

# ON SOME ASYMPTOTIC PROPERTIES OF $U$ STATISTICS AND ONE-SIDED ESTIMATES

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Let  $\{X_i, 1 \leq i \leq n\}$  be independent and identically distributed random variables. For a symmetric function  $h$  of  $m$  arguments, with  $\theta = Eh(X_1, \dots, X_m)$ , we propose estimators  $\theta_n$  of  $\theta$  that have the property that  $\theta_n \rightarrow \theta$  almost surely (a.s.) and  $\theta_n \geq \theta$  a.s. for all large  $n$ . This extends the results of Gilat and Hill, who proved this result for  $\theta = Eh(X_1)$ . The proofs here are based on an almost sure representation that we establish for  $U$  statistics. As a consequence of this representation, we obtain the Marcinkiewicz-Zygmund strong law of large numbers for  $U$  statistics and for a special class of  $L$  statistics.

**1. Introduction.** Let  $X, X_1, X_2, \dots$  be a sequence of independent and identically distributed (iid) observations from a distribution with finite mean  $\mu$ . The usual estimate  $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$  has the property that  $\bar{X}_n \rightarrow \mu$  a.s. and it fluctuates around  $\mu$ . However, in many practical situations it may be desirable to have an estimate  $\mu_n$  of  $\mu$  that is conservative in the sense that  $\mu_n \rightarrow \mu$  a.s. and  $\mu_n \geq \mu$  a.s. for all large  $n$ . We will then say  $\mu_n$  converges to  $\mu$  from above and write  $\mu_n \rightarrow_+ \mu$  a.s. ( $\mu_n \rightarrow_- \mu$  is defined in a similar manner). A candidate estimator for the convergence from above is one that puts more weight to the higher order statistics. Consider then the following estimator

$$(1.1) \quad \hat{X}_n = \sum_{i=1}^n \left( \frac{1}{n} - \frac{n+1}{2n^\alpha} + \frac{i}{n^\alpha} \right) X_{(i)},$$

where  $X_{(1)} \leq \dots \leq X_{(n)}$  is the order statistics of  $X_1, \dots, X_n$  and  $\alpha > 2$  is an appropriate constant. The following theorem was proved in Gilat and Hill (1992) (hereafter referred to as GH).

**THEOREM 1.1** [Gilat and Hill (1992)]. *If  $E|X|^{1+\gamma} < \infty$  for some  $\gamma > 0$ , then for any  $\alpha, 2 < \alpha < \min(2 + \gamma/(1 + \gamma), 5/2)$ ,  $\hat{X}_n \rightarrow_+ \mu$  a.s.*

The proof in GH is based on the following facts:

1. If  $E|X|^{1+\gamma} < \infty$  for some  $0 < \gamma < 1$ , then  $\bar{X}_n - \mu = o(n^{-\gamma/(1+\gamma)})$  a.s. This is known as the Marcinkiewicz-Zygmund strong law of large numbers; see, for example, Chow and Teicher (1978).

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2. If  $E|X| < \infty$ , then

$$(1.2) \quad \lim_{n \rightarrow \infty} \frac{2}{n^2} \sum_{i=1}^n iX_{(i)} = E(\max(X_1, X_2)) \quad \text{a.s.}$$

(This is Theorem 1.1 of GH with  $k = 1$ .)

Observe that  $\hat{X}_n$  may be expressed as

$$(1.3) \quad \hat{X}_n = \bar{X}_n + \frac{1}{4n^\alpha} \sum_{i,j=1}^n |X_i - X_j|$$

and hence may also be identified as a  $U$  statistic, with varying kernel, of order 2.

We prove an almost sure representation theorem for  $U$  statistics (see Theorem 2.1), and use it to establish the Marcinkiewicz–Zygmund strong law of large numbers for  $U$  statistics (see Remark 2.1) and a special class of  $L$  statistics (see Remark 2.2). This may be used to give a different proof of Theorem 1.1. It may also be used to obtain the almost sure rate of convergence in Theorem 1.1 of GH (see Remark 2.2). In fact, representation (1.3) also suggests a way of extending the results of GH to the following situation.

Suppose we wish to estimate  $\theta = Eh(X_1, \dots, X_m)$ , where  $h$  is a symmetric function of its arguments. For instance, when  $\theta$  is the population variance, we let  $m = 2$  and  $h(x_1, x_2) = (x_1 - x_2)^2/2$ . We will use the notation  $U_n(g)$  to denote the  $U$  statistics based on  $X_1, \dots, X_n$  corresponding to the *symmetrized* version of the kernel function  $g$ . Consider the function

$$D(x_1, \dots, x_{2m}) = |h(x_1, \dots, x_m) - h(x_{m+1}, \dots, x_{2m})|.$$

Define the estimator

$$(1.4) \quad \theta_n = U_n(h) + a_n U_n(D),$$

where  $a_n$  is an appropriate sequence of positive constants converging to zero. We shall show that  $\theta_n \rightarrow_+ \theta$  a.s. (see Theorem 2.2).

It is interesting to note that the smaller the value of  $a_n$ , the lesser is the bias of the estimator. We will allow a wider choice of  $a_n$  than allowed by GH. In fact, as the proofs will show, our choice of  $a_n$  is rather tight.

We use a similar idea for the problem of quantile estimation. For a suitably constructed empirical distribution function  $G_n$ , we show that  $G_n(x) \rightarrow_+ F(x)$  a.s., where  $F$  is the cumulative distribution function of  $h(X_1, \dots, X_m)$ . The  $p$ th quantile of  $G_n$  is shown to be an upper estimate of the  $p$ th quantile of  $F$  (see Theorem 2.3).

In a subsequent paper, we will report finer asymptotic properties of our estimators.

**2. Main results.**

2.1. *An almost sure representation for  $U$  statistics.* If  $U_n(h)$  is a  $U$  statistic with kernel  $h$  and  $Eh^2 < \infty$ , then it is known that

$$(2.1) \quad U_n = \widehat{U}_n + R_n,$$

where  $\widehat{U}_n$  is the usual projection of  $U_n$  and  $R_n = o(n^{-1}(\log n)^\delta)$  a.s. for all  $\delta > 1/2$ ; see, for example, Serfling [(1980), page 189]. We establish a similar representation under the weaker assumption  $E|h|^{1+\gamma} < \infty$  for some  $\gamma > 0$ . This result is of independent interest. It will be used to obtain the Marcinkiewicz–Zygmund strong law of large numbers (to be abbreviated as MZSLLN) for  $U$  statistics and for a special class of  $L$  statistics. To state our result on  $U$  statistics, we will adopt the notations of Serfling (1980). Also,  $C$  will denote a generic positive constant throughout the paper.

**THEOREM 2.1.** *Suppose  $U_n$  is the  $U$  statistic based on the symmetric kernel  $h$ , where  $E|h(X_1, \dots, X_m)|^{1+\gamma} < \infty$  for some  $0 < \gamma \leq 1$ . Then*

$$(2.2) \quad U_n - \theta = \widehat{U}_n - \theta + R_{2n} + \dots + R_{mn},$$

where

$$(2.3) \quad R_{jn} = o(n^{-j\gamma/(1+\gamma)}(\log n)^{1/(1+\gamma)}(\log \log n)^\delta) \quad \text{a.s.}$$

for any  $\delta > 1/(1+\gamma)$ . Further, if for some  $c > 1$ ,  $\zeta_1 = \dots = \zeta_{c-1} = 0$ , then

$$\widehat{U}_n - \theta = R_{2n} = \dots = R_{c-1,n} = 0 \quad \text{a.s.}$$

**PROOF.** Define as in Serfling [(1980), page 177]

$$(2.4) \quad \widetilde{h}_1(x) = Eh(x, X_2, \dots, X_m) - \theta$$

and  $\widehat{U}_n - \theta = (m/n)\sum_{i=1}^n \widetilde{h}_1(X_i)$ . Note that

$$U_n - \theta = \widehat{U}_n - \theta + R_n,$$

where

$$R_n = \sum_{j=2}^m \binom{m}{j} \binom{n}{j}^{-1} S_{jn} = \sum_{j=2}^m R_{jn}$$

is also a  $U$  statistic with kernel

$$H(x_1, \dots, x_m) = h(x_1, \dots, x_m) - \sum_{i=1}^n \widetilde{h}_1(x_i) - \theta$$

and for each  $j = 2, \dots, m$ ,

$$S_{jn} = \sum_{1 \leq i_1 < \dots < i_j \leq n} g_j(X_{i_1}, \dots, X_{i_j}), \quad n \geq j,$$

is a martingale and  $Eg_j(x_1, \dots, x_{j-1}, X_j) = 0$ . See Serfling [(1980), page 178] for the definition of the  $g_j$ 's.

We first establish (2.3) for  $j = 2$ . Define

$$\lambda_n = n^{2\gamma/(1+\gamma)}(\log n)^{-1/(1+\gamma)}(\log \log n)^{-\delta}.$$

It is enough to show that for any  $\varepsilon > 0$ ,

$$P(\lambda_n n^{-2} |S_{2n}| > \varepsilon \text{ i.o.}) = 0.$$

Since  $\lambda_n$  is nondecreasing for large  $n$ , it suffices to show that

$$\sum_{k=1}^{\infty} P(B_k) = \sum_{k=1}^{\infty} P\left(\lambda_{2^{k+1}} \max_{2^k \leq n \leq 2^{k+1}} n^{-2} |S_{2n}| > \varepsilon\right) < \infty.$$

For any  $n \geq 2$ ,

$$\begin{aligned} E|S_{2n}|^{1+\gamma} &= E \left| \sum_{1 \leq i_1 < i_2 \leq n} g_2(X_{i_1}, X_{i_2}) \right|^{1+\gamma} = E \left| \sum_{i_2=2}^n \sum_{i_1=1}^{i_2-1} g_2(X_{i_1}, X_{i_2}) \right|^{1+\gamma} \\ &= E \left| \sum_{i_2=2}^n D_2(i_2) \right|^{1+\gamma}, \end{aligned}$$

where  $D_2(i_2) = \sum_{i_1=1}^{i_2-1} g_2(X_{i_1}, X_{i_2})$ ,  $2 \leq i_2 \leq n$ , is a martingale difference sequence.

By Burkholder's inequality, the above expectation is bounded by

$$CE \left| \sum_{i_2=2}^n D_2^2(i_2) \right|^{(1+\gamma)/2} \leq C \sum_{i_2=2}^n E|D_2(i_2)|^{1+\gamma}.$$

Now observe that for every fixed  $i_2$ ,  $g_2(X_{i_1}, X_{i_2})$ ,  $1 \leq i_1 \leq i_2 - 1$ , is a martingale difference. Thus, using the same argument again,

$$(2.5) \quad E|S_{2n}|^{1+\gamma} \leq C \sum_{i_2=2}^n \sum_{i_1=1}^{i_2-1} E|g_2(X_{i_1}, X_{i_2})|^{1+\gamma} \leq Cn^2$$

since  $E|h|^{1+\gamma} < \infty$  implies  $E|g_2|^{1+\gamma} < \infty$ .

Using (2.5) and the maximal inequality for the martingale  $S_{2n}$ ,

$$\begin{aligned} \sum_{k=1}^{\infty} P(B_k) &\leq \sum_{k=1}^{\infty} P\left(\lambda_{2^{k+1}} \sup_{2^k \leq n \leq 2^{k+1}} |S_{2n}| \geq \varepsilon 2^{2k}\right) \\ &\leq C \sum_{k=1}^{\infty} (\lambda_{2^{k+1}})^{1+\gamma} (2^{2k} \varepsilon)^{-(1+\gamma)} (2^{k+1})^2 \\ &\leq C \sum_{k=1}^{\infty} k^{-1} (\log k)^{-\delta(1+\gamma)} < \infty, \end{aligned}$$

since  $\delta > 1/(1 + \gamma)$ . This completes the proof for  $j = 2$ .

A similar argument shows that

$$E|S_{jn}|^{1+\gamma} \leq Cn^j.$$

Using the martingale property again, we have the required order for  $R_{jn}$ . The second part of the theorem is trivial.  $\square$

REMARK 2.1. Note that  $(\hat{U}_n - \theta)$  is a mean of iid random variables with zero mean and finite  $(1 + \gamma)$ th absolute moment. Thus from Theorem 2.1 it follows that:

(i) If  $0 < \gamma < 1$  and  $\zeta_1 > 0$ , then  $U_n - \theta = o(n^{-\gamma/(1+\gamma)})$  a.s. This may be termed as the MZSLLN for  $U$  statistics.

(ii) If  $0 < \gamma < 1$  and  $\zeta_1 = \dots = \zeta_{c-1} = 0$ ,  $\zeta_c > 0$ , for some  $c > 1$ , then

$$U_n - \theta = o(n^{-c\gamma/(1+\gamma)}(\log n)^{1/(1+\gamma)}(\log \log n)^\delta) \quad \text{a.s.}$$

for any  $\delta > 1/(1 + \gamma)$ .

(iii) If  $\gamma = 1$  and  $\zeta_1 > 0$ , then using the LIL for iid random variables, one may obtain

$$U_n - \theta = O(n^{-1/2}(\log \log n)^{1/2}) \quad \text{a.s.}$$

REMARK 2.2. GH have shown that if  $E|X| < \infty$ , then for any nonnegative integer  $k$ ,

$$L_n(k) = \frac{k+1}{n^{k+1}} \sum_{i=1}^n i^k X_{(i)} \rightarrow E(\max(X_1, \dots, X_k)) = M_k \quad \text{a.s.},$$

where  $X_{(1)} \leq \dots \leq X_{(n)}$  are the order statistics of  $X_1, \dots, X_n$ . See also Helmers (1977) and van Zwet (1980).

Note that for each  $k$ ,  $L_n(k)$  is an  $L$  statistic and  $L_n(0)$  is simply the sample mean. Given the MZSLLN for the sample mean, it is natural to ask if a similar result may be proved for  $L_n(k)$ .

(a) Remark 2.1 may be used to show that if  $E|X|^{1+\gamma} < \infty$ , for some  $0 < \gamma < 1$ , then

$$(2.6) \quad L_n(k) - M_k = o(n^{-\gamma/(1+\gamma)}) \quad \text{a.s.}$$

This may be proved as follows. First let  $k = 1$ . Consider the kernel  $h(x_1, x_2) = \max(x_1, x_2)$  and the corresponding  $U$  statistic  $U_n(h)$ . From Remark 2.1 it follows that

$$(2.7) \quad U_n(h) - M_2 = o(n^{-\gamma/(1+\gamma)}) \quad \text{a.s.}$$

On the other hand,

$$\begin{aligned} U_n(h) &= \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \max(X_{(i)}, X_{(j)}) \\ &= \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} X_{(j)} \\ &= \binom{n}{2}^{-1} \sum_{1 \leq j \leq n} X_{(j)}(j-1) \\ &= \frac{n}{n-1} L_n(1) - \frac{2}{n-1} L_n(0) \end{aligned}$$

and hence (2.6) follows from the usual MZSLLN for sample mean and (2.7). For general  $k$ , it may be shown by algebraic calculations that

$$L_n(k) = U_n(h) + R_n,$$

where  $h(x_1, \dots, x_k) = \max(x_1, \dots, x_k)$  and  $R_n$  involves linear combinations of  $L_n(s)$ ,  $1 \leq s \leq k-1$ , with coefficients of smaller order. Hence by induction, the result follows for all  $k$ .

(b) when  $\gamma = 0$ , the above arguments can be used, along with the strong law of large numbers for  $U$  statistics (see Serfling (1980), Theorem A, page 190] to show that  $L_n(k) - M_k = o(1)$  a.s. This is precisely Theorem 1.1 of GH.

2.2. *One sided convergence.* Recall the estimate

$$\theta = U_n(h) + a_n U_n(D).$$

Define  $\Delta = E|h(X_1, \dots, X_m) - h(X_{m+1}, \dots, X_{2m})|$  and note that  $\Delta > 0$  whenever  $F$ , the distribution of  $h(X_1, \dots, X_m)$ , is nondegenerate.

THEOREM 2.2. Suppose  $E|h(X_1, \dots, X_m)|^{1+\gamma} < \infty$  for some  $0 < \gamma \leq 1$ .

- (i) If  $\gamma < 1$ , then  $\theta_n \rightarrow_+ \theta$  a.s. provided  $\liminf a_n n^{\gamma/(1+\gamma)} > 0$ .
- (ii) If  $\gamma = 1$ , then  $\theta_n \rightarrow_+ \theta$  a.s. provided  $\liminf a_n n^{1/2} (\log \log n)^{-1/2} = \infty$ .

REMARK 2.3. Note that taking  $m = 1$  and  $h(x) = x$ , we essentially get the estimator  $\widehat{X}_n$  of GH with  $a_n = n^{-(\alpha-2)}$ , where  $\alpha$  is as in Theorem 1.1. Theorem 2.2 is a stronger assertion than GH for any  $0 < \gamma \leq 1$ .

PROOF. When  $F$  is degenerate, there is nothing to prove. When  $F$  is not degenerate, write

$$\theta_n - \theta = U_n(h) - \theta + a_n (U_n(D) - \Delta) + a_n \Delta.$$

From Remark 2.1, when  $\gamma < 1$ ,

$$U_n(h) - \theta = o(n^{-\gamma/(1+\gamma)}) \quad \text{a.s.}$$

and by the strong law of large numbers for  $U$  statistics,

$$U_n(D) - \Delta = o(1) \quad \text{a.s.}$$

and hence (i) follows from the condition on the sequence  $a_n$ .

When  $\gamma = 1$ , using the LIL for  $U$  statistics,

$$|U_n(h) - \theta| = O(n^{-1/2}(\log \log n)^{1/2}) \quad \text{a.s.}$$

Thus again the result follows by using the given condition on  $a_n$ .  $\square$

REMARK 2.4. When  $\zeta_1 = \dots = \zeta_{c-1} = 0$  the range of values for  $a_n$  may be extended by using Remark 2.1.

REMARK 2.5. It may be noted that there are other estimators that will achieve positive convergence. In general, any estimator of the form

$$\theta_{1n} = U_n(h) + a_n E_n,$$

where  $E_n$  is such that  $E_n - E = o(1)$  a.s. for some  $E > 0$ , will converge from above to  $\theta$ . Some possible choices are  $U_n(D^*)$  for a kernel  $D^*$  such that  $E(D^*) > 0$ ,  $\bar{X}_n^2$  when  $E(X) \neq 0$ ,  $U_n(|h|)$  and so forth. The choice  $U_n(D)$  that we have used has the advantage that it has the same order of moments as  $U_n(h)$ , provides a  $U$  statistics representation for  $\theta_n$  and also has the appeal that it is a dispersion index. Even though our later asymptotic results will be stated and proved for the estimator  $\theta_n = U_n(h) + a_n U_n(D)$ , it will be clear from the proofs that many of these results remain valid for  $\theta_n$  with  $D$  replaced by any other suitable kernel  $D^*$ .

The optimality of  $U_n(h)$  as an estimate of  $\theta$  is well known. It will be an interesting problem to obtain guidelines for the choice of the perturbation that is added to it to obtain convergence from above. Our choice  $U_n(D)$  may play a significant role in this respect.

2.3. *Estimation of quantiles.* Let  $F$  be the distribution of  $h(X_1, \dots, X_m)$ , where  $h$  is a symmetric kernel, and let  $F_n$  be the empirical distribution function that puts equal mass at each  $W_i$ , the  $N = \binom{n}{m}$  values of  $h(X_{i_1}, \dots, X_{i_m})$ .

Define

$$(2.8) \quad G_n = F_n - a_n F_n(1 - F_n).$$

Observe that

$$(2.9) \quad G_n(x) = \frac{1}{N} \sum_{j=1}^N I(W_{(j)} \leq x) \left[ 1 - a_n \left( 1 + \frac{1}{N} - \frac{2j}{N} \right) \right],$$

where  $W_{(1)} \leq W_{(2)} \leq \dots \leq W_{(N)}$  are the ordered values of the  $W_i$ 's,  $1 \leq i \leq N$ . Note that  $G_n$  is an empirical distribution function that gives more weight to the higher order statistics and less to the lower ones. It may be noted that the estimator  $G_n$  resembles the estimator given in (12) of GH when  $m = 1$ .

In the following lemma we derive a probabilistic bound on the Kolmogorov-Smirnov statistic  $D_n$  for the empirical distribution of a  $U$  statistic. This bound, which will be used in the proof of Theorem 2.3 on the upper convergence of quantiles, is slightly weaker than the best possible bound known for the empirical distribution of iid observations.

LEMMA 2.1. *Let  $F_n$  be the empirical distribution function of a  $U$  statistic with kernel  $h(x_1, \dots, x_m)$ . Define  $D_n = \sup_x |F_n(x) - F(x)|$ , where  $F$  is the distribution function of  $h(x_1, \dots, x_m)$ . Then*

$$(2.10) \quad P(D_n \geq t) \leq C_1 t^{-1} \exp(-C_2 n t^2)$$

for some constants  $C_1$  and  $C_2$ .

PROOF. For any integer  $s$ , let  $x_{s,k} = \phi(k/s)$ , where  $\phi(u) = \inf\{x: F(x) \geq u\}$  and define

$$\begin{aligned} D_{s,n} &= \max_{1 \leq k \leq s} \max \{ |F_n(x_{s,k}) - F(x_{s,k})|, |F_n(x_{s,k-}) - F(x_{s,k-})| \} \\ &= \max_{1 \leq k \leq s} \max \{ \alpha_{n,s,k}, \beta_{n,s,k} \} \end{aligned}$$

as in Billingsley [(1991), page 276].

Then  $D_n \leq D_{s,n} + 1/s$ . Choosing  $s = [2/t] + 1$ , we have

$$(2.11) \quad P(D_n \geq t) \leq P(D_{s,n} \geq t - 1/s) \leq P(D_{s,n} \geq t/2).$$

For each  $s$ , the terms  $\alpha_{n,s,k}$  and  $\beta_{n,s,k}$  are  $U$  statistics with a kernel that is bounded by 1.

Using Theorem A of Serfling [(1980), page 201], for each  $t$ ,

$$(2.12) \quad P(\alpha_{n,s,k} \geq t) \leq 2 \exp(-2[n/m]t^2).$$

Using Bonferroni's inequality and relations (2.11) and (2.12),

$$P(D_n \geq t) \leq 4 \left( [2/t] + 1 \right) \exp(-2[n/m]t^2).$$

This proves the lemma.  $\square$

For any  $p$ , let

$$\xi_p = \inf\{x: F(x) \geq p\}$$

be the  $p$ th quantile of  $F$  and let

$$\widehat{\xi}_{p_n} = \inf\{x: G_n(x) \geq p\}$$

be the  $p$ th quantile of  $G_n$ . Also let

$$\psi_n(x) = x - a_n x(1-x), \quad 0 \leq x \leq 1, \quad \text{and} \quad p_n = \psi_n^{-1}(p).$$



We prove the following theorem.

**THEOREM 2.3.** *Let  $a_n$  be such that  $\liminf a_n n^{1/2} (\log \log n)^{-1/2} = \infty$ . Then:*

- (i)  $\lim_n \sup_x |G_n(x) - F(x)| = 0$  a.s.
- (ii)  $G_n(x) \rightarrow_- F(x)$  a.s. for every  $x$ .
- (iii)  $\widehat{\xi}_{p_n} \rightarrow \xi_p$  a.s. for all  $0 \leq p \leq 1$ . Further, if for all sufficiently large  $n$ ,

$$a_n p(1-p) > (3m/4)^{1/2} n^{-1/2} (\log n)^{1/2},$$

then  $\widehat{\xi}_{p_n} \rightarrow_+ \xi_p$  a.s. for all  $0 < p < 1$ .

- (iv) Mean of  $G_n \rightarrow$  mean of  $F$  a.s.

**PROOF.** The first part (i) follows from the observation that  $\|G_n - F_n\| \leq a_n$  and the Glivenko–Cantelli theorem for  $U$  statistics, which says that  $\|F_n - F\| \rightarrow 0$  a.s.

- (ii) Note that

$$G_n - F = (1 - a_n)(F_n - F) + a_n(F_n^2 - F^2) - a_n F(1 - F).$$

For each fixed  $x$  the first term is  $o(a_n)$  a.s. from the LIL for  $U$  statistics. The second term is obviously  $o(a_n)$ . Hence (ii) follows.

(iii) By using the Borel–Cantelli lemma, it easily follows from Lemma 2.1 that almost surely,

$$(2.13) \quad |D_n| \leq C_0 n^{-1/2} (\log n)^{1/2},$$

where  $C_0$  may be chosen to be any number greater than  $(3m/4)^{1/2}$ ; see the proof of Lemma 2.1. Note that  $G_n = \psi_n(F_n)$  and for every  $n$ ,  $\psi_n(x)$  is strictly increasing in  $x$ .

Using Lemma (iii) of Serfling [(1980), page 3], first observe that

$$(2.14) \quad \begin{aligned} G_n^{-1}(p) = \widehat{\xi}_{p_n} \geq \xi_p & \quad \text{iff } F_n^{-1}(\psi_n^{-1}(p)) \geq F^{-1}(p) \\ & \quad \text{iff } F(F_n^{-1}(\psi_n^{-1}(p))) \geq p. \end{aligned}$$

Using the bound (2.13) on  $D_n$  and Lemma (ii) of Serfling [(1980), page 3], we get

$$(2.15) \quad \begin{aligned} F(F_n^{-1}(\psi_n^{-1}(p))) & \geq F_n(F_n^{-1}(\psi_n^{-1}(p))) - C_0 n^{-1/2} (\log n)^{1/2} \\ & \geq \psi_n^{-1}(p) - C_0 n^{-1/2} (\log n)^{1/2}. \end{aligned}$$

Solving the quadratic equation  $\psi_n(x) = p$ , one gets

$$(2.16) \quad \psi_n^{-1}(p) = p + a_n p(1-p) + O(a_n^2).$$

Using the condition on  $a_n$ , the result follows from (2.14)–(2.16).

(iv) Note that

$$E_{G_n}(Y) = U_n(h) - \frac{a_n}{2N^2} \sum_{j=1}^N \left( \frac{N+1}{2} - j \right) W_{(j)}.$$

Thus

$$\begin{aligned} |E_{G_n}(Y) - U_n(h)| &\leq \frac{a_n}{N^2} \sum_{j=1}^N \left| \frac{N+1}{2} - j \right| |W_{(j)}| \\ &\leq C \frac{a_n}{2N} \sum_{j=1}^N |W_{(j)}| = C a_n U_n(|h|). \end{aligned}$$

Observe that  $a_n \rightarrow 0$  and thus using the SLLN for  $U$  statistics, (iv) follows.  $\square$

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