx\, dk\, F(|Q+x-z_o| \leq L(t+s)^{1-\sigma}) \cdot \int dx\, dk\, |\langle\eta_{xk}|f\rangle|^2 \\
&= \int dQ\, F(|Q-z_o| \leq L(t+s)^{1-\sigma}) \int_{B(as^{\beta})} dx \int_E dk \quad \text{by the change of variable } (Q,x) \to (Q+x,x) \\
&\leq K(t+s)^{n(1-\sigma)} s^{n\beta} . \qquad (4.29)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int dQ\{ \int_{B(as^{\beta})\times E} dx\, dk |\langle\eta_{xk}|f\rangle|\, |x|\, F(|Q+x-z_o| \leq L(t+s)^{1-\sigma}\}^2 \\
&\leq K(t+s)^{n(1-\sigma)} s^{(n+2)\beta} . \qquad (4.30)
\end{aligned}
$$

By choosing M large enough in (4.28) and using (4.29), (4.30) and (4.7), (4.8) we have

$$||\{W^L(Q,P)-W^L(X_\xi^!(t+s,P),P)\}\varphi(P)Z_{t+s}\ T(B(as^\beta)\times E)||$$

$$\leq \quad K\{t+s)^{-2} + (t+s)^b\}$$

where

$$b = -1-\delta-\tfrac{1}{2}n + \tfrac{1}{2}n(1-\sigma) + \max\{\mu(2) + \tfrac{1}{2}n\beta,\ \tfrac{1}{2}(n+2)\beta\}\ .$$

The result follows if $b < -1$. This can be achieved when $\beta < (n+4\delta)/(2n+4)$ by choosing σ close to $\frac{1}{2}$, $\sigma < \frac{1}{2}$ and $\mu(2)$ near 0, $\mu(2) > 0$.

Q.E.D.

Lemma 4.5 Let φ,β,E be as in Lemma 4.4. Then

$$\lim_{s\to\pm\infty} ||(\Omega_\pm\ Z_s^* U_s - 1)Z_s\varphi(P)T(B(a|s|^\beta)\ \times\ E)|| = 0\ .$$

Proof We prove only for the positive sign. As in Lemma 4.3 and 4.4 we can assume that $\varphi \in C_0^\infty(G)$. In such a case for s large and $t \geq 0$, by (2.17)

$$\frac{d}{dt}\{V_{t+s}^*\ Z_{t+s}\varphi(P)\} = i\ V_{t+s}^*\{W^S(Q,P)+W^L(Q,P)-W^L(X_\xi^!(t+s,P),P)\}Z_{t+s}\varphi(P)\ .$$

This together with Lemma 4.3 and 4.4 yields

$$\lim_{s\to\infty}\ \sup_{t\geq 0}\ ||(V_{t+s}^*\ Z_{t+s} - V_s^* Z_s)\varphi(P)T(B(as^\beta)\ \times\ E)|| = 0$$

so that

$$\lim_{s\to\infty} ||(\Omega_+ - V_s^* Z_s)\varphi(P)T(B(as^\beta)\ \times\ E)|| = 0$$

Since $V_s\ \Omega_+ = \Omega_+\ U_s$ the result follows.

Q.E.D.

Theorem 4.6 Let $\varphi \in C_0^\infty(G)$, $\beta \in (0,(n+4\delta)/(2n+4))$ and $a > 0$. Then

$$\lim_{t\to\pm\infty} ||(\Omega_\pm U_t - Z_t)\varphi(P)F(|Q|\leq a|t|^\beta)|| = 0.$$

Proof We prove only for the positive sign and write Ω for Ω_+. Since supp φ is compact we can find (E_1,c_1), (E_2,c_2), ... (E_r,c_r) such that E_j is bounded, open, the c_j neighbourhood of E_j is in G for c_j in (0,1/3);

on this neighbourhood $\xi \to \nabla h_0(\xi)$ is a diffeomorphism; and $\operatorname{supp} \varphi \subset E_1 \cup E_2 \cup \ldots \cup E_r = E$. Now choose $c = \min\{c_1, c_2, \ldots, c_r, \operatorname{dist}(R^n \setminus E, \operatorname{supp} \varphi)\}$ which is in $(0,1/3)$. Then by Lemma 4.5 and Corollary 3.2 we get

$$\lim_{t \to \infty} \|(\Omega U_t - Z_t)\varphi(P)T(B(at^\beta) \times E)\| = 0 . \tag{4.31}$$

By the choice of φ, E, c and using Lemma 3.1(v) one sees

$$\varphi(P)\ T(B(at^\beta) \times R^n \setminus E) = 0 .$$

Since T is a measure this together with (4.31) yields

$$\lim_{t \to \infty} \|(\Omega U_t - Z_t)\varphi(P)\ T(B(at^\beta) \times R^n)\| = 0 .$$

An application of Lemma 3.1 (vi) yields the result.

Q.E.D.

Corollary 4.7 : Let $f \in \mathcal{H}$ be such that for some $\varphi \in C_0^\infty(G)$, $\underset{t \to \pm\infty}{\text{s-lim}} \{1-\varphi(P)\}V_t f = 0$. Then $f \in$ Range $\Omega_\pm$ iff for every $a > 0$ and for some β in $(0,(n+4\delta)/(2n+4))$ we have

$$\underset{t \to \pm\infty}{\text{s-lim}}\ F(|Q| \geq a|t|^\beta) Z_t^*\ V_t\ f = 0$$

Proof : We prove only for the positive sign and write Ω for Ω_+. Necessity follows from Lemma 2.5(iii). For sufficiency, note that by hypothesis and Theorem 4.6

$$\underset{t \to \infty}{\text{s-lim}}\ (\Omega U_t\ Z_t^* - 1)\varphi(P)V_t\ f = 0 .$$

Since $\underset{t \to \infty}{\text{s-lim}}\ (1-\varphi(P))V_t\ f = 0$ we get

$$\underset{t \to \infty}{\text{s-lim}}\ (\Omega\ U_t\ Z_t^* - 1)V_t\ f = 0 .$$

The result follows by Lemma 2.5 (ii) .

Q.E.D.

<u>Lemma 4.8</u> *Let f be such that for some real valued* $\varphi \in C_o^\infty(G)$,

$\underset{t\to\infty}{\text{s-lim}}\ \{1-\varphi(P)\}V_t f = 0$; for some $\beta > 1-\delta$ and for every $a > 0$

$\underset{t\to\infty}{\text{s-lim}}\ F(|Q| \geq at^\beta)U_t^* V_t f = 0$. Then $\underset{t\to\infty}{\text{s-lim}}\ F(|Q| \geq at^{\beta_1})Z_t^* V_t f = 0$ for every $\beta_1 > \beta$ and for every $a > 0$.

<u>Proof</u> : Certainly it is sufficient to prove $\underset{t\to\infty}{\text{s-lim}}\ F(|Q| \geq 2at^{\beta_1})\varphi(P)Z_t^* V_t f = 0$

which will follow if $\lim_{t\to\infty} \|F(|Q| \geq 2at^{\beta_1})\varphi(P)Z_t^* U_t F(|Q| \leq at^\beta)\| = 0$.

$$\|F(|Q| \geq 2at^{\beta_1})\varphi(P)Z_t^* U_t F(|Q| \leq at^\beta)\|$$

$$\leq K t^{-\beta_1} \|\ |Q|F(|Q| \geq 2at^{\beta_1})\ \varphi(P)Z_t^* U_t F(|Q| \leq at^\beta)\|$$

$$\leq K \sum_j t^{-\beta_1} \|F(|Q| \geq 2at^{\beta_1}) [Q_j, \varphi(P)Z_t^* U_t] F(|Q| \leq at^\beta)\| +$$

$$K \sum_j t^{-\beta_1} \|F(|Q| \geq 2at^{\beta_1})\varphi(P)Z_t^* U_t Q_j F(|Q| \leq at^\beta\ \|$$

$$\leq K \sum_j t^{-\beta_1}\{\|\partial\varphi(P)/\partial P_j\| + \|\varphi(P)\partial\{X(t,P)-th_o(P)\}/\partial P_j\| + t^\beta\}$$

$$\leq K t^{-\beta_1}\{1+t^{1-\delta} + t^\beta\} \qquad \text{by (2.19) for t large .}$$

Q.E.D.

The result follows since $\beta_1 > \beta > 1-\delta$.

<u>Remark 4.9</u>: The results of this section can be easily seen to be true when (i) $h_o(\xi)$ is a real valued C^∞ function with polynomial growth in ξ and $G = \{\xi : |\nabla h_o(\xi)| + |\det h_o''(\xi)| \neq 0\}$ is an open set whose complement has Lebesgue measure zero and (ii) ξ^α for $|\alpha| \leq m$ are replaced by C^∞ functions of ξ.

§ 5. ABSENCE OF POSITIVE EIGENVALUES

Let $b_1,\ldots,b_n: R^n \to R$ be n functions each of which are C^1 and let $b = (b_1,\ldots,b_n)$. Define curl b to be the matrix whose (j,k)th element is $D_jb_k - D_kb_j$ i.e.,

$$\text{curl } b = ((D_jb_k - D_kb_j)). \tag{5.1}$$

For our static electro-magnetic model, one can show, using the results of [40] that there are no positive eigenvalues.

<u>Theorem 5.1</u> : Let $q, b_j: R^n \to R$ be bounded, C^1 functions such that their derivatives are also bounded. Furthermore, assume that for some Λ in R there exist β,ε,L such that $0 \leq \beta < \frac{2}{3}$, $0 \leq \varepsilon < \frac{2}{3} - \beta$, $L > 0$ so that

$$2\Sigma x_jD_jq + 4(1-\beta-\varepsilon)q + \frac{1}{\varepsilon}|(\text{curl } b)x|^2 \leq 4(1-\beta-\varepsilon)\Lambda \quad \text{for all } |x| \geq L. \tag{5.2}$$

Then

(i) $H = \frac{1}{2}(P-b)^2 + q$ is self adjoint with $D(H) = D(\frac{1}{2}P^2)$,

(ii) H has no eigenvalues in (Λ,∞).

In particular if

$$\lim_{|x|\to\infty} \{|q(x)| + |x| \sum_{j=1}^{n} |(D_j q)(x)| + |x| \sum_{j,k=1}^{n} |(D_jb_k)(x)|\} = 0 \qquad \ldots\ldots(5.3)$$

then (iii) H does not have any eigenvalue in $(0,\infty)$

Proof (i) is trivial.

(ii) Refer [40].

(iii) follows from (i).

Q.E.D.

Corollary 5.2 : Let $h_o(\xi) = \frac{1}{2}\xi^2$, $H = H_o + \sum_{|\alpha|\leq 1} W_\alpha(Q)P^\alpha$ where W_α are real valued potentials on R^n satisfying (2.4) and $\sum_{j=1}^{n} \partial W_j(x)/\partial x_j = 0$. Then

(i) $(\sum_{|\alpha|\leq 1} W_\alpha(Q)P^\alpha)(H_o+1)^{-1}$ is compact ,

(ii) H is self adjoint on $D(H) = D(H_o)$,

(iii) $\sigma_c(H) \subset [0,\infty)$,

(iv) H does not have any positive non-zero eigenvalue i.e. $(0,\infty) \subset \sigma_c(H)$.

Proof : (i) Follows from Corollory 1.8.

(ii) By (i) and remark 1.2 the relative bound of $\Sigma W_\alpha(Q)P^\alpha$ with respect to H_o is zero. The result follows by Theorem 1.1.

(iii) By Theorem 1.6, $\sigma_{ess}(H) = \sigma_{ess}(H_o) = [0,\infty)$. The result follows by noting $\sigma_c(H) \subset \sigma_{ess}(H)$.

(iv) If we write H in the form in Theorem 5.1 we get

$$b_j = -W_j \quad ,$$

$$q = W_o - \left(\tfrac{1}{2} \sum_{j=1}^{n} W_j^2\right) .$$

Using (2.4) it is easily seen that (5.3) is satisfied and so the result follows from Theorem 5.1 (iii).

Q.E.D.

§ 6. ASYMPTOTIC EVOLUTION AND STRONG ASYMPTOTIC COMPLETENESS FOR LONG RANGE HAMILTONIAN WITH STATIC ELECTRO-MAGNETIC FIELD

The material of this section closely follows [23]. Let

$$H_o = -\tfrac{1}{2}\Delta = \tfrac{1}{2} P^2 \qquad \text{on } L^2(R^n),$$

and

$$H = H_o + \sum_{|\alpha|\leq 1} W_\alpha(Q)P^\alpha$$

be as in Corollary 5.2. For potentials a_j and V the operator $\tfrac{1}{2}\sum_j (P_j - a_j(Q))^2 + V(Q)$ is the Hamiltonian for a particle in a static electro-magnetic field and can be reduced to $H_o + \sum_{|\alpha|\leq 1} W_\alpha(Q)P^\alpha$.

The aim of this section is to prove

(i) for $\beta > 1-\delta$, $F(|Q| \geq |t|^\beta)U_t^* V_t f \to 0$ strongly as $t \to \pm\infty$ for f in $\mathcal{H}_c(H)$,

(ii) strong asymptotic completeness for H by using (i) and the results of § 4.

For proving (i) the techniques of [23] will be employed.

Define the unitary group of dilations on $L^2(R^n)$ by

$$(Y_t f)(x) = \exp(-nt/4) f(x \exp(-\tfrac{1}{2} t)) \text{ for } f \text{ in } L^2(R^n).$$

It is clear that Y_t is a unitary strongly continuous representation of the dilation group. Let A be its generator so that

$$Y_t = \exp(-itA) \quad \text{for } t \text{ in } R.$$

A simple calculation shows that

$$A = (PQ+QP)/4 \qquad \text{on } S(R^n).$$

Also A and log H_o are Weyl conjugate to each other; i.e., for all α, t in R

$$e^{itA} e^{-i\alpha \log H_o} e^{-itA} = e^{-i\alpha \log H_o} e^{i\alpha t}$$

which formally implies

$$[A,H_o] = iH_o \ , \quad [A, \log H_o] = i \ .$$

Further it is easy to see that

$$Y_t^* P Y_t = Pe^{-\frac{1}{2}t}, \quad Y_t^* H_o Y_t = H_o e^{-t} \ .$$

We have seen in Corollary 5.2 that

$$(0,\infty) \subset \sigma_c(H) = \text{spec } H_c \subset [0,\infty) \ . \qquad (6.1)$$

So, we set

$$G_o = (0,\infty) \ ,$$

By (6.1), $\{f \in \mathcal{H}_c(H) : H \text{ spectral supp } f \text{ is compact in } G_o\}$ is easily seen to be dense in $\mathcal{H}_c(H)$. Further for any bounded continuous real valued function with support in G_o one gets $\varphi(H) = \varphi(H_c)$.

Since $D(H) = D(H_o)$, it follows that $(H+i)^{-1}(H_o+i)$, $(H_o+i)^{-1}(H+i)$, $(H+i)(H_o+i)^{-1}$, $(H_o+i)(H+i)^{-1}$ are all bounded operators.

For real t, $\{U_t^*(A/t)U_t - H_o\}f = (A/t)f \to 0$ as $t \to \pm\infty$ if $f \in S(R)$. i.e. we can expect the scaled observable A/t under the free evolution to behave like the free Hamiltonian as $t \to \pm\infty$. A similar result may be expected for Hamiltonians with potentials vanishing at ∞, only if we look at the continuous subspace of that Hamiltonian.

A look at (6.2) (below) together with RAGE Theorem [6] gives that $V_t^*(A/t)V_t f \to Hf$ if $f \in D(A) \cap D(H_c)$. We can (and infact do) show that $D(A) \cap D(H_c)$ is dense in $D(H_c)$. If further the eigenvectors of H are in D(A) then it is clear that $V_t^*(A/t)V_t f \to H_c f$. Since we do not have any information about eigenvectors of H we go in a round about manner viz prove that $\underset{t\to\infty}{\text{s-lim}} \ V_t^* Y_{v/t} V_t = e^{-iv H_c}$. For this we need

Lemma 6.1 Let $f \in D(H) \cap \mathcal{H}_c(H)$ and $u \in R$. Then

(i) $\sup_{|t| \geq 1} \sup_{s \in [0,u]} ||(H+i)^{-1}\{(V^*_{t+s}AV_{t+s}-A)t^{-1}-H\}(H+i)^{-1}|| < \infty$,

(ii) $\sup_t ||(H+i)^{-1}\{(V^*_tAV_t-A)t^{-1} - H\}(H+i)^{-1}|| < \infty$,

(iii) $\lim_{|t|\to\infty} \sup_{s \in [0,u]} ||(H+i)^{-1}(V^*_{t+s}AV_{t+s}-A)t^{-1}-H\}f|| = 0$,

Further, for any $g \in L^2(R^n)$ and any $t \in R$

(iv) $||\{(H_o+1)^{-1} Y_t(H_o+1)-1\}g|| \leq K\{|e^t-1| \, ||g|| + ||(Y_t-1)g||\}$,

(v) $||\{(H_o+1)Y_t(H_o+1)^{-1}-1\}g|| \leq K\{|e^{-t}-1| \, ||g|| + ||(Y^*_t-1)g||\}$.

Proof By the fundamental theorem of calculus

$V^*_tAV_t-A$

$$= i\int_0^t dy\, V^*_y\{[H_o,A] + \sum_{j=1}^n [W_j P_j,A] + [W_o,A]\}V_y$$

$$= t H - \int_0^t V^*_y\{\tfrac{1}{2}\sum_{j,k=1}^n Q_k(D_k W_j)P_j+\tfrac{1}{2}\sum_{j=1}^n W_jP_j + \tfrac{1}{2}\sum_{j=1}^n Q_j D_jW_o + W_o\}V_y$$

So, $V^*_tAV_t-A-tH$

$$= -\tfrac{1}{2}\int_0^t dy\, V^*_y\{\sum_{j,k=1}^n Q_k(D_kW_j)P_j + \sum_{j\neq 0} W_jP_j + \sum_{j\neq 0} Q_jD_jW_o + 2W_o\}V_y \qquad (6.2)$$

$$= \int_0^t dy\, V^*_y\{\sum_j \Gamma_j(Q)P_j + \Gamma_o(Q)\}V_y \qquad (6.3)$$

where Γ_j, Γ_o are some potentials satisfying (2.4).

(i) follows from (6.3).

(ii) follows from (i).

(iii) $[\Sigma\, \Gamma_j(Q_j)P_j + \Gamma_o(Q)](H+i)^{-1}$ is compact.

The result follows by RAGE Theorem [6] and (6.3).

(iv) $\|\{(H_o+1)^{-1} Y_t(H_o+1)-1\}g\|$

$\leq \|\{H_o+1)^{-1} Y_t(H_o+1)Y_t^*-1\}\|\|g\| + \|(Y_t-1)g\|$

$\leq \|\{(H_o+1)^{-1} (e^t H_o+1)-1\}\| \|g\| + \|(Y_t-1)g\|$

$\leq |e^t-1| \|g\| + \|(Y_t-1)g\|$.

(v) Similar to (iv) .

Q.E.D.

Theorem 6.2 $\text{s-lim}_{t \to \pm\infty} V_t^* Y_{u/t} V_t = \exp(-iu H_c)$ for $u \in R$.

Proof : Since the limit is unitary and the sequence of operators are contractions, it suffices to prove weak convergence on a total set. This too we prove only for $t \to \infty$, the other case being similar.

If $f \in L^2(R^n)$ is an eigenvector for H, then $\|V_t^* Y_{u/t} V_t f - \exp(-iuH_c)f\| = \|(Y_{u/t}-1)f\|$ and the result follows for f. So the result is proved on $\mathcal{H}_p(H)$.

Now let $f \in D(H) \cap \mathcal{H}_c(H)$. Then,

$$\lim_{t\to\infty} \|(H+i)^{-1}(V_t^* Y_{u/t} V_t - V_u)f\|$$

$$\leq \lim_{t\to\infty} \|(H+i)^{-1}(V_t^* Y_{u/t} V_t - V_u Y_{u/t})f\| + \lim_{t\to\infty} \|(Y_{u/t}-1)f\|$$

$$\leq \|(H+i)^{-1}(H_o+1)\| \cdot \lim_{t\to\infty} \|(H_o+1)^{-1} Y_{u/t}(H_o+1)\|.$$

$$\lim_{t\to\infty} \|(H_o+1)^{-1}(V_t - Y^*_{u/t} V_{u+t} Y_{u/t})f\|$$

$$\leq K \lim_{t\to\infty} \|(H_o+1)^{-1}(V_t - Y^*_{u/t} V_{u+t} Y_{u/t})f\| \quad \text{by } D(H)=D(H_o) \text{ and Lemma 6.1(iv)}$$

$$\leq K \lim_{t\to\infty} \|(H_o+1)^{-1} \int_0^u ds\, D_s(Y^*_{s/t} V_{s+t} Y_{s/t} f)\| \quad \text{where } D_s = d/ds$$

$$= K \lim_{t\to\infty} \|\int_0^u ds (H_o+1)^{-1} Y^*_{s/t} V_{s+t}\{(V^*_{s+t} A V_{s+t} - A)t^{-1} - H\} Y_{s/t} f\|$$

$$\leq \quad K|u| \lim_{t \to \infty} \sup_{s \in [0,u]} ||(H_o+1)^{-1} Y^*_{s/t}(H_o+1)||.$$

$$\lim_{t \to \infty} \sup_{s \in [0,u]} ||(H_o+1)^{-1}\{(V^*_{s+t}AV_{s+t}-A)t^{-1}-H\}Y_{s/t}f||$$

$$\leq \quad K|u| \lim_{t \to \infty} \sup_{s \in [0,u]} ||(H_o+1)^{-1}\{(V^*_{s+t}\, AV_{s+t}-A)t^{-1}-H\}f|| +$$

$$K|u| \lim_{t \to \infty} \sup_{s \in [0,u]} ||(H_o+1)^{-1}\{(V^*_{s+t}AV_{s+t}-A)t^{-1}-H\}(Y_{s/t}-1)f||$$

by Lemma 6.1 (iv)

$$\leq \quad K|u| \lim_{t \to \infty} \sup_{s \in [0,u]} ||(H_o+1)^{-1}\{(V^*_{s+t}AV_{s+t}-A)t^{-1}-H\}(H_o+1)^{-1}||.$$

$$\lim_{t \to \infty} \sup_{s \in [0,u]} ||(H_o+1)(Y_{s/t}-1)(H_o+1)^{-1}(H_o+1)f||$$ by Lemma 6.1(iii) and $\mathcal{D}(H) = \mathcal{D}(H_o)$

$$\leq \quad K|u| \lim_{t \to \infty} \sup_{s \in [0,u]} ||\{(H_o+1)Y_{s/t}(H_o+1)^{-1}-1\}(H_o+1)f||$$ by Lemma 6.1(i) and $\mathcal{D}(H) = \mathcal{D}(H_o)$

$\leq \quad 0$ by Lemma 6.1 (v)

Q.E.D.

<u>Corollary 6.3</u> If $f \in \mathcal{H}_c(H)$, then $\operatorname{w-lim}_{t \to \pm\infty} V_t f = 0$

<u>Proof</u> Refer [36].

Q.E.D.

Further we have

$$V^*_tU_tQ_jU^*_tV_t - Q_j$$

$$= i \int_0^t ds\, V^*_s[\sum_k W_kP_k + W_o,\, Q_j-sP_j]V_s$$

$$= \int_0^t ds\, V^*_s\{W_j + s\sum_k(D_jW_k)P_k + s\, D_jW_o\}V_s$$

$$= \int_0^t ds\, V^*_s\{W_j + si\sum_k(D_kD_jW_k) + s\sum_k P_k(D_jW_k) + sD_jW_o\}V_s \qquad (6.4)$$

If $||(1+|Q|)^{-\gamma} V_sf|| = O(s^{-\gamma})$ for some "nice" f, for $0 \leq \gamma \leq 1+\delta$ then by (6.4), we have $||(|P|+1)^{-1} |Q|U^*_t V_t f|| = O(t^{1-\delta})$, from which, we can deduce that, for all f in $\mathcal{H}_c(H)$ and for all $\beta > 1-\delta$, $F(|Q| \geq a|t|^\beta)U_t^* V_t f \to 0$ strongly as $t \to \pm\infty$. In Lemma 6.4 we reduce the problem of proving $||(1+|Q|)^{-\gamma} V_s f|| = O(s^{-\gamma})$ to proving $||(1+|A|)^{-\gamma} V_s f|| = O(s^{-\gamma})$.

We give a heuristic argument how one can expect these results. By Theorem 6.2, $V_t^* AV_t/t$ behaves like H on $\mathcal{H}_c(H)$. So it is reasonable to expect that, for $f \in \mathcal{H}_c(H) \cap D(H^{-2})$, $t^\gamma V_t^* A^{-\gamma} V_t f$ behaves like $H^{-\gamma}f$ for $0 \leq \gamma \leq 2$. That it is so will be proved rigorously for (i) $0 \leq \gamma \leq 1$ in Lemma 6.5, (ii) $0 \leq \gamma \leq 1+\delta$ in Lemma 6.6 by developing a Taylor's expansion around $t = \pm\infty$.

Recall that A^2+P^2+1, A^4+P^6+1 have self adjoint extensions by Theorem 1.5.

<u>Lemma 6.4</u> Let $\varphi \in S(R)$, $-k$ real, $\in$ Res(H) = resolvent set of H. Then

(i) $A\varphi(H)\,(A^2+P^2+1)^{-\frac{1}{2}}$ is bounded,

(ii) $A^2\varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}$ is bounded,

(iii) $A(H+k)^{-1}\,(1+|Q|)^{-1}$ is bounded,

(iv) $A(H+k)^{-2}\,(1+|Q|)^{-1}$ is bounded,

(v) $A^2(H+k)^{-2}\,(1+Q^2)^{-1}$ is bounded,

(vi) The family $(1+|Q|)^{-\gamma}\,\varphi(H)(1+|A|)^\gamma$ is uniformly bounded for $0 \leq \gamma \leq 2$,

(vii) The family $(1+|Q|)^{-\gamma}\,P_j\varphi(H)(1+|A|)^\gamma$ is uniformly bounded for $0 \leq \gamma \leq 1$ and for each j,

(viii) $Q_j\varphi(H)(Q^2+P^2+1)^{-\frac{1}{2}}$ is bounded for each j.

<u>Proof</u> : B will denote any bounded operator; B(j,t) any operator family depending on t with $||B(j,t)|| \leq K_j(1+|t|)^j$; $\Gamma_0, \Gamma_1, \ldots$ for functions of x which are bounded with all their derivatives. Further we shall very frequently use the formula $XY = [X,Y]+ YX$ and that H_0, H as well as $|P|$, $|H|^{\frac{1}{2}}$

are bounded with respect to each other.

(i) $A\varphi(H)$

$$= \int dt\ \hat{\varphi}(t)[A, V_t^*] + B(A+i)$$

$$= \int dt\ \hat{\varphi}(t)\{-tHV_t^* + B(1,t)(|P|+1)\} + B(A+i) \quad \text{by (6.3)}$$

$$= B + B(|P| + 1) + B(A+i).$$

The result follows immediately since $|P|(A^2+P^2+1)^{-\frac{1}{2}}$ and $A(A^2+P^2+1)^{-\frac{1}{2}}$ are both bounded.

(ii) $A^2\varphi(H)$

$$= \int dt\ \hat{\varphi}(t)[A^2, V_t^*] + \varphi(H)A^2 \qquad (6.5)$$

$$[A^2, V_t]$$

$$= A[A, V_t] + [A, V_t]A$$

$$= t\ AV_tH + \int_0^t ds\ AV_{t-s}\{\sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)\}V_s + tV_tHA + B(1,t)(|P|+1)A \quad \text{by (6.3)} \qquad (6.6)$$

$$\int_0^t AV_{t-s}\{\sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)\}V_s$$

$$= \int_0^t ds[A, V_{t-s}]\{\sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)\}V_s + \int_0^t ds\ V_{t-s}[A, \sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)]V_s +$$
$$\int_0^t ds\ V_{t-s}\{\sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)\}[A, V_s] + B(1,t)(|P|+1)A$$

$$= \int_0^t ds(t-s)V_{t-s}\ H\{\sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)\}V_s \ +$$
$$\int_0^t ds \int_0^{t-s} du\ V_{t-s-u}\{\sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)\}V_u\{\sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)\}V_s + B(1,t)(|P|+1) +$$
$$\int_0^t ds\ V_{t-s}\{\sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)\}\{sHV_s + \int_0^s du\ V_{s-u}\{\sum_j \Gamma_j(Q)P_j + \Gamma_0(Q)\}V_u +$$
$$B(1,t)(|P|+1)A \qquad \text{by (6.3)}$$

$$= B(2,t)(|P|^3+1) + B(1,t)(|P|+1)A \qquad (6.7)$$

(In the last step we have used (i) by commutation rules the boundedness of $H\{\sum_j \Gamma_j(Q)P_j + \Gamma_o(Q)\}(|P|^3+1)^{-1}$; (ii) $P_j\ V_u(|P|)+1)^{-1}$ is uniformly bounded in u) Substituting (6.7) in (6.6),

$$[A^2,V_t]$$

$$= tAV_tH + tV_tHA + B(2,t)(|P|^3+1) + B(1,t)(|P|+1)A \quad . \tag{6.8}$$

Substituting (6.8) in (6.5) and carrying out the integration, we get for some ψ in $S(R)$,

$$A^2\varphi(H) = \varphi(H)A^2 + A\psi(H) + \psi(H)A + B(|P|^3+1) + B(|P|+1)A \ .$$

The result follows as in (i), by using (i), if we can show the boundedness of $(H_o+1)^{\frac{1}{2}} A(A^4+P^6+1)^{-\frac{1}{2}}$. For this it is enough to show the boundedness of $(A^4+P^6+1)^{-\frac{1}{2}} A(H_o+1)A(A^4+P^6+1)^{-\frac{1}{2}}$ which is clear since $[A,H_o] = iH_o$.

(iii) $$-[A,(H+k)^{-1}] = (H+k)^{-1}[A,H](H+k)^{-1}$$
$$= (H+k)^{-1}\{iH_o + \sum_j \Gamma_j(Q)P_j + \Gamma_o(Q)\}\ (H+k)^{-1}$$
$$= B \ . \tag{6.9}$$

$$A(H+k)^{-1}(1+|Q|)^{-1} = B + (H+k)^{-1} A(1 + |Q|)^{-1} \quad \text{by } (6.9)$$
$$= B + B(H_o+1)^{-1} A(1+|Q|)^{-1}$$
$$= B \ .$$

(iv) $$[A,(H+k)^{-2}] = (H+k)^{-1}[A,(H+k)^{-1}] + [A,(H+k)^{-1}](H+k)^{-1}$$
$$= B \qquad \text{by } (6.9) \ .$$

The result follows as in (iii).

(v) In what follows absolute constants will not matter; for example we shall write H_o for iH_o. In the first step we use $D(H^2) = D(H_o^2)$ which is proved in Lemma 7.2.

$$A^2(H+k)^{-2}(1+Q^2)^{-1}$$

$$= [A^2,(H+k)^{-2}](1+Q^2)^{-1} + B(H_o+1)^{-2}A^2(1+Q^2)^{-1}$$

$$= (H+k)^{-2}[A^2,(H+k)^2](H+k)^{-2}(1+Q^2)^{-1} + B$$

$$= (H+k)^{-1}[A^2,H](H+k)^{-2}(1+Q^2)^{-1} + (H+k)^{-2}[A^2,H](H+k)^{-1}(1+Q^2)^{-1} + B$$

$$= (H+k)^{-1}\{H_o+H_oA + \sum_j \Gamma_j(Q)P_j + \sum_j \Gamma_j(Q)P_jA+\Gamma_o(Q)+\Gamma_o(Q)A\}(H+k)^{-2}(1+Q^2)^{-1} +$$
$$(H+k)^{-2}\{H_o+H_oA + \sum_j \Gamma_j(Q)P_j + \sum \Gamma_j(Q)P_jA + \Gamma_o(Q)+\Gamma_o(Q)A\}(H+k)^{-1}(1+Q^2)^{-1}+B$$

$$= (H+k)^{-1}\{\sum \Gamma_j(Q)P_j + \sum \Gamma_j(Q)P_jB\}+B+(H+k)^{-2}\{\sum \Gamma_j(Q)P_j+\sum \Gamma_j(Q)P_jB\}$$

by (iii) and (iv)

$$= (H+k)^{-1}\sum_j P_j B+B + (H+k)^{-2}(\sum P_jB + B)+B$$ [by using commutation relations between P,Q]

$$= B .$$

(vi) It is enough to prove the boundedness $(1+Q^2)^{-1}\varphi(H)A^2$ and then interpolate.Choose k as in (v). Then

$$A^2\varphi(H)(1+Q^2)^{-1}$$

$$= A^2\varphi(H)(H+k)^2(A^4+P^6+1)^{-\frac{1}{2}}(A^4+P^6+1)^{+\frac{1}{2}}(H+k)^{-2}(1+Q^2)^{-1}$$

By (ii) it is enough to prove the boundedness of $(A^4+P^6+1)^{\frac{1}{2}}(H+k)^{-2}(1+Q^2)^{-1}$. This is so if

$$(1+Q^2)^{-1}(H+k)^{-2}(A^4+P^6+1)(H+k)^{-2}(1+Q^2)^{-1} \text{ is bounded.}$$

Certainly it is enough to prove $(1+Q^2)^{-1}(H+k)^{-2}A^2$ is bounded which is guaranteed by (v).

(viii) Similar to (vi). Let k be as in (vi)

$$A\varphi(H)P_j(1+|Q|)^{-1}$$

$$= A\varphi(H)(H+k)(A^2+P^2+1)^{-\frac{1}{2}}(A^2+P^2+1)^{\frac{1}{2}}(H+k)^{-1}P_j(1+|Q|)^{-1}.$$

As in (vi), by using (i), it is sufficient to prove the boundedness of $(1+|Q|)^{-1} P_j (H+k)^{-1}(A^2+P^2+1)(H+k)^{-1} P_j(1+|Q|)^{-1}$ which follows if $A(H+k)^{-1} P_j(1+|Q|)^{-1}$ is bounded. Now

$$\begin{aligned} & A(H+k)^{-1} P_j(1+|Q|)^{-1} \\ = \; & [A,(H+k)^{-1}P_j](1+|Q|)^{-1} + B(|P|+1)^{-1} A(1+|Q|)^{-1} \\ = \; & (H+k)^{-1} [A,H] (H+k)^{-1} P_j (1+|Q|)^{-1} + (H+k)^{-1} [A,P_j] (1+|Q|)^{-1} + B \\ = \; & B . \end{aligned}$$

(viii) Similar to (i) by noting $[Q_j,V_t] = B(1,t)(|P|+1)$.

Q.E.D.

<u>Lemma 6.5</u> Let $\varphi \in C_o^\infty (G_o)$ and $u,t \in R$.

(i) If u,t are of the same sign and $|t| \geq 1$, then

$$||(Y^*_{u/t} V^*_t Y_{u/t} V_t - V_u)\varphi(H)|| \leq K u^2(1+|t|)^{-1} ,$$

(ii) If u,t are of opposite signs, $|t| \geq 1$ and $t_1 \in (0,t]$, then

$$||(H_o+1)^{-1} Y_{u/t_1}(Y^*_{u/t} V^*_t Y_{u/t} V_t - V_u)\varphi(H)|| \leq K u^2 (1+|t|)^{-1},$$

(iii) $$||AV_t \varphi(H)(A^2+ P^2+1)^{-\frac{1}{2}}|| \leq K(1+|t|),$$

(iv) For $|t| \geq 1$, $||(H_o+1)^{-1}(V^*_t Y_{u/t} V_t - V_u)\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| \leq K(1+u^2)(1+|t|)^{-1}$,

(v) For $\psi \in S(R)$, $||(H_o+1)^{-1} V^*_t\{\psi(A/t)-\psi(H)\}V_t \varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| \leq K(1+|t|)^{-1}$,

(vi) For $0 \leq \gamma \leq 1$, $||(1+|Q|)^{-\gamma} V_t\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| \leq K(1+|t|)^{-\gamma}$,

(vii) For $0 \leq \gamma \leq 1$ and each j, $||(1+|Q|)^{-\gamma} P_j V_t\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| \leq K(1+|t|)^{-\gamma}$.

<u>Proof</u> (i) Since $\varphi \in C_0^\infty(G_0)$, $\varphi(H) = \varphi(H_c)$. So, by Theorem 6.2 $\underset{t\to\pm\infty}{\text{s-lim}}\ Y^*_{u/t} V^*_t Y_{u/t} V_t \varphi(H) = V_u \varphi(H)$. By the Fundamental Theorem of calculus, denoting d/ds by D_s,

$$(Y^*_{u/t} V^*_t Y_{u/t} V_t - V_u)\varphi(H)$$

$$= -\int_t^\infty ds\, D_s(Y^*_{u/s} V^*_s Y_{u/s} V_s)\varphi(H)$$

$$= -\int_t^\infty ds\, D_s[\{\exp(isY^*_{u/s} HY_{u/s})\}V_s]\ \varphi(H)$$

$$= -\int_t^\infty ds\, D_s\{\exp(isH_0\ e^{-u/s} + isY^*_{u/s}[\sum_j W_j(Q)P_j + W_0]Y_{u/s}).V_s\}\varphi(H)$$

$$= -i\int_t^\infty ds\, \exp(\ldots)D_s\{sH_0 \exp(-u/s) - sH_0 + sY^*_{u/s}(\sum_j W_jP_j + W_0)Y_{u/s} - s(\sum_j W_jP_j + W_0)\}V_s\varphi(H$$

$$= -i\int_t^\infty ds\, Y^*_{u/s} V^*_s Y_{u/s} D_s(s[\exp(-u/s)-1])H_0\varphi(H)V_s -$$

$$i\int_t^\infty ds\, Y^*_{u/s} V^*_s Y_{u/s} D_s(s[Y^*_{u/s} \sum_j W_j P_j Y_{u/s} - \sum_j W_j P_j])V_s\varphi(H) -$$

$$i\int_t^\infty ds\, Y^*_{u/s} V^*_s Y_{u/s} D_s (s[Y^*_{u/s} W_0 Y_{u/s} - W_0])V_s\varphi(H)\ . \tag{6.10}$$

We now assume $u \geq 0$ and $t \geq 0$. The proof when $u \leq 0$ and $t \leq 0$ is similar. We handle (6.10) term by term. For the first term

$$D_s\ \{s[(\exp -u/s)-1]\}$$

$$= D_s\ \{s\int_0^u dp\ D_p \exp(-p/s)\}$$

$$= s^{-2}\int_0^u dp\ p\exp(-p/s)\ . \tag{6.11}$$

Since $u \geq 0$, $s \geq t \geq 0$, by (6.11) we have

$$|D_s(s[\exp(-u/s)-1])| \leq u^2 s^{-2}\ , \tag{6.12}$$

Norm of the first term of R.H.S. of (6.10)

$$\leq \int_t^\infty ds\ u^2 s^{-2} ||H_o\ \varphi(H)||$$

$$\leq K\ u^2\ (1+|t|)^{-1}.$$

For the second term,

$$D_s(s[Y^*_{u/s} \sum_j W_jP_j\ Y_{u/s} - \sum_j W_j\ P_j])$$

$$= D_s\ s \int_0^u dp\ D_p(Y^*_{p/s}\ \sum_j W_jP_j\ Y_{p/s})$$

$$= i\ D_s \int_0^u dp\ Y^*_{p/s}\ [A, \sum_j W_jP_j]Y_{p/s}$$

$$= s^{-2} \int_0^u dp\ p\ Y^*_{p/s}\ [A,[A, \sum_j W_jP_j]]Y_{p/s}\ . \tag{6.13}$$

Following the generic notation Γ as in Lemma 6.4

$$2[A, \sum_j W_jP_j] = \sum_{k,j} [Q_k\ P_k, W_jP_j] = \sum_j \Gamma_j(Q)P_j\ .$$

$$2[A, \sum_j W_jP_j) = \sum_{k,j} [P_kQ_k, W_jP_j] = \sum_j P_j\ \Gamma_j(Q) + \Gamma_o(Q).$$

So

$$[A,[A, \sum_j W_jP_j]] = \sum_j \Gamma_j(Q)P_j \quad (\Gamma_j \text{ generic})\ . \tag{6.14}$$

and

$$[A,[A, \sum_j W_jP_j]] = \sum_j P_j\Gamma_j(Q)+\Gamma_o(Q) \quad (\Gamma_j, \Gamma_o \text{ generic})\ . \tag{6.15}$$

By using (6.13) and (6.14), we have,

norm of the second term of R.H.S. of (6.10)

$$\leq K \int_t^\infty ds\ s^{-2} \int_0^u dp\ p \sum_j ||P_j\ Y_{p/s}\ \varphi(H)||$$

$$= K \sum_j \int_t^\infty ds\ s^{-2} \int_0^u dp\ p\ ||Y_{p/s}\ \exp(-p/(2s))P_j\varphi(H)||$$

$$\leq K \int_t^\infty ds\ s^{-2}\ u^2 \leq K\ u^2(1+|t|)^{-1}$$

(We have used the fact that under the assumptions $\exp(-p/(2s)) \leq 1$).

Regarding the third term, as proceeding for the second term

$$D_s(s[Y^*_{u/s} W_o Y_{u/s} - W_o])$$

$$= s^{-2} \int_0^u dp\, p\, Y^*_{p/s} [A,[A,W_o]]Y_{p/s}$$

$$= s^{-2} \int_0^u dp\, p\, Y^*_{p/s} \Gamma_o(Q)Y_{p/s} . \tag{6.16}$$

Norm of the third term of R.H.S. of (6.10), using (6.16),

$$\leq K \int_t^\infty ds\, s^{-2} \int_0^u dp\, p \leq Ku^2(1+|t|)^{-1}.$$

Since all the three terms of R.H.S. of (6.10) have been shown to have the desired growth rate in u and decay rate in t, the proof of (i) is complete.

(ii) As proceeding in (i) one gets,

$$(H_o+1)^{-1} Y_{u/t_1} (Y^*_{u/t} V^*_t Y_{u/t} V_t - V_u)\varphi(H)$$

$$= - i \int_t^\infty ds(H_o+1)^{-1} Y_{u/t_1-u/s} V^*_s Y_{u/s} H_o D_s(s[\exp(-u/s)-1])\varphi(H)V_s -$$

$$i \int_t^\infty ds(H_o+1)^{-1} Y_{u/t_1-u/s} V^*_s Y_{u/s} D_s(s[Y^*_{u/s} \sum_j W_j P_j Y_{u/s} - \sum_j W_j P_j])V_s\varphi(H) -$$

$$i \int_t^\infty ds(H_o+1)^{-1} Y_{u/t_1-u_s} V^*_s Y_{u/s} D_s(s[Y^*_{u/s} W_o Y_{u/s} - W_o])V_s\varphi(H). \tag{6.17}$$

We assume $u \leq 0$, $t \geq 0$; the proof for the other case is similar. As in (i) the terms of R.H.S. of (6.17) will be handled one by one. Using (6.11), norm of the first term of R.H.S. of (6.17)

$$\leq K \int_t^\infty ds\, s^{-2} \int_0^u dp\, p \exp((u-p)/s)\|(H_o+1)^{-1} Y_{u/t_1-u/s} V^*_s H_o Y_{u/s}\|$$

$$\leq K \int_t^\infty ds\, s^{-2} \int_0^u dp\, p\|(H_o+1)^{-1} Y_{u/t_1-u/s} V^*_s H_o\| \text{ since } (u-p)/s \leq 0 .$$

. . . . . . (6.18)

$$||(H_o+1)^{-1} Y_{u/t_1-u/s} V_s^* H_o||$$

$$\leq ||(H_o+1)^{-1}(Y_{u/t_1-u/s})(H_o+1)(H_o+1)^{-1} V_s^*(H+i)(H+i)^{-1} H_o||$$

$$\leq K||(H_o+1)^{-1} Y_{u/t_1-u/s} (H_o+1)||$$

$$= K\{||(H_o+1)^{-1} H_o|| \exp(u/t_1-u/s) + ||(H_o+1)^{-1}||\}$$

$$\leq K\{1 + \exp(u/t_1-u/s)\} \leq K. \tag{6.19}$$

In the last step, since $s \geq t \geq t_1 \geq 0$ and $u \leq 0$ we have $(u/t_1-u/s) \leq 0$. Substituting (6.19) in (6.18) one gets the result for the first term of R.H.S. of (6.17).

Norm of the second term of R.H.S. of (6.17), by (6.13) and (6.15),

$$\leq K \sum_j \int_t^\infty ds\, s^{-2} \int_0^u dp\, p\{||(H_o+1)^{-1} Y_{u/t_1-u/s} V_s^* Y_{u/s-p/s} P_j|| + K\}$$

$$\leq K \sum_j \int^\infty ds\, s^{-2} \int_0^u dp\, p\{\exp[(u-p)/(2s)]||(H_o+1)^{-1} Y_{u/t_1-u/s} V_s^* P_j|| + K\}$$

$$\leq K u^2(1+|t|)^{-1} \qquad \text{using } (u-p)/s \leq 0 \text{ and } (6.19).$$

The third term is handled as in (i).

(iii) $||AV_t\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}||$

$$\leq ||[A,V_t]\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| + K \quad \text{by Lemma 6.4(i)}$$

$$\leq K\{|t|\ ||H\varphi(H)|| + |t|\ ||(|P|+1)\varphi(H)|| + 1\} \quad \text{by (6.3)}$$

$$\leq K.$$

(iv) Putting $t_1 = t$ in (ii), from (i) and (ii) it is clear that

$$||(H_o+1)^{-1}(V_t^* Y_{u/t} V_t - Y_{u/t} V_u)\varphi(H)|| \leq Ku^2 (1+|t|)^{-1}$$

So the result will follow if

$$||(Y_{u/t}-1)\ V_u\ \varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| \leq K\ u^2(1+|t|)^{-1}$$

which is guaranteed by (iii).

(v) When $|t| \geq 1$,

L.H.S.

$$= \quad ||\int du\ \hat{\psi}(u)(H_o+1)^{-1}(V_t^*\ Y_{u/t}^*\ V_t - V_u^*)\ \varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}||$$

$$\leq \quad K\int du\ |\hat{\psi}(u)|(1+u^2)\ (1+|t|)^{-1} \quad \text{by} \quad (iv)$$

$$\leq \quad K(1+|t|)^{-1}.$$

When $|t| \leq 1$, the result is obvious.

(vi) We need to choose ψ appropriately in (v) and use Lemma 6.4 (vi). Choose $\psi \in C_o^\infty(R)$ such that $0 \leq \psi \leq 1$, $\psi(0) = 1$ in some neighbourhood of 0 and $\psi = 0$ on support of φ. For such a ψ, we have $\psi(H)\varphi(H) = 0$ and by (v)

$$||(H+i)^{-1}\psi(A/t)V_t\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| \leq K(1+|t|)^{-1} \tag{6.20}$$

and clearly

$$||(1+|A|)^{-\gamma}\ \{1-\psi(A/t)\}|| \leq K(1+|t|)^{-\gamma}. \tag{6.21}$$

Now choose $\varphi_1 \in C_o^\infty(R)$ such that $0 \leq \varphi_1 \leq 1$ and $\varphi_1 = 1$ on supp φ so that $\varphi\ \varphi_1 = \varphi$. Then we have

$$||(1+|Q|)^{-\gamma}\ V_t\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}||$$

$$\leq \quad ||(1+|Q|)^{-\gamma}\ \varphi_1(H)(H+i)(H+i)^{-1}\psi(A/t)V_t\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| +$$

$$||(1+|Q|)^{-\gamma}\varphi_1(H)(1+|A|)^{\gamma}\ (1+|A|)^{-\gamma}\{1-\psi(A/t)\}V_t\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}||$$

$$\leq \quad K||(H+i)^{-1}\ \psi(A/t)\ V_t\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| +$$

$$K||(1+|Q|)^{-\gamma}\ \varphi_1(H)(1+|A|)^{\gamma}||\cdot||(1+|A|)^{-\gamma}\ \{1-\psi(A/t)\}||.$$

The result follows by (6.20); Lemma 6.4 (vi) and (6.21).

(vii) Similar to (vi). Instead of Lemma 6.4 (vi) use Lemma 6.4 (vii).

Q.E.D.

By using, fundamental theorem of calculus as in (6.12),

$$|D_s\{s[(\exp(-u/s))-1]-(u^2/(2s))\}| \leq K|u|^3 \; |s|^{-3} \quad \text{when } us \geq 0 \, , \tag{6.22}$$

and

$$|\exp(u/s)D_s\{s[\exp(-u/s)-1]-(u^2/(2s))\}| \leq K|u|^3 \; |s|^{-3} \quad \text{when } us \leq 0 \tag{6.23}$$

We apply fundamental theorem of calculus repeatedly to $D_s[s\{Y^*_{u/s}(\Sigma W_j(Q)P_j)Y_{u/s} - \Sigma W_j(Q)P_j\}]$ to get (remembering Γ is a generic notation)

$$D_s\{s(Y^*_{u/s} \sum_j W_jP_j \, Y_{u/s} - \sum_j W_jP_j)\}$$

$$= \quad s^{-2} \int_0^u dp \; p \; Y^*_{p/s}(\sum_j \Gamma_j(Q)P_j)Y_{p/s} \quad \text{same as (6.13) with (6.14)}$$

$$= \quad s^{-2} \int_0^u dp \; p \; (\sum_j \Gamma_j(Q)P_j) + s^{-2} \int_0^u dp \; p \int_0^p D_q \; Y^*_{q/s}(\sum_j \Gamma_j(Q)P_j)Y_{q/s} \, dq$$

$$= \quad u^2 s^{-2}(\sum_j \Gamma_j(Q)P_j) + s^{-3} \int_0^u dp \; p \int_0^p Y^*_{q/s}(\sum_j \Gamma_j(Q)P_j)Y_{q/s} \, dq \tag{6.24}$$

Similarly using (6.15) instead of (6.14) one gets

$$D_s\{s(Y^*_{u/s} \sum_j W_j \, P_j \, Y_{u/s} - \sum_j W_j \, P_j)\}$$

$$= \quad u^2 s^{-2}(\sum_j P_j \, \Gamma_j + \Gamma_0) + s^{-3} \int_0^u dp \; p \int_0^p dq \; Y^*_{q/s}(\sum_j P_j \, \Gamma_j + \Gamma_0)Y_{q/s}. \tag{6.25}$$

Very easily one gets

$$D_s\{s(Y^*_{u/s} \, W_0 \, Y_{u/s} - W_0)\}$$

$$= \quad s^{-2} \, u^2 \, \Gamma_0(Q) + s^{-3} \int_0^u dp \; p \int_0^p dq \; Y^*_{q/s} \, \Gamma_0(Q)Y_{q/s} \, . \tag{6.26}$$

Now we shall find out the second term of Taylor's formula and prove $||(1+|Q|)^{-1-\delta} \; V_t\varphi(H) \; (A^4+P^6+1)^{-\frac{1}{2}}|| = O(t^{-1-\delta})$. The method of proof is one of iteration. Even the sequence of results will be similar to the sequence of results of Lemma 6.5.

<u>Lemma 6.6</u> : Let $\varphi \in C_0^\infty(G_0)$ and $u,t \in R$.

(i) If u,t are of the same sign and $|t| \geq 1$, then

$$||\{Y^*_{u/t}V^*_t Y_{u/t}V_t - V_u - i(u^2/(2t))HV_u\}\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}|| \leq K(1+|u|)^4 (1+|t|)^{-1-\delta},$$

(ii) If u,t are of opposite signs, $|t| \geq 1$ and $t_1 \in (0,t]$ then

$$||(H_0+1)^{-1} Y_{u/t_1}\{Y^*_{u/t}V^*_t Y_{u/t}V_t - V_u - i(u^2/(2t))HV_u\}\varphi(H)(A^2+P^2+1)^{-\frac{1}{2}}||$$

$$\leq K(1+|u|)^4 (1+|t|)^{-1-\delta},$$

(iii) $||A^2 V_t\varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}|| \leq K(1+|t|)^2$,

(iv) $(A^2+P^2+1)(A^4+P^6+1)^{-\frac{1}{2}}$ is bounded ,

(v) For $|t| \geq 1$,

$$||(H_0+1)^{-1}\{V^*_t Y_{u/t}V_t - (1-iuA/t)V_u - i(u^2/(2t))HV_u\}\varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}||$$

$$\leq K(1+|u|)^4 (1+|t|)^{-1-\delta} ,$$

(vi) Let $\psi \in S(R)$ be such that $\psi'\varphi = 0$. Then

$$||(H_0+1)^{-1} V^*_t\{\psi(A/t)-\psi(H)\}V_t \varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}|| \leq K(1+|t|)^{-1-\delta} ,$$

(vii) For $0 \leq \gamma \leq 1+\delta$

$$||(1+|Q|)^{-\gamma} V_t\varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}|| \leq K(1+|t|)^{-\gamma} .$$

<u>Proof</u> : We prove the results for $t \geq 0$. For $t \leq 0$, it is similar.

(i) As proceeding in Lemma 6.5 (i), but using (6.22), (6.24), (6.26) and writing $H_0 = H - \sum_j W_j P_j - W_0$, we get

$$(Y^*_{u/t}\, V^*_t\, Y_{u/t}\, V_t - V_u)\, \varphi(H)$$

$$= -i \int_t^\infty ds\, Y^*_{u/s} V^*_s Y_{u/s}\, D_s\{s[\exp(-u/s)-1] - (u^2/(2s))\}\, H_o \varphi(H) V_s +$$

$$i(u^2/2) \int_t^\infty ds\, s^{-2}\, (Y^*_{u/s} V^*_s Y_{u/s} V_s - V_u) H\varphi(H) + i(u^2/2) \int_t^\infty ds\, s^{-2}\, V_u\, H\varphi(H) -$$

$$i(u^2/2) \int_t^\infty ds\, s^{-2}\, Y^*_{u/s}\, V^*_s\, Y_{u/s}\, (\sum_j W_j P_j + W_o) V_s \varphi(H) -$$

$$u^2 \int_t^\infty ds\, s^{-2}\, Y^*_{u/s}\, V^*_s\, Y_{u/s}\, (\sum_j \Gamma_j(Q) P_j) V_s\, \varphi(H) -$$

$$\int_t^\infty ds\, s^{-3}\, Y^*_{u/s}\, V^*_s Y_{u/s} \int_0^u dp\, p \int_0^G dq\, Y^*_{q/s} (\sum_j \Gamma_j(Q) P_j) Y_{q/s}\, \varphi(H) V_s -$$

$$u^2 \int_t^\infty ds\, s^{-2}\, Y^*_{u/s}\, V^*_s\, Y_{u/s}\, \Gamma_o(Q)\, \varphi(H) V_s -$$

$$\int_t^\infty ds\, s^{-3} \int_0^u dp\, p \int_0^p dq\, Y^*_{q/s}\, \Gamma_o(Q) Y_{q/s}\, \varphi(H) V_s \,. \qquad (6.27)$$

Now transfering third term to the left we get

L.H.S. of (i)

$$\leq \quad K \int_t^\infty ds\, |D_s\{s[\exp(-u/s)-1] - (u^2/(2s))\}| +$$

$$K u^2 \int_t^\infty ds\, s^{-2}\, \|(Y^*_{u/s}\, V^*_s\, Y_{u/s}\, V_s - V_u) H\varphi(H) (A^2+P^2+1)^{-\frac{1}{2}}\| +$$

$$K u^2 \int_t^\infty ds\, s^{-2} \|(1+|Q|)^{-\delta}\, P_j V_s\, \varphi(H) (A^2+P^2+1)^{-\frac{1}{2}}\| +$$

$$K u^2 \int_t^\infty ds\, s^{-2}\, \|(1+|Q|)^{-\delta}\, V_s \varphi(H)\, (A^2+P^2+1)^{-\frac{1}{2}}\| +$$

$$K \int_t^\infty ds\, s^{-3} \int_0^u dp\, p \int_0^p dq \sum_j \|P_j\, Y_{q/s}\, \varphi(H)\| +$$

$$K \int_t^\infty ds\, s^{-3} \int_0^u dp\, p \int_0^p dq \,. \qquad (6.28)$$

We show that each term of R.H.S. of (6.28) has the desired growth in u and decay in t.

For the first term use (6.22); for the second, use Lemma 6.5 (i); for the third, use Lemma 6.5 (vii); for the fourth, use Lemma 6.5 (vi); for the fifth note that $||P_j\, Y_{q/s}\varphi(H)|| = \exp(-q/2s)||Y_{q/s}P_j\varphi(H)|| \leq K$; for the sixth term, simply carry out the integration.

(ii) As in (i) we get an expression similar to (6.27) but this time using (6.25) instead of (6.24). Note that $t \geq 0$ and $u \leq 0$.

$$(Y^*_{u/t}\, V^*_t\, Y_{u/t}\, V_t - V_u)\varphi(H)$$

$$= -i\int_t^\infty ds\, Y^*_{u/s}\, V^*_s\, Y_{u/s}\, H_o\, D_s\{s[\exp(-u/s)-1] - (u^2/(2s))\}\varphi(H)V_s +$$

$$i(u^2/2)\int_t^\infty ds\, s^{-2}\, (Y^*_{u/s}\, V^*_s\, Y_{u/s}\, V_s - V_u)H\varphi(H) +$$

$$i(u^2/2)\int_t^\infty ds\, s^{-2}\, V_u\, H\varphi(H) -$$

$$i(u^2/2)\int_t^\infty ds\, s^{-2}\, Y^*_{u/s}\, V^*_s\, Y_{u/s}\, (\sum_j W_j\, P_j + W_o)\varphi(H)V_s -$$

$$u^2\int_t^\infty ds\, s^{-2}\, Y^*_{u/s}\, V^*_s\, Y_{u/s}\, (\sum_j P_j\Gamma_j(Q)+\Gamma_o)\varphi(H)V_s -$$

$$\int_t^\infty ds\, s^{-3}\, Y^*_{u/s}\, V^*_s\, Y_{u/s}\int_0^u dp\, p\int_0^p dq\, Y^*_{q/s}(\sum_j P_j\Gamma_j + \Gamma_o)Y_{q/s}\, V_s\varphi(H) -$$

$$u^2\int_t^\infty ds\, s^{-2}\, Y^*_{u/s}\, V^*_s\, Y_{u/s}\, \Gamma_o(Q)\varphi(H)V_s -$$

$$\int_t^\infty ds\, s^{-3}\int_0^u dp\, p\int_0^p dq\, Y^*_{q/s}\, \Gamma_o(Q)Y_{q/s}\varphi(H)V_s. \qquad (6.29)$$

Now transfering third term to the left, we get,

L.H.S. of (i)

$$\leq K \int_t^\infty ds \| (H_o+1)^{-1} Y_{u/t_1-u/s} V_s^* Y_{u/s} H_o \| \cdot |D_s \{s[\exp(-u/s)-1] - (u^2/(2s))\}| +$$

$$Ku^2 \int_t^\infty ds\ s^{-2} \| (H_o+1)^{-1} Y_{u/t_1} (Y_{u/s}^* V_s^* Y_{u/s} V_s - V_u) H\varphi(H) \| +$$

$$Ku^2 \int_t^\infty ds\ s^{-2} \| (1+|Q|)^{-\delta} P_j \varphi(H) V_s (A^2+P^2+1)^{-\frac{1}{2}} \| +$$

$$Ku^2 \int_t^\infty ds\ s^{-2} \| (1+|Q|)^{-\delta} \varphi(H) V_s (A^2+P^2+1)^{-\frac{1}{2}} \| +$$

$$Ku^2 \int_t^\infty ds\ s^{-2} \sum_j \| (H_o+1)^{-1} Y_{u/t_1-u/s} V_s^* Y_{u/s} P_j \| \cdot \| (1+|Q|)^{-\delta} V_s \varphi(H) (A^2+P^2+1)^{-\frac{1}{2}} \| +$$

$$Ku^2 \int_t^\infty ds\ s^{-3} \sum_j \| (H_o+1)^{-1} Y_{u/t_1-u/s} V_s^* \int_0^u dp\ p \int_0^p dq\ Y_{(u-q)/s} P_j \| +$$

$$K \int_t^\infty ds\ s^{-3} |u|^3 . \qquad (6.30)$$

As in (i) we show that each term of R.H.S. of (6.30) has the desired growth in u and decay in t. For the first term use (6.19) and (6.23); for the second, use Lemma 6.5 (ii); for the third, use Lemma 6.5 (vii); for the fourth, use Lemma 6.5 (vi); for the fifth, use $Y_{u/s} P_j = P_j Y_{u/s} \exp(u/(2s))$ and use (6.19), also use Lemma 6.5 (vi); for the sixth term use (6.19) and $Y_{(u-q)/s} P_j = P_j Y_{(u-q)/s} \exp((u-q)/2s)$; for the seventh term simply carry out the integration.

(iii) Since $HA = AH - iH + \sum_j \Gamma_j(Q) P_j + \Gamma_o(Q)$ and since $(|P|+1) A \varphi(H) (A^4+P^6+1)^{-\frac{1}{2}}$ is bounded, using (6.8), and Lemma 6.5 (iii), we get

$$[A^2, V_t]\ \varphi(H)\ (A^4+P^6+1)^{-\frac{1}{2}}$$

$$= \quad t\,B(1,t) + t\,B + B(2,t) + B(1,t) \cdot$$

The result follows by noting $A^2 V_t = [A^2, V_t] + V_t A^2$ and using Lemma 6.4 (ii) .

(iv) Obvious.

(v) Similar to Lemma 6.5 (iv). Putting $t_1 = t$ in (ii) and using (iv),(i),(ii),

$$||(H_o+1)^{-1}\{V_t^* Y_{u/t}V_t - Y_{u/t}V_u - i(u^2/(2t))Y_{u/t}V_u H\}\ \varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}||$$

$$\leq K\ (1+|u|)^4\ (1+|t|)^{-1-\delta}\ .$$

The result follows if $||\{Y_{u/t}-(1-iuA/t)\}V_u\ \varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}||$ and $||(1-Y_{u/t})\ V_u\ H\varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}||$ are bounded by $(1+|u|)^4\ (1+|t|)^{-1-\delta}$ and $(1+|u|)^2\ (1+|t|)^{-\delta}$ respectively. For the first one,

$$||\{Y_{u/t} - (1-iuA/t)\}V_u\varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}||$$

$$\leq K(u^2/t^2)\ ||A^2\ V_u\varphi(H)(A^4+P^6+1)^{-\frac{1}{2}}||$$

$$\leq K(1+|u|)^4\ (1+|t|)^{-2} \quad \text{by (iii).}$$

The second one follows similarly by Lemma 6.5 (iii).

(vi) Similar to Lemma 6.5 (v) noting that if $\psi'\varphi = 0$ then $\psi''\varphi = 0$.

(vii) Similar to Lemma 6.5 (vi).

Q.E.D.

<u>Theorem 6.7</u> Let $f \in \mathcal{H}_c(H)$, $a > 0$ and $\beta > 1-\delta$. Then

$$\underset{t\to\pm\infty}{\text{s-lim}}\ F(|Q| \geq a|t|^{\beta})\ U_t^*\ V_t\ f = 0$$

<u>Proof</u> (Step i) Let $g \in S(R^n)$, $\varphi \in C_o^\infty(G_o)$ and $f = \varphi(H)g$. Since $\varphi \in C_o^\infty(G_o)$, $\varphi(H) = \varphi(H_c)$. So $f \in \mathcal{H}_c(H)$. Further, by Lemma 6.4 (viii), $f \in D(Q_j)$ for each j. By Lemma 6.6 (vii) for γ in $[0,1+\delta]$ we get $||(1+|Q|)^{-\gamma}V_s f|| \leq K(1+|s|)^{-\gamma}$. So by (6.4)

$$||(H+i)^{-\frac{1}{2}}\ (V_t^*\ U_t\ Q_j\ U_t^*\ V_t\ -\ Q_j)f||$$

$$\leq K \int_0^t ds\{(1+|s|)^{-\delta} + |s|(1+|s|)^{-1-\delta}\} \leq K(1+|t|)^{1-\delta}$$

Since $f \in D(Q_j)$ and $D(H) = D(H_o)$ we have

$$\lim_{|t|\to\infty} ||(H_o+i)^{-\frac{1}{2}} Q_j U_t^* V_t |t|^{-\beta} f|| = 0 \text{ for every } j \qquad (6.31)$$

Since $[(H_o+1)^{-\frac{1}{2}}, Q_j]$ is a bounded operator, by (6.31),

$$\operatorname{s-lim}_{|t|\to\infty} |t|^{-\beta} |Q_j| U_t^*(H_o+i)^{-\frac{1}{2}} V_t f = 0 \text{ for every } j$$

so that

$$\lim_{|t|\to\infty} F(|Q| \geq a|t|^{\beta}) U_t^* (H_o+i)^{-\frac{1}{2}} V_t f = 0 \qquad (6.32)$$

Since $f \in \mathcal{H}_c(H)$, by Corollary 6.3 $\operatorname{w-lim}_{|t|\to\infty} V_t f = 0$. Since $(H_o+i)^{-\frac{1}{2}}-(H+i)^{-\frac{1}{2}}$ is compact $\operatorname{s-lim}_{|t|\to\infty} \{(H_o+i)^{-\frac{1}{2}} - (H+i)^{-\frac{1}{2}}\}V_t f = 0$. Then by (6.32)

$$\lim_{|t|\to\infty} F(|Q| \geq a|t|^{\beta}) U_t^* V_t f = 0 \qquad (6.33)$$

<u>(Step 2)</u>. The result follows for $f \in \mathcal{H}_c(H)$ since the family $F(|Q| \geq a|t|^{\beta}) U_t^* V_t$ is norm bounded and $\{\varphi(H)g : \varphi \in C_0^{\infty}(G_0), g \in S(R^n)\}$ is a total subset of $\mathcal{H}_c(H)$.

Q.E.D.

<u>Theorem 6.8</u> Let $\delta \in (\frac{1}{2},1)$. Then the wave operators

$$\Omega_{\pm} = \operatorname{s-lim}_{t\to\pm\infty} V_t^* Z_t$$

exist and strong asymptotic completeness holds. i.e.

$$\text{Range } \Omega_{\pm} = \mathcal{H}_c(H).$$

<u>Proof</u> (for + ve sign only) Existence is proved in Theorem 2.3 for $\delta > 0$. For asymptotic completeness it is enough to prove $f \in \text{Range } \Omega$ (where $\Omega_+ = \Omega$) whenever $f \in \mathcal{H}_c(H)$ has H-support compact in $(0,\infty)$. For such an f choose $\varphi \in C_0^{\infty}(0,\infty)$ such that $\varphi(H)f = f$.

Since $\varphi(H) - \varphi(H_o)$ is compact, by Corollary 6.3,

$$\operatorname*{s-lim}_{t \to \infty} \{1-\varphi(H_o)\}V_t f = 0.$$

Since $\delta \in (\frac{1}{2},1)$ we can choose β, β_1 such that $1-\delta < \beta < \beta_1 < (n+4\delta)/(2n+4)$.

By Theorem 6.7 and Lemma 4.8

$$\operatorname*{s-lim}_{t \to \infty} F(|Q| \geq at^{\beta_1}) Z_t^* V_t f = 0.$$

The result follows by Corollary 4.7.

Q.E.D.

<u>Remark 6.9</u> : Let $H_o = P_1^{2r} + P_2^{2r} + \ldots + P_n^{2r}$, $H = H_o + \sum_{|\alpha| \leq 2r-1} W_\alpha(Q)P^\alpha$ be self adjoint where W_α satisfy (2.4). Then taking $A=(PQ+QP)/(4r)$ one can try to extend the results of this section to prove strong asymptotic completeness for H when $\delta > \frac{1}{2}$.

§ 7 ASYMPTOTIC COMPLETENESS FOR TOTAL HAMILTONIAN OF THE STATIC ELECTROMAGNETIC FIELD WITH SHORT RANGE POTENTIALS

Let

$$H_o = -\tfrac{1}{2}\Delta ,$$

$$H_L = H_o + \sum_{|\alpha|\le 1} W_\alpha^L(Q)\, P^\alpha ,$$

and

$$H = H_L + \sum_{|\alpha|\le 1} W_\alpha^S(Q)\, P^\alpha \quad \text{on } L^2(R^n),$$

where the potentials W_α^L, W_α^S are real valued and satisfy

(i) $\sum_{j=1}^{n} \partial W_j^L(x)/\partial x_j = 0,$

(ii) $\sum_{j=1}^{n} \partial W_j^S(x)/\partial x_j = 0$ in the sense of distributions,

(iii) W_α^L satisfy (2.4) for each α with $\delta > \frac{1}{2}$,

(iv) For some $\lambda < 2$, $W_j^S \in M_\lambda$ for each $j = 0,1,\ldots,n$.

(for the definition of M_λ see Theorem 1.3)

(v) For some $r > \frac{1}{2}n$, $W_j(x)(1+|x|)^r \in L^2(R^n)$ for each $j = 0,1,\ldots,n$.

<u>Lemma 7.1</u> : Let the conditions (i), (ii), (iii) and (iv), hold. Then H is self adjoint with $D(H) = D(H_L) = D(H_o)$.

<u>Proof</u> : By Corollory 5.2, H_L is self adjoint with $D(H_L) = D(H_o)$. By condition (ii), (iv) $\sum_\alpha W_\alpha^S(Q)P^\alpha$ is a symmetric operator on $D(H_o)$.

By (iv) and Theorem 1.3 $\rho(\sum_\alpha W_\alpha^S(Q)P^\alpha, H_o) = 0$. So,

$$\rho(\sum_\alpha W_\alpha^S(Q)P^\alpha,\ H_L)$$

$$\leq \quad \rho(\sum_\alpha W_\alpha^S(Q)P^\alpha,\ H_o)\ \rho(H_o,L)$$

$$\leq \quad 0.1 \qquad\qquad \text{since } D(H_o) = D(H_L)$$

$$= \quad 0$$

Applying Theorem 1.1 the result follows.

Q.E.D.

We prove asymptotic completeness for H by using trace conditions. As a first step we have

<u>Lemma 7.2</u> Let (i) and (iii) hold. Then $D(H_o^M) = D(H_L^M)$ for all integers $M \geq 0$.

<u>Proof</u> : By induction on M it is easy verify that

$$H_L^M = H_o^M + \sum_{|\theta|\leq 2M-1} \Gamma_\theta(Q)P^\theta$$

where Γ_θ are infinitely differentiable bounded functions with bounded derivatives. The result follows from the above equality.

Q.E.D.

<u>Theorem 7.3</u> (Birman's theorem) [6]. Let A,B be self adjoint operators with $F(E|A)$, $F(E|B)$ the corresponding spectral projections. If

(i) $F(I|A)(A-B)F(I|B)$ is a trace class operator for every bounded interval I, and

(ii) there exist real valued functions f_1, f_2, f_3, f_4 such that $f_j(x) \geq 1$ for all x, $\lim_{|x|\to\infty} f_j(x) = \infty$ for $j = 1,2,3,4$ such that $f_1(A)f_2(B)^{-1}$ and $f_3(B)\ f_4(A)^{-1}$ are bounded , then the wave operators

$$\Omega_\pm(A,B) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itA}\ e^{-itB}\ E_{ac}(B)$$

exist and asymptotic completeness holds, i.e.,

$$\text{Range } \Omega_+(A,B) = \mathcal{H}_{ac}(A) = \text{Range } \Omega_-(A,B).$$

The theorem is the same as Theorem XI.10 of [6].

<u>Corollary 7.4</u> Suppose A and B are self adjoint operators such that

(i) $(A+i)^{-1}(A-B)(B+i)^{-M}$ is trace class for some $M \geq 0$

(ii) $D(A) = D(B)$

Then $\Omega_{\pm}(A,B)$ exist and asymptotic completeness holds.

<u>Proof</u> It is sufficient to verify hypothesis of Theorem 7.3. Since for any bounded interval I, $F(I|A)(A+i)$ and $(B+i)^M F(I|B)$ are both bounded (i) of Corollary 7.4 yields (i) of Theorem 7.3. For $f_j(x) = 1+|x|$ for $j = 1,2,3,4$, (ii) of Theorem 7.3 is seen to hold by using condition (ii) of Corollary 7.4.

Q.E.D.

<u>Theorem 7.5</u> Let (i), (ii), (iii), (iv) and (v) hold. Then the wave operators

$$\Omega_{\pm} = \underset{t \to \pm\infty}{\text{s-lim}} \; e^{itH} e^{-iX(t,P)}$$

exist and

$$\text{Range } \Omega_{\pm} = \mathcal{H}_{ac}(H)$$

<u>Proof</u> We prove only for the positive sign. Let

$$\Omega(H_L,H_o) = \underset{t \to \infty}{\text{s-lim}} \; e^{itH_L} e^{-iX(t,P)}$$

Then by Theorem 6.8 $\Omega(H_L,H_o)$ exists and Range $\Omega(H_L,H_o) = \mathcal{H}_{ac}(H_L)$.

By Lemma 7.1, $D(H) = D(H_L)$. By Lemma 7.2 $(H_o+i)^M(H_L+i)^{-M}$ is bounded. By condition (v) and Theorem 1.7 $(H-H_L)(H_o+i)^{-M}$ is trace class for $M > \frac{1}{2}(n+1)$. So, $(H-H_L)(H_L+i)^{-M}$ is trace class. Applying Corollary 7.4, the wave operator

$$\Omega(H,H_L) = \operatorname*{s-lim}_{t \to \infty} e^{itH} e^{-itH_L} E_{ac}(H_L)$$

exists and Range $\Omega(H,H_L) = \mathcal{H}_{ac}(H)$

By the chain rule [1,2,3,6] for wave operators,

$$\Omega(H,H_L)\,\Omega(H_L,H_o) = \Omega = (\Omega_+)$$

and so

$$\text{Range}\,\Omega = \text{Range}\;\Omega(H,H_L) = \mathcal{H}_{ac}(H).$$

Q.E.D.

<u>Remark 7.6</u> Theorem 7.5 does not say anything about the equality of $\mathcal{H}_c(H) = \mathcal{H}_{ac}(H)$. This is a limitation of all proofs of asymptotic completeness involving trace methods.

<u>Remark 7.7</u> In § 4 of [23] when $W_j^L = 0$ for $j = 1,2,\ldots,n$ it has been proved that for ε in $(0,1)$, $\varphi \in C_o^\infty(0,\infty)$ and $t \gtrless 0$

$$\|F(A \gtrless 0)\exp(itH_L)\,\varphi(H_L)(1+|A|)^{-1-\varepsilon}\| \le K(1+|t|)^{-1-\frac{1}{2}\varepsilon\delta}.$$

In § 6, we could have proved a similar result from which strong asymptotic completeness would follow for $H = H_o + \sum_{|\alpha|\le 1} (W_\alpha^L(Q) + W_\alpha^S(Q)P^\alpha$ where in addition to (i) to (iv) we require that $W_\alpha^S(Q)(1+|Q|)^{1+\varepsilon}(1+|P|)^{-1}$ is bounded for some $\varepsilon > 0$ and all α(instead of (v)). So as to keep this thesis with in reasonable length we have not ventured in this direction.

APPENDIX

In this appendix we state the results on the growth properties of composition of certain functions and on the stationary phase estimates, which were frequently used in Sections 2 and 4. Proofs of these results can be found in [28].

Lemma A.1 : Assume that $\psi : R^{n_1} \to R^{n_2}$ and $\varphi : R^{n_2} \to R^{n_3}$ are C^∞ germs at x and at $y = \psi(x)$, depending on a parameter t, which satisfy estimates of the form (remembering that K is a generic constant),

$$|D^\alpha \psi(x)| \leq K_\alpha t^{a(|\alpha|)} , \quad |D^\beta \varphi(y)| \leq K_\beta t^{b(|\beta|)} \quad t > 1$$

for all α, β. If a and b are convex sequences, it follows that

$$|D^\gamma(\varphi \cdot \psi)(x)| \leq K_\gamma t^{c(|\gamma|)} \quad \gamma \neq 0, \; t > 1$$

where $c(k) = \max \{b(1) + a(k), b(k) + ka(1)\}$.

Moreover

$$|D^\gamma(\varphi \cdot \psi)(x) - \varphi'(y) D^\gamma \psi(x)| \leq K_\gamma t^{d(|\gamma|)} \quad |\gamma| > 1, \quad t > 1$$

where

$$d(k) = \text{Max} \{b(2)+a(1)+a(k-1); \; b(k)+ka(1)\}$$

Proof Same as Lemma 3.6 of [28] Q.E.D.

Lemma A.2 Assume that $\psi : R^n \to R$ is a C^∞ germ at x depending on a parameter t satisfying the estimates

$$|D^\alpha \psi(x)| \leq K_\alpha t^{1+\nu(|\alpha|)} \quad t > 1$$

where ν is as in Theorem 2.2. Let $\varphi : R^n \to R$ be given by $\varphi(x) = \exp(i\psi(x))$. Then the C^∞ germ φ satisfies

$$|D^\alpha \varphi(x)| \leq K_\alpha t^{(1-\delta)|\alpha|} \quad t > 1$$

Proof : If $\alpha = 0$ the result is clear. Let $\alpha \neq 0$. Define $\varphi_1, \varphi_2 : R \to R$ by $\varphi_1(x) = \cos x, \varphi_2(x) = \sin x$. Then by Lemma A.1 $x \to \varphi_1(\psi(x))$ and $x \to \varphi_2(\psi(x))$ are germs satisfying

$$|D^\alpha \varphi_1\psi(x)| + |D^\alpha \varphi_2\psi(x)| \leq K_\alpha t^{(1-\delta)|\alpha|}$$

The result follows by noting $\varphi(x) = \varphi_1(\psi(x)) + i\, \varphi_2(\psi(x))$.

Q.E.D.

Lemma A.3 Let C be a compact subset of R^n, G_o a neighbourhood of C and $f \in C^{k+1}(G_o,R)$ a function with $f' \neq 0$ in C. Then we have for ω in R and u in $C_o^k(C)$

$$|\int dx\, u(x)\exp(i\omega f(x))| \leq K_M(1+|\omega|)^{-M} \sum_{|\alpha|\leq M} ||D^\alpha u||_\infty$$

The same constant K_M can be used for all f in a compact subset of $C^{k+1}(G_c)$ satisfying the hypotheses.

Proof Same as Lemma A.1 of [28].

Q.E.D.

Lemma A.4 Let A be any real symmetric nondegenerate matrix of signature σ. Then the following identity is valid when $\omega > 0$, $u \in S(R^n)$

$$\int dx\, u(x)e^{i\omega<Ax,x>/2} = |\det 2\pi\omega A|^{-\frac{1}{2}} e^{\pi i\sigma/4} \int d\xi\, \hat{u}(\xi)\, e^{-i<A^{-1}\xi,\xi>/(2\omega)}$$

For every integer $k > 0$ and every $s > 2k+\frac{1}{2}n$ we have when $u \in S(R^n)$,

$$|\int dx\, u(x)e^{i\omega<Ax,x>/2} - |\det \omega A/2\pi|^{-\frac{1}{2}} e^{\pi i\sigma/4} \sum_{0}^{k-1} (2i\omega)^j (j!)^{-1}(<A^{-1}P,P>^j u)(0)|$$

$$\leq K_{k,s}\, \omega^{-k-\frac{1}{2}n} ||u||_{(s)} \qquad \omega > 1$$

Here $K_{k,s}$ is a continuous function of A and s, and

$$||u||_{(s)} = \{(2\pi)^{-n} \int d\xi\, |\hat{u}(\xi)|^2 (1+|\xi|^2)^s\}^{\frac{1}{2}}$$

Proof : Same as Lemma A.2 of [28].

Q.E.D.

Lemma A.5 Let $f \in C^\infty(G_o)$ be a real valued function in a neighbourhood G_o of 0 in R^n and assume that $f'(0) = 0$ and that $A = f''(0)$ is non-singular. Then there exist neighbourhoods G_1, G_2 of 0 in R^n and a diffeomorphism $X: G_1 \to G_2$ such that $X(x)-x = O(|x|^2)$ as $x \to 0$ and $f(x) = f(0)+\langle AX,X\rangle/2$ for $x \in G_1$. Further there exist differential operators $L_{f,j}$ of order $2j$ such that when $u \in C_o^\infty(C)$, C compact in G_1,

$$\left|\int dx\, u(x)e^{i\omega f(x)} - \sum_{j=0}^{k-1} |\det f''(0)/2\pi|^{-\frac{1}{2}}\, e^{i(\omega f(0)+\pi\sigma/4)}\omega^{-j-\frac{1}{2}n}\,(L_{f,j}\,u)(0)\right|$$

$$\leq K\,\omega^{-k-\frac{1}{2}n} \sum_{|\alpha|\leq s} \|D^\alpha u\|_\infty \quad \text{provided } s > 2k+\tfrac{1}{2}n \text{ and } \omega > 1$$

Proof : Same as Lemma A.3, A.4 of [28]. Q.E.D.

Lemma A.6 Let f be a C^∞ germ at x_o with $f'(x_o) = 0$ and $A = f''(0)$ non singular. Assume that for some $t > 1$

$$|D^\alpha f(x_o)| \leq K_\alpha\, t^{a(|\alpha|)} \qquad |\alpha| > 1$$

where $a(k)$ is a convex sequence with $a(1) = a(2) = 0$. Then there is a diffeomorphism ψ of a neighbourhood of 0 on a neighbourhood of x_o such that

$$f(\psi(y)) = f(x_o) + \langle Ay,y\rangle/2$$

$$|D^\alpha \psi(0)| \leq K_\alpha^*\, t^{a(|\alpha|+1)}$$

where K_α^* depends only on the constants K and on a bound for A^{-1}.

Proof : Same as Lemma A.6 of [28]. Q.E.D.

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