Pergamon

Pattern Recognition, Vol. 28, No. 12, pp. 1955–1963, 1995
Elsevier Science Ltd

0031-3203/95 $9.50+.00

0031-3203(95)00058-5

A NEW APPROACH TO COMPUTING THE EULER CHARACTERISTIC

P. K. SAHA* and B. B. CHAUDHURI

Electronics and Communication Sciences Unit, Indian Statistical Institute, 203 Barrackpur Trunk Road, Calcutta 700035, India

(*Received* 30 *August* 1994; *in revised form* 23 *March* 1995; *received for publication* 7 *April* 1995)

Abstract—In this paper we describe a new approach to computing the Euler characteristic of a three dimensional digital image. Our approach is based on computing the change in numbers of black components, tunnels and cavities in $3 \times 3 \times 3$ neighborhood of an object (black) point due to its deletion. The existing algorithms to computing the Euler characteristic of a 3D digital image are based on counting the numbers of all k-dimensional elements ($0 \leq k \leq 3$) in a polyhedral representation of the image. Our approach can be modified for (6, 26), (18, 6), (6, 18) and other connectivity relations of grid points. A parallel implementation of the algorithm is described using the concept of sub-fields.

3D digital topology   Betti numbers   Tunnels   Binary transformation
Euler characteristic   Sub-fields

1. INTRODUCTION

Three dimensional (3D) digital images result directly from laser range scanner, computer aided tomography (CAT) and magnetic resonance imaging (MRI) devices. Indirectly, computer vision algorithms generate 3D digital representation from 2D images and camera calibration information. The study of 3D digital topology is an important aspect of 3D image processing.[1–4] Topological information is useful in object classification, thinning and skeletonization of objects,[5–7] segmentation by parts,[8] and many other problems. Most of the publications on digital topology deal with binary images. In this paper we concentrate on the problems of computing the Euler characteristic of a 3D binary image. The Euler characteristic is an important topological feature that may be used in pattern classification. It is also interesting to study the behaviour of the Euler characteristic in a 3D digital space.

Several publications on computation of the Euler characteristic of digital image are found in the literature.[9–12] Dyer[11] proposed an interesting algorithm for computing the Euler characteristic of a 2D digital image from its quad-tree representation. In reference (9), a recursive algorithm is proposed for computing the Euler characteristic and other additive functionals of a n-dimensional digital image from its array representation. Bieri[10] modified their previous algorithm for computing the Euler characteristic and other additive functionals of a n-dimensional digital image from its n-tree representation. In reference (12), Voss proposed an interesting approach for computing the Euler characteristic of an object in n-dimensional homogeneous grid.

In this paper we describe a different approach and its parallel implementation to computing the Euler characteristic of a 3D digital image from its array representation. While other algorithms work on the number of all i-dimensional elements; $0 \leq i \leq n$ in a polyhedral representation of a digital object, our algorithm is based on finding the change in numbers of black components, tunnels and cavities in $3 \times 3 \times 3$ window of a point due to its deletion. In Section 2, general definitions and notations related to 3D digital topology are presented. Theoretical aspects of the Euler characteristic as well as a recursive equation for its computation are discussed in Section 3. In that connection a simple expression for the number of tunnels in $3 \times 3 \times 3$ neighborhood of a point is established in Section 3. An efficient algorithm to compute the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood of a point due to its deletion is described in Section 3. In Section 3, a parallel algorithm is proposed to compute the Euler characteristic of an image.

2. GENERAL DEFINITIONS AND NOTATIONS

At first we present a few definitions related to 3D digital topology frequently used in this paper. We consider 3D cubic grid[3] to represent a 3D digital image. In subsequent discussions, points refer to digital grid points unless stated otherwise. We follow the conventional definition of *α-neighborhood* or *α-adjacency* of points, where $\alpha \in \{6, 18, 26\}$. Two non-empty sets of points S_1 and S_2 are said to be α-adjacent if at least one point of S_1 is α-adjacent to at least one

* Author to whom correspondence should be addressed.

point of S_2. Let S be a non-empty set of points. An *α-path* between two points p, q in S means a sequence of distinct points $p = p_0, p_1, \ldots, p_n = q$ in S such that p_i is α-adjacent to p_{i+1}, $0 \le i < n$. An α-path $p_0, p_1, \ldots, p_n$ is an α-closed path if p_0 is α-adjacent to p_n. An *α-closed curve* is an α-closed path π such that every point of π is α-adjacent to exactly two other points of π. An α-closed curve with more than three points is called a *non-trivial α-closed curve*. Let $\pi_0, \pi_1, \ldots, \pi_n$ denote $n+1$ non-trivial α-closed curve. We say that they are independent if there exists no π_i such that

$$\pi_i \subseteq \bigcup_{j \neq i} \pi_j$$

Two points $p, q \in S$ are *α-connected* in S if there exists an α-path from p to q in S. An *α-component* of S is a maximal subset of S where each pair of points is α-connected.

A 3D digital image $\not{p}$ is defined as a quadruple $(v, \alpha, \beta, \mathscr{B})$. Here v is the image space which is a set of all grid points (i, j, k) where i, j, k are integers and $i_{min} \le i \le i_{max}, j_{min} \le j \le j_{max}, k_{min} \le k \le k_{max}$. In other words, v is a set of all cubic grid points in a finite rectangular parallelepiped. Also, $\mathscr{B}$ is the set of black points in $\not{p}$. In addition, α-adjacency and β-adjacency are used for finding α-components and β-components in $\mathscr{B}$ and $v-\mathscr{B}$, respectively. Note that $v-\mathscr{B}$ denotes the set of white points in $\not{p}$. In this paper we consider 26-adjacency for black points and 6-adjacency for white points. Obviously, a 26-component of $\mathscr{B}$ is a *black component* of $\not{p}$ while a 6-component of $v-\mathscr{B}$ is a *white component* of $\not{p}$. Since v denotes a set of grid points in a finite rectangular parallelepiped, we can define both interior and border of v. A point $p \in v$ is an *interior point* of v if all 26-neighbors of p are included in v. Similarly, a point $p \in v$ is a *border point* of v if all 26-neighbors of p are not included in v. The set of all border points of v is called the *border* of v and is denoted as v^*. A border point of v is called a *6-border point* of v if it has exactly five 6-neighbors included in v. A *cavity* in $\not{p}$ is a white component of $\not{p}$ surrounded by a black component. According to our convention for v a *cavity* may be defined as a white component of $\not{p}$ containing no border point of v.

Shrinking is a process of sequential transformation of black simple points to white in an image as long as the image contains at least one black simple point. A point is a *simple point* if its binary transformation does not change the image topology. See references (6, 7, 13) for more on simple point.

In the following discussions, $\mathscr{N}(p)$ is used to denote the set of 27 points in $3 \times 3 \times 3$ neighborhood of a point p including p itself. The set of points of $\mathscr{N}(p)$ excluding p is denoted as $\mathscr{N}^*(p)$. Note that $\mathscr{N}^*(p)$ is the border of $\mathscr{N}(p)$. We classify the points of $\mathscr{N}^*(p)$ according to their adjacency relations with p.

(1) An *s-point* is 6-adjacent p.

(2) An *e-point* is 18-adjacent but not 6-adjacent to p.

(3) A *v-point* is 26-adjacent but not 18-adjacent to p.

Nomenclature of the points of $\mathscr{N}(p)$ is explained in Fig. 1. In Fig. 1, E, W, S, N, T, B denote *east*, *west*, *south*, *north*, *top*, and *bottom points*, respectively. Similarly, TE denotes *top-east point* and so on. Let $x = (k_0, k_1, k_2)$ be an s-point of $\mathscr{N}(p)$ where $p = (l_0, l_1, l_2)$ i.e. $|k_i - l_i| = 1$ for some i and $k_j = l_j$ for all $i \neq j$. We define *surface*(x, p) as the set of points $(m_0, m_1, m_2) \in \mathscr{N}(p)$ such that $m_i = k_i$. It may be noted that a surface(x, p) contains exactly one s-point of $\mathscr{N}(p)$. Let $x = (k_0, k_1, k_2)$ be an e-point of $\mathscr{N}(p)$ where $p = (l_0, l_1, l_2)$ i.e $|k_i - l_i| = 1$ and $|k_j - l_j| = 1$

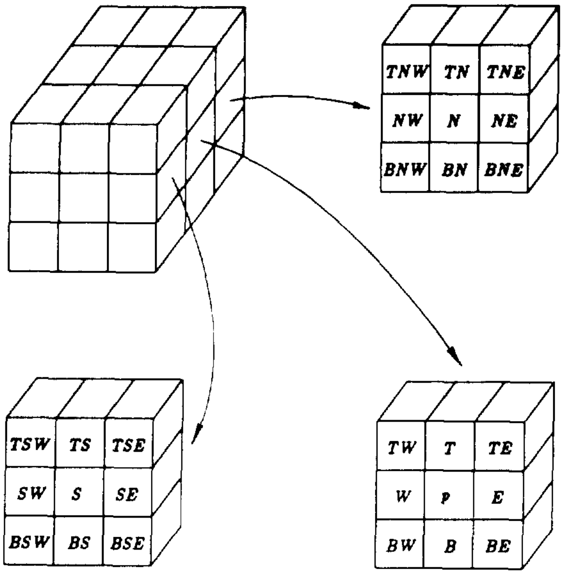

Fig. 1. Nomenclature of the points in $3 \times 3 \times 3$ neighborhood of a point p. Clockwise from top-left corner—neighborhood representation; back vertical plane; middle vertical plane; front vertical plane.

for some distinct i, j and $k_h = l_h$ for $h \neq i$, $h \neq j$. We define *edge* (x, p) as the set of points $(m_0, m_1, m_2) \in \mathcal{N}(p)$ such that $m_i = k_i$ and $m_j = k_j$. It may be noted that an edge(x, p) contains exactly one *e*-point of $\mathcal{N}(p)$.

Two *s*-points $a, b \in \mathcal{N}(p)$ are called *opposite* if they are not 26-adjacent. Otherwise, they are called *non-opposite* *s*-points. Let a, b, c denote three non-opposite *s*-points of $\mathcal{N}(p)$. Then we define the following two functions.

$e(a, b, p) = q | q \in \mathcal{N}^*(p)$ and 6-adjacent to a, b;

$v(a, b, c, p) = q | q \in \mathcal{N}^*(p)$ and 6-adjacent to $e(a, b, p)$, $e(b, c, p), e(c, a, p)$.

For example, if a, b, c denote the points N, T, E in $\mathcal{N}(p)$ then according to the above definitions $e(a, b, p)$ and $v(a, b, c, p)$ will denote the points TN and TNE, respectively. It may be noted that $e(a, b, p)$ is an *e*-point while $v(a, b, c, p)$ is a *v*-point of $\mathcal{N}(p)$.

3. THEORETICAL DISCUSSION

The Euler characteristic of a polyhedral set $S \subset \mathcal{R}^3$, denoted as $\chi(S)$ is defined by the following axioms.[1]

(1) $\chi(S) = 0$ if $S = \phi$;
(2) $\chi(S) = 1$ if S is non-empty and convex;
(3) for any two polyhedra X and Y, $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$.

For any arbitrary triangulation of a set S, the value of $\chi(S)$ is equal to the following alternating sum.[1]

$\chi(S) =$ number of points in S $-$ number of edges in S $+$ number of triangles in S $-$ number of tetrahedra in S.

The Euler characteristic of a polyhedral set $S \subset \mathcal{R}^3$ is also equal to the number of connected components in S minus the number of tunnels in S plus the number of cavities in S.[1] For example, the Euler characteristic of a hollow cube is two since it has one component, one cavity and no tunnels; the Euler characteristic of the border of a rectangle is zero since it has one component, one tunnel and no cavities. A component of a set S is defined as a maximal connected subset of S and a cavity in S is a component of $\mathcal{R}^3$–S surrounded by a component of S. Although it is quite difficult to define a tunnel in S, the number of tunnels has some precise definition in $\mathcal{R}^3$. The number of tunnels in a polyhedral set $S \subset \mathcal{R}^3$ is defined as the rank of its first homology group.[1,14] On the development of the Euler characteristic see references (15, 16).

An analogous definition of the Euler characteristic is introduced for a digital image $\mathcal{P}$ which is denoted as $\chi(\mathcal{P})$. To do so, each digital image is associated with a polyhedral set $C(\mathcal{P})$, defined as a continuous analog of $\mathcal{P}$.[3] The Euler characteristic $\chi(\mathcal{P})$ of a digital picture is defined as $\chi(\mathcal{P}) = \chi(C(\mathcal{P}))$.[1] We use the following axioms to compute the Euler characteristic of a 3D digital image.

Axiom 1. *For a 3D digital image $\mathcal{P}$, the Euler characteristic $\chi(\mathcal{P})$ equals the number of black components minus the number of tunnels plus the number of cavities in $\mathcal{P}$.*

Axiom 2. *Let $\mathcal{P} = (v, 26, 6, \mathcal{B})$ be a 3D digital image and let $p \in \mathcal{B}$ be a black point in $\mathcal{P}$. Under this assumption, the Euler characteristic of $\mathcal{P}$ is equal to the Euler characteristic of $(v, 26, 6, \mathcal{B} - \{p\})$ plus the change in the Euler characteristic in $\mathcal{N}(p)$ due to the deletion of p.*

Both Axiom 1 and Axiom 2 are important in our approach to computing the Euler characteristic of a 3D digital image. While Axiom 1 is motivated by above discussion Axiom 2 is discussed by other authors.[1,17] Let us consider a 3D digital image $\mathcal{P} = (v, 26, 6, \mathcal{B})$. To compute the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood of a point p we define two images in $\mathcal{N}(p)$ as follows.

$$\hat{\mathcal{N}}(p) = (\mathcal{N}(p), 26, 6, (\mathcal{N}(p) \cap \mathcal{B}) - \{p\})$$
$$\dot{\mathcal{N}}(p) = (\mathcal{N}(p), 26, 6, (\mathcal{N}(p) \cap \mathcal{B}) \cup \{p\})$$

Thus, $\hat{\mathcal{N}}(p)$ and $\dot{\mathcal{N}}(p)$ are two 3D digital images with image space as $\mathcal{N}(p)$. Moreover, p is always white in $\hat{\mathcal{N}}(p)$ (i.e. p is deleted) while p is always black in $\dot{\mathcal{N}}(p)$ (i.e. p is added). For any other point of $\mathcal{N}(p)$, its color in $\hat{\mathcal{N}}(p)$ and $\dot{\mathcal{N}}(p)$ is the same as that of corresponding point in $\mathcal{P}$. Therefore, the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood of a black point p due to its deletion is equal to the Euler characteristic of $\dot{\mathcal{N}}(p)$ minus the Euler characteristic of $\hat{\mathcal{N}}(p)$. We then define the following recursive relation of $\chi(v, 26, 6, \mathcal{B})$.

(1) $\chi(v, 26, 6, \mathcal{B}) = 0$ if $\mathcal{B} = \phi$;
(2) for any point $p \in \mathcal{B}$, $\chi(v, 26, 6, \mathcal{B})$ (1)
$= \chi(v, 26, 6, \mathcal{B} - \{p\}) + \chi(\dot{\mathcal{N}}(p)) - \chi(\hat{\mathcal{N}}(p))$.

From Saha and Chaudhuri[7] we state the following results on $\dot{\mathcal{N}}(p)$.

(1) $\dot{\mathcal{N}}(p)$ contains exactly one black component;
(2) $\dot{\mathcal{N}}(p)$ contains no tunnel;
(3) $\dot{\mathcal{N}}(p)$ contains no cavity.

Hence, $\chi(\dot{\mathcal{N}}(p)) = 1 - 0 + 0 = 1$. Thus our work boils down to the computation of the Euler characteristic of $\hat{\mathcal{N}}(p)$ which then leads to the estimation of the numbers of black components, tunnels and cavities in $\hat{\mathcal{N}}(p)$. Definitions of cavities and black components are well established. Moreover, it follows from the structure of $\mathcal{N}(p)$ that $\hat{\mathcal{N}}(p)$ may contain at most one cavity that occurs only when all *s*-points of $\mathcal{N}(p)$ are black.[7] Here we provide a formal definition of the number of tunnels in $\hat{\mathcal{N}}(p)$.

Number of tunnels in $\hat{\mathcal{N}}(p)$

It follows from the discussion by Kong *et al.*[3] that if a 3D digital image $\mathcal{P} = (v, 26, 6, \mathcal{B})$ contains no hollow torus then the number of tunnels in the image equals to the number of solid handles. Let us consider an image

$\mathscr{I} = (v, 26, 6, \mathscr{B})$ where $\mathscr{B} \subset v^*$ (in other words all black points lie on the border of the rectangular parallelepiped v). Since the set of black point in $\mathscr{I}$ is a subset of the border of a rectangular parallelepiped it cannot contain a hollow torus. Otherwise at least one of the interior points must be black. Thus, the number of tunnels in $\mathscr{I}$ is equal to the number of solid handles in the image. Again each solid handle leads to an independent non-trivial 26-closed curve in its shrunk version. Moreover, $\mathscr{I}$ can contain at most one cavity that occurs only when all of its 6-border points are black. In that case all black points of $\mathscr{I}$ are 26-connected and also $\mathscr{I}$ cannot contain any solid handle i.e. $\mathscr{I}$ cannot contain any tunnel. In a shrunk version of $\mathscr{I}$, each simply connected black component (i.e. a black component containing neither a cavity nor a tunnel) leads to a black point that has no black 26-neighbor. Thus, we have the following results for a 3D digital image $\mathscr{I} = (v, 26, 6, \mathscr{B})$ where $\mathscr{B} \subset v^*$ and its shrunk version is $\mathscr{I}' = (v, 26, 6, \mathscr{B}')$.

Result 1. *$\mathscr{I}$ can contain at most one cavity that occurs only when $\mathscr{B}'$ consists of all 6-border points of v. In that case the number of tunnels in $\mathscr{I}$ is zero.*

Result 2. *When $\mathscr{I}$ contains no cavity, the number tunnels in $\mathscr{I}$ is equal to the number of independent non-trivial 26-closed curves in $\mathscr{B}'$.*

Result 3. *When $\mathscr{I}$ contains no cavity, for any point $p \in \mathscr{B}'$, if $\mathscr{N}^*(p) \cap \mathscr{B}'$ contains exactly two 26-components then we can find a set of n independent non-trivial 26-closed curves $\{\pi_0, \pi_1, \ldots, \pi_{n-1}\}$ in $\mathscr{B}'$ such that exactly one of them contains p and $\bigcup_{i=0}^{n-1} \pi_i = \mathscr{B}'$* (i.e. *there exist no more independent non-trivial 26-closed curves in $\mathscr{B}'$). In other words, removal of p from $\mathscr{B}'$ removes exactly one independent non-trivial 26-closed curve from $\mathscr{B}'$.*

In $\hat{\mathscr{N}}(p)$, p is the only interior point of the image space $\mathscr{N}(p)$ and p is always white in $\hat{\mathscr{N}}(p)$. Thus, in $\hat{\mathscr{N}}(p)$ all black points lie on the border of $\mathscr{N}(p)$. Hence, $\hat{\mathscr{N}}(p)$ belongs to the class of images discussed above. Again, the set of s-points of $\mathscr{N}(p)$ is the set of 6-border points of $\mathscr{N}(p)$. Using Results 1–2 we state the definition of the number of tunnels in $\hat{\mathscr{N}}(p)$ as follows.

Definition 1. *The number of tunnels in $\hat{\mathscr{N}}(p)$ is zero when all s-points are black* (i.e. *$\hat{\mathscr{N}}(p)$ contains a cavity*), *otherwise the number of tunnels in $\hat{\mathscr{N}}(p)$ is equal to the number of independent non-trivial 26-closed curves in X; where X is the set of black points in a shrunk version of $\hat{\mathscr{N}}(p)$.*

Using Definition 1 we develop a simpler definition of the number of tunnels in $\hat{\mathscr{N}}(p)$ as follows. In a previous publication[7] we have established the following theorem for the existence of tunnels in $\hat{\mathscr{N}}(p)$.

Theorem 1. *$\hat{\mathscr{N}}(p)$ contains no tunnel iff the set of white s-points is 6-connected in the set of white s-points and e-points.*

We use the following notations necessary for future development.

- $\mathscr{B}(p)$: the set of black points in $\hat{\mathscr{N}}(p)$;
- $W_s(p)$: the set of white s-points in $\hat{\mathscr{N}}(p)$;
- $W_e(p)$: the set of white e-points in $\hat{\mathscr{N}}(p)$;
- $W_{se}(p)$: $W_s(p) \cup W_e(p)$;
- $\hat{\mathscr{N}}'(p)$: a shrunk version of $\hat{\mathscr{N}}(p)$;
- $\mathscr{B}'(p)$: the set of black points in $\hat{\mathscr{N}}(p)$;
- $W'_s(p)$: the set of white s-points in $\hat{\mathscr{N}}'(p)$;
- $W'_e(p)$: the set of white e-points in $\hat{\mathscr{N}}'(p)$;
- $W'_{se}(p)$: $W'_s(p) \cup W'_e(p)$;

Proposition 1. *In $\hat{\mathscr{N}}(p)$, if two opposite s-points are white and other four s-points are black then the number of tunnels in $\hat{\mathscr{N}}(p)$ is exactly one.*

Proof. Let $(a, d), (b, e), (c, f)$ denote three distinct unordered pairs of opposite s-points of $\mathscr{N}(p)$ and let a, d be white while b, e, c, f be black in $\hat{\mathscr{N}}(p)$. In that case all e-points and v-points are simple points of $\hat{\mathscr{N}}(p)$ [see Saha et al.[6,7,13] for the characterization of simple point]. Thus, a shrunk version of $\hat{\mathscr{N}}(p)$ contains the set of black points $\{b, e, c, f\}$. Now b, c, e, f is one and only independent non-trivial 26-closed curve of black points in the shrunk version of $\hat{\mathscr{N}}(p)$. Therefore, according to Definition 1 $\hat{\mathscr{N}}(p)$ contains exactly one tunnel. □

Proposition 2. *In $\hat{\mathscr{N}}(p)$, if two non-opposite s-points $x, y \in W_s(p)$ are not 6-connected in $W_{se}(p)$ then transformation of the e-point $e(x, y, p)$ to white removes exactly one tunnel from $\hat{\mathscr{N}}(p)$.*

Proof. Before we enter the proof let us develop some necessary background. Let (a, d), (b, e), (c, f) denote three distinct unordered pairs of opposite s-points of $\mathscr{N}(p)$. As mentioned earlier shrinking is a process of sequential deletion of black simple points. Let $\mathscr{B}_i(p)$ and $W^i_{se}(p)$ denote the set of black points and the set of white s-points and e-points, respectively in $\mathscr{N}(p)$ after the completion of ith step of the shrinking process of $\hat{\mathscr{N}}(p)$ (at each step single black simple point is deleted).

Let p' be a 26-neighbor of p. We define a 3D digital image as follows.

$$\hat{\mathscr{N}}_i(p, p') = (\mathscr{N}(p'), 26, 6, \mathscr{B}_{i-1}(p) \cap \mathscr{N}^*(p'));$$

Thus $\hat{\mathscr{N}}_2(p, p')$ is a digital image with $\mathscr{N}(p')$ as the image space and the set of black point as $\mathscr{B}_{i-1}(p) \cap \mathscr{N}^*(p')$. Now p' is removable at ith step of shrinking iff p' is a simple point in $\mathscr{B}_{i-1}(p)$. Using the characterization of simple point by Saha et. al.[6,7,13] it can be inferred that p' is removable at ith step of shrinking iff $\mathscr{B}_{i-1}(p) \cap \mathscr{N}^*(p')$ contains exactly one simply connected black component (i.e. a black component containing no cavity or tunnel). By definition of shrinking we have

$$\mathscr{B}_i(p) \subset \mathscr{B}_{i-1}(p), \quad \text{and} \quad \mathscr{B}_{i-1}(p) - \mathscr{B}_i(p) = \{q\};$$

where q is deleted at ith step. Also,

if q is a v-point then $W^{i-1}_{se}(p) = W^i_{se}(p)$,
otherwise, $W^{i-1}_{se}(p) \subset W^i_{se}(p)$, and
$W^i_{se}(p) - W^{i-1}_{se}(p) = \{q\}$.

Now, we enter into the proof. Two non-opposite s-points $x, y \in W_s(p)$ are not 6-connected in $W_{se}(p)$ implies that the e-point $e(x, y, p) \in \mathscr{B}(p)$. Using Definition 1 we can establish the proposition by showing that removal of $e(x, y, p)$ from $\mathscr{B}'(p)$ removes exactly one independent non-trivial 26-closed curve from $\mathscr{B}'(p)$. For that purpose we shall first establish that x, y are not 6-connected in $W'_{se}(p)$. If this is not true, let us assume that x, y are 6-connected in $W'_{se}(p)$. Since, they are not 6-connected in $W_{se}(p)$ but are 6-connected in $W'_{se}(p)$, there must be some i such that x, y are not 6-connected in $W^{i-1}_{se}(p)$ but are 6-connected in $W^i_{se}(p)$. If this is true then $\mathscr{B}_{i-1}(p) - \mathscr{B}_i(p)$ must contain an e-point or an s-point.

At first we consider the case that $\mathscr{B}_{i-1}(p) - \mathscr{B}_i(p)$ contains an e-point, say $e(a, b, p)$. Since, x, y are not 6-connected in $W^{i-1}_{se}(p)$ but are 6-connected in $W^i_{se}(p)$ i.e. $W^{i-1}_{se}(p) \cup \{e(a, b, p)\}$, each 6-path between x, y in $W^i_{se}(p)$ contains $e(a, b, p)$. Again a 6-path in $W^i_{se}(p)$ between x, y through $e(a, b, p)$ must contain the points a, b (no e-point of $\mathscr{N}(p)$ is 6-adjacent to $e(a, b, p)$ and a, b are only s-points which are 6-adjacent to $e(a, b, p)$) i.e. $a, b \notin \mathscr{B}_{i-1}(p)$. Since the 6-path between x, y in $W^i_{se}(p)$ contains a, $e(a, b, p)$, b, without loss of generality let us assume that x is 6-connected to a and b is 6-connected to y in $W^{i-1}_{se}(p)$. Now, according to our assumption, x is not 6-connected to y in $W^{i-1}_{se}(p)$ which implies that a is not 6-connected to b in $W^{i-1}_{se}(p)$. Thus, both the sets $\{e(a, c, p),\ c,\ e(b, c, p)\}$ and $\{e(a, f, p), f, e(b, f, p)\}$ intersect with $\mathscr{B}_{i-1}(p)$. Otherwise, $a, e(a, c, p), c, e(b, c, p), b$ or $a, e(a, f, p), f, e(b, f, p),\ b$ will lead to a 6-path in $W^{i-1}_{se}(p)$. Now, the set of black points in $\hat{\mathscr{N}}_i(p, e(a, b, p))$ is a subset of $\{v(a, b, c, p),\ e(a, c, p), c, e(b, c, p),\ v(a, b, f, p), e(a, f, p),\ f, e(b, f, p)\}$. This is because

$$\mathscr{B}_{i-1}(p) \subset \mathscr{N}^*(p);$$

$$\mathscr{N}^*(p) \cap \mathscr{N}^*(e(a, b, p)) = \{a, b, v(a, b, c, p), e(a, c, p), c, e(b, c, p), v(a, b, f, p), e(a, f, p), f, e(b, f, p)\};$$

and

$$a, b \notin \mathscr{B}_{i-1}(p);$$

imply

$$\mathscr{B}_{i-1}(p) \cap \mathscr{N}^*(e(a, b, p)); \subset \mathscr{N}^*(p) \cap \mathscr{N}^*(e(a, b, p)) - \{a, b\};$$

or

$$\mathscr{B}_{i-1}(p) \cap \mathscr{N}^*(e(a, b, p)) \subset \{v(a, b, c, p), e(a, c, p) c, e(b, c, p), v(a, b, f, p), e(a, f, p), f, e(b, f, p)\}.$$

Now, from the set of black points of $\hat{\mathscr{N}}_i(p, e(a, b, p))$ we can find two non-empty subsets $(\mathscr{B}_{i-1}(p) \cap \mathscr{N}^*(e(a, b, p))) \cap \{e(a, c, p), c, e(b, c, p)\}$ and $(\mathscr{B}_{i-1}(p) \cap \mathscr{N}^*(e(a, b, p))) \cap \{e(a, f, p), f, e(b, f, p)\}$ such that no two points, one from each subset are 26-connected in $\{v(a, b, c, p), e(a, c, p), c, e(b, c, p), v(a, b, f, p), e(a, f, p), f, e(b, f, p)\}$ and hence they are not 26-connected in a smaller (or equal) set $\mathscr{B}_{i-1}(p) \cap \mathscr{N}^*(e(a, b, p))$. Thus, the black points of $\hat{\mathscr{N}}_i(p, e(a, b, p))$ are not 26- connected and hence $e(a, b, p)$ is not a simple point in $\hat{\mathscr{N}}_i(p, e(a, b, p))$. Contradiction!!

Now, let us consider the situation that $\mathscr{B}_{i-1}(p) - \mathscr{B}_i(p)$ contains an s-point say a ($a \neq x$; $a \neq y$). Thus, every 6-path between x, y in $W^i_{se}(p)$ contains a. Again, a 6-path in $W^i_{se}(p)$ from x to y through a must contain one of the following two types of sequences.

Sequence 1. $b, e(a, b, p), a, e(a, c, p), c$
(here, b, c are non-opposite).
Sequence 2. $b, e(a, b, p), a, e(a, e, p), e$
(here, b, e are opposite).

Following the same approach as above it can be shown that $\mathscr{B}_{i-1}(p) \cap \mathscr{N}^*(a)$ is not 26-connected for both the sequences. In other words a is never a simple point in $\hat{\mathscr{N}}_i(p, a)$ which leads to the same contradiction as above.

Hence, x, y are not 6-connected in $W'_{se}(p)$. Since x, y are non-opposite s-points of $\mathscr{N}(p)$, we can rename three distinct unordered pairs of opposite s-points of $\mathscr{N}(p)$ as (u, v), (w, x) and (y, z). Since x, y are not 6-connected in $W'_{se}(p)$, we infer the following three results.

(1) $e(x, y, p) \in \mathscr{B}'(p)$; otherwise, $x, e(x, y, p), y$ will lead to a 6-path in $W'_{se}(p)$.

(2) $\{e(u, x, p), u, e(u, y, p)\} \cap \mathscr{B}'(p) \neq \phi$; otherwise, $x, e(u, x, p), u, e(u, y, p), y$ will lead to a 6-path in $W'_{se}(p)$.

(3) $\{e(v, x, p), v, e(v, y, p)\} \cap \mathscr{B}'(p) \neq \phi$; otherwise, $x, e(v, x, p), v, e(v, y, p), y$ will lead to a 6-path in $W'_{se}(p)$.

It is easy to show that $\mathscr{B}'(p) \cap \mathscr{N}^*(e(x, y, p)) \subseteq \{v(u, x, y, p), e(u, x, p), u, e(u, y, p), v(v, x, y, p), e(v, x, p), v, e(v, y, p)\}$. Now every two points of $\{v(u, x, y, p), e(u, x, p), u, e(u, y, p)\}$ are 26-adjacent and the same is true for $\{v(v, x, y, p), e(v, x, p), v, e(v, y, p)\}$. Again, $\mathscr{B}'(p) \cap \{v(u, x, y, p), e(u, x, p), u, e(u, y, p)\}$ and $\mathscr{B}'(p) \cap \{v(v, x, y, p), e(v, x, p), v, e(v, y, p)\}$ are two non-empty subsets of $\mathscr{B}'(p) \cap \mathscr{N}^*(e(x, y, p))$ such that no two points, one from each subset are 26-connected in $\{v(u, x, y, p), e(u, x, p), u, e(u, y, p), v(v, x, y, p), e(v, x, p), v, e(v, y, p)\}$ and hence they are not 26-connected in a smaller set $\mathscr{B}'(p) \cap \mathscr{N}^*(e(x, y, p))$. Thus $\mathscr{B}'(p) \cap \mathscr{N}^*(e(x, y, p))$ contains exactly two 26-components $\mathscr{B}'(p) \cap \{v(u, x, y, p), e(u, x, p), u, e(u, y, p)\}$ and $\mathscr{B}'(p) \cap \{v(v, x, y, p), e(v, x, p), v, e(v, y, p)\}$. Therefore, according to Result 3 removal of $e(x, y, p)$ from $\mathscr{B}'(p)$ removes exactly one independent non-trivial 26-closed curve from $\mathscr{B}'(p)$.

□

Here, we state the definition of the number of tunnels in $\hat{\mathscr{N}}(p)$ which is one of the most important results of this work.

Theorem 2. *The number of tunnels in $\hat{\mathscr{N}}(p)$ is one less than the number of 6-components of $W_s(p)$ in $W_{se}(p)$ when $W_s(p)$ is non-empty. Otherwise the number of tunnels in $\hat{\mathscr{N}}(p)$ is zero.*

Proof. The "otherwise" part of the theorem when $W_s(p)$ is empty (i.e. all s-points are black) follows from Definition 1. To prove the theorem for $W_s(p) \neq \phi$ we use the method of induction. Let n denote the number of 6-components of $W_s(p)$ in $W_{se}(p)$. It follows from Theorem 1 that for $n = 1$, $\hat{\mathcal{N}}(p)$ contains no tunnel. Now, we shall show that for $n = 2$, $\hat{\mathcal{N}}(p)$ contains exactly one tunnel and for $n > 2$ removal of one tunnel from $\hat{\mathcal{N}}(p)$ leaves exactly $n - 1$ number of 6-components of $W_s(p)$ in $W_{se}(p)$.

For $n = 2$, two situations may arise. They are classified as follows.

Case 1. $W_s(p)$ contains exactly two opposite s-points which are not 6-connected in $W_{se}(p)$,
Case 2. $W_s(p)$ contains at least two non-opposite s-points which are not 6-connected in $W_{se}(p)$.

By Proposition 1, $\hat{\mathcal{N}}(p)$ contains exactly one tunnel in Case 1. For Case 2, let us consider two non-opposite s-points a and b which are not 6-connected in $W_{se}(p)$. Since a, b are not 6-connected in $W_{se}(p)$, the e-point $e(a,b,p)$ is black i.e. $e(a,b,p) \in \mathscr{B}(p)$. Using Proposition 2, removal of the e-point $e(a,b,p)$ from $\mathscr{B}(p)$ removes exactly one tunnel from $\hat{\mathcal{N}}(p)$. Again removal of $e(a,b,p)$ from $\mathscr{B}(p)$ leaves $W_s(p)$ to be 6-connected in $W_{se}(p) \cup \{e(a,b,p)\}$. Thus, $\hat{\mathcal{N}}(p)$ contains exactly one tunnel.

To prove the induction part, let $n > 2$ 6-components of $W_s(p)$ in $W_{se}(p)$ be $W_1, W_2, \ldots, W_n$. Let us consider that the theorem is true for $n - 1$. Using Proposition 2 we need to establish following two statements to prove this theorem for n.

(1) there exists an e-point $e(a,b,p) \in \mathscr{B}(p)$ such that the s-points $a \in W_i$ and $b \in W_j$ for some $i \neq j$.

(2) the number of 6-components of $W_s(p)$ in $W_{se}(p) \cup \{e(a,b,p)\}$ is $n - 1$.

Since $n > 2$, let us consider three s-points $a \in W_1$, $b \in W_2$, $c \in W_s$. Then at least two s-points among a, b, c are mutually non-opposite. Let us consider that a, b are mutually non-opposite. Moreover, a, b belonging to different 6-components of $W_s(p)$ in $W_{se}(p)$ they are not 6-connected in $W_{se}(p)$. Thus the e-point $e(a,b,p) \notin W_{se}(p)$ i.e. $e(a,b,p) \in \mathscr{B}(p)$. Also, none of the points of $W_{se}(p)$ except a, b is 6-adjacent to $e(a,b,p)$. Thus, $W_1 \cup W_2, W_3, \ldots, W_n$ are 6-components of $W_s(p)$ in $W_s(p) \cup \{e(a,b,p)\}$. Hence, the number 6-components of $W_s(p)$ in $W_s(p) \cup \{e(a,b,p)\}$ is exactly $n - 1$.

□

From the above theorem the corrollary given below immediately follows.

Corollary 1. *The number of tunnels in $\hat{\mathcal{N}}(p)$ is independent of the color of v-points.*

The Euler characteristic in $3 \times 3 \times 3$ neighborhood

Here, we develop an efficient algorithm to compute the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood of a point due to its deletion. For that purpose we shall first establish a few interesting propositions on the numbers black components, tunnels and cavities in $\hat{\mathcal{N}}(p)$. Let *components* $(\hat{\mathcal{N}}(p))$, *tunnels* $(\hat{\mathcal{N}}(p))$, and *cavities* $(\hat{\mathcal{N}}(p))$ denote the numbers of black components, tunnels, and cavities in $\hat{\mathcal{N}}(p)$, respectively. To compute components $(\hat{\mathcal{N}}(p))$, tunnels $(\hat{\mathcal{N}}(p))$, and cavities $(\hat{\mathcal{N}}(p))$ we use following two important and useful properties of $\mathcal{N}(p)$.

Property 1. *Let x be an s-point of $\mathcal{N}(p)$ and y be a point in surface (x,p). Then for any point $q \in \mathcal{N}(p)$, q is 26-adjacent to y implies that q is 26-adjacent to x.*
Property 2. *Let x be an e-point of $\mathcal{N}(p)$ and y be a point in edge (x,p). Then for any point $q \in/\mathcal{N}(p)$, q is 26-adjacent to y implies that q is 26-adjacent to x.*
Proposition 3. *If an s-point x is black in $\hat{\mathcal{N}}(p)$ then components $(\hat{\mathcal{N}}(p))$ is independent of the color of other points of surface (x,p).*

Proof. Let S denote the set of black points in $\hat{\mathcal{N}}(p)$ and let $y \neq x$ be a point in surface (x,p). To establish the proposition we shall show that the number of 26-components of $S - \{y\}$ and that of $S \cup \{y\}$ are the same. If this is not true then one of the following two cases must occur.

Case 1. None of the points of $S - \{y\}$ is 26-adjacent to y. In other words, the number of 26-components of $S \cup \{y\}$ is greater than that of $S - \{y\}$.
Case 2. Two or more 26-components of $S - \{y\}$ are 26-adjacent to y. In other words, the number of 26-components of $S - \{y\}$ is greater than that of $S \cup \{y\}$.

By assumption x is black i.e. $x \in S - \{y\}$ and $y \in \text{surface}(x,p)$ i.e. y is 26-adjacent to x. Thus, Case 1 never occurs. About Case 2, let us assume that two points $q, r \in S - \{y\}$ belong to two different 26-components of $S - \{y\}$ and q, r are 26-adjacent to y. Since, q, r are 26-adjacent to y they are also 26-adjacent to x (by Property 1). Thus, q, r are 26-connected in $S - \{y\}$ by the 26-path q, x, r. Hence the contradiction that q, r belong to two different 26-components of $S - \{y\}$. Thus, neither Case 1 nor Case 2 may occur and hence the number of 26-components of $S - \{y\}$ and that of $S \cup \{y\}$ are the same.

□

Proposition 4. *If an s-point x is black in $\hat{\mathcal{N}}(p)$ then the number of tunnels in $\hat{\mathcal{N}}(p)$ is independent of the color of other points of surface (x,p).*

Proof. According to Corollary 1, the number of tunnesl in $\hat{\mathcal{N}}(p)$ is independent of the color of v-points of $\hat{\mathcal{N}}(p)$. Thus, to establish the proposition we shall show that the number of tunnels in $\hat{\mathcal{N}}(p)$ is independent of the color of e-points of surface (x,p). Let S denote the set of white s-points in $\hat{\mathcal{N}}(p)$ and let S' denote the set of white s-points and e-points in $\hat{\mathcal{N}}(p)$. Let y be an e-point in surface (x,p). According to Theorem 2 the number of tunnels in $\hat{\mathcal{N}}(p)$ is one less than to the number of 6-components of S in S'. We shall show that the number of 6-components of S in $S' - \{y\}$ is the same as that of S in $S' \cup \{y\}$.

It may be noted from $\mathcal{N}(p)$ that no two s-points are 6-adjacent and also no two e-points are 6-adjacent. Thus, a 6-path of s-points and e-points must be an alternating sequence of s-points and e-points. Hence, a 6-path of s-points and e-points between two s-points through the e-point y must contain two s-points of $\mathcal{N}(p)$ which are 6-adjacent to y. Moreover, it follows from $\mathcal{N}(p)$ that $\mathcal{N}(p)$ has exactly two s-points which are 6-adjacent to y. By assumption x is black i.e. $x \notin S' \cup \{y\}$ and $y \in \text{surface}(x, p)$ i.e. y is 6-adjacent to x. Thus, there exists no 6-path in $S' \cup \{y\}$ between two white s-points through y. Hence, two s-points of S are 6-connected in $S' \cup \{y\}$ implies that they are 6-connected in $S' - \{y\}$. Also, it is obvious that two s-points of S are 6-connected in $S' - \{y\}$ implies that they are 6-connected in $S' \cup \{y\}$. Hence the number of 6-components of S in $S' - \{y\}$ is the same as that of S in $S' \cup \{y\}$. □

As described earlier, the number of cavities in $\hat{\mathcal{N}}(p)$ may be formulated as follows.

$$\text{cavities}(\hat{\mathcal{N}}(p)) = \begin{cases} 1 & \text{if six } s\text{-points are black;} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Thus, cavities($\hat{\mathcal{N}}(p)$) is a function of s-point configuration. According to Proposition 3 and Proposition 4, components($\hat{\mathcal{N}}(p)$) as well as tunnels($\hat{\mathcal{N}}(p)$) are independent of the color of the points of surface(x, p) when the s-point x is black. We define a surface (x, p) as a *dead-surface* of $\mathcal{N}(p)$ if the s-point x is black. A v-point or an e-point is called an *effective* point of $\mathcal{N}(p)$ if it does not belong to any dead-surface. Using Propositions 3, Proposition 4, equation 2 and these definitions we state the following corollary.

Corollary 2. *With a known s-point configuration of $\mathcal{N}(p)$ the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood of p may be computed from the effective point configuration of $\mathcal{N}(p)$.*

Proposition 5. *If an e-point x is black in $\hat{\mathcal{N}}(p)$ then components($\hat{\mathcal{N}}(p)$) is independent of the color of other points of the edge (x, p).*

Proof. Using Property 2 the proof is similar to that of Proposition 3.

According to Corollary 1, equation 2 and Proposition 5 we see that components($\hat{\mathcal{N}}(p)$), tunnels($\hat{\mathcal{N}}(p)$), cavities($\hat{\mathcal{N}}(p)$) are independent of the color of other points of edge(x, p) when the e-point x is black. An edge(x, p) is defined as a *dead-edge* of $\mathcal{N}(p)$ if the e-point x is black. A v-point is called an *isolated point* if it neither belongs to a dead-surface of $\mathcal{N}(p)$ nor it belongs to a dead-edge of $\mathcal{N}(p)$. The following corollary is a straight-forward consequence of this definition.

Corollary 3. *Let y be a black isolated point of $\mathcal{N}(p)$. Then $\{y\}$ is a black component of $\hat{\mathcal{N}}(p)$.*

The corollary given below follows from Proposition 5 and Corollary 3.

Corollary 4. *Let all the six s-points of $\mathcal{N}(p)$ be white. Then components($\hat{\mathcal{N}}(p)$) is equal to the number of 26-components of $B_e(p)$ plus the number of black isolated points of $\mathcal{N}(p)$ (where $B_e(p)$ is the set of black e-points of $\mathcal{N}(p)$).*

Now we shall describe all possible geometric classes of s-point configurations. Two configurations belong to the same geometric class iff one can be transformed to the other by three dimensional rotation in multiples of 90° about different axes (with p as origin). Possible geometric classes of s-point configurations and corresponding number of effective points (n_e) are as follows.

Class 0: Six s-points are black ($n_e = 0$).
Class 1: Five s-points are black ($n_e = 0$).
Class 2: Two pairs of opposite s-points are black ($n_e = 0$).
Class 3: One pair of opposite s-points and two non-opposite s-points are black ($n_e = 1$).
Class 4: One pair of opposite s-points and another s-point are black ($n_e = 2$).
Class 5: Three non-opposite s-points are black ($n_e = 4$).
Class 6: One pair of opposite s-points are black ($n_e = 4$).
Class 7: Two non-opposite s-points are black ($n_e = 7$).
Class 8: One s-point is black ($n_e = 12$).
Class 9: No s-point is black ($n_e = 20$).

After finding the s-point configuration of a point $p \in \mathscr{B}$ we can compute the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood of p from the configuration of effective points only. It is easy to note that the maximum value of components($\hat{\mathcal{N}}(p)$) is "8" and it occurs only when all v-points of $\mathcal{N}(p)$ are black while all other points of $\mathcal{N}(p)$ are white. The maximum value of tunnels($\hat{\mathcal{N}}(p)$) is "5" that occurs only when all s-points of $\mathcal{N}(p)$ are white while all e-points are black. Also, maximum value of cavities($\hat{\mathcal{N}}(p)$) is "1" that occurs only when all s-points of $\mathcal{N}(p)$ are black i.e. s-point configuration belongs to Class 0. Hence, the maximum change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood can be easily accomodated in single byte. Thus, after finding the s-point configuration of $\mathcal{N}(p)$ we consider one of the three cases given below to compute the change in the Euler characteristic in $\mathcal{N}(p)$ due to the deletion of p.

Case 1. (s-point configuration belonging to Class 0–2)

Here the number of effective points is zero. Thus we can at once know the values of components($\hat{\mathcal{N}}(p)$), tunnels($\hat{\mathcal{N}}(p)$), cavities($\hat{\mathcal{N}}(p)$). These values are as follows.

Class 0: components($\hat{\mathcal{N}}(p)$) = 1, tunnels($\hat{\mathcal{N}}(p)$) = 0, cavities($\hat{\mathcal{N}}(p)$) = 1;

Class 1: components($\hat{\mathcal{N}}(p)$) = 1, tunnels($\hat{\mathcal{N}}(p)$) = 0, cavities($\hat{\mathcal{N}}(p)$) = 0;

Class 2: components($\hat{\mathcal{N}}(p)$) = 1, tunnels($\hat{\mathcal{N}}(p)$) = 1, cavities($\hat{\mathcal{N}}(p)$) = 0.

The change in the Euler characteristic in $\mathcal{N}(p)$ due to the deletion of p for Class 0, Class 1, and Class 2 are $-1, 0, 1$, respectively.

Case 2. (s-point configuration belonging to Class 3–8) Here we use a look_up_table. Since different s-point configurations may belong to the same geometric class, we consider an s-point configuration $base_i$ (a set of black s-points) for each Class i. Let Class i has n_i number of effective points. An ordered set EFO ($base_i$) of these n_i effective points is defined as

$$EFO(base_i) = \{e_0, e_1, \ldots, e_{n_i}\}$$

An effective point configuration for the s-point configuration $base_i$ is denoted by a n_i bit binary number where jth bit denotes the color of e_j. The look_up_table needs 2^{n_i} entries i.e. 2^{n_i} bytes. Each entry needs one byte which contains the change in the Euler characteristic in $\mathcal{N}(p)$ due to the deletion of p. Only one look_up_table is needed for all s-point configurations belonging to the same geometric class. To illustrate the fact let us consider an s-point configuration γ belonging to Class i such that

$$\gamma = Rot(\mu, base_i)$$

where $Rot(\mu, base_i)$ is a function that generates a set from $base_i$ such that the jth element of the set is obtained from the jth element of $base_i$ after the rotation μ with p as origin (here, μ is a sequence of rotations about different axes in integral multiples of 90°). Then the same look_up_table may be used for the s-point configuration γ with its ordered set of effective point $EFO(\gamma)$ as

$$EFO(\gamma) = Rot(\mu, EFO(base_i))$$

Case 3. (s-point configuration belonging to Class 9) Here, a straight-forward application of the look_up_table described in Case 2 needs 2^{20} bytes. Thus, we modify the form of the look_up_table. Let X be an ordered set of all e-points of $\mathcal{N}(p)$. According to Corollary 4 components($\hat{\mathcal{N}}(p)$) is equal to the number of 26-components of $X \cap \mathcal{B}$ plus the number of black isolated points of $\mathcal{N}(p)$. Also, tunnels($\hat{\mathcal{N}}(p)$) is computable from the configuration of X according to Corollary 1. Here, cavities($\hat{\mathcal{N}}(p)$) is always zero according to equation 2. Let Y be an ordered set of all v-points of $\mathcal{N}(p)$. A look_up_table is used where each entry contains two bytes and the ith entry corresponds to the configuration value of X as i. The look_up_table needs 2^{12} entries i.e. 8 kbytes. For ith entry, jth bit of the first byte informs whether jth point of Y is an isolated point while the second byte contains

$$1 - \text{the number of 26-components in } X \cap \mathcal{B} + \text{tunnels}(\hat{\mathcal{N}}(p)).$$

To get the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood a one byte word w is generated to denote the configuration of Y. Then the number of 1's in the bitwise 'AND' between w and the first byte gives the number of black isolated points of $\mathcal{N}(p)$. The change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood is calculated as

$$\text{the value of the second byte} - \text{the number of black isolated points.}$$

The Euler characteristic of digital image

In Section 3 we have described an efficient algorithm to compute the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood of a point p that occurs due to the deletion of p. Using this algorithm the Euler Characteristic of a 3D digital image may be computed according to equation 1. A parallel implementation of the method is possible using the concept of subfields.[18] The parallelization is based on the following concept.

If two points p, $q \in \mathcal{B}$ are not 26-adjacent then

$$\chi(v, 26, 6, \mathcal{B}) = \chi(v, 26, 6, \mathcal{B} - \{p, q\}) + \chi(\dot{\mathcal{N}}(p)) - \chi(\hat{\mathcal{N}}(p)) + \chi(\dot{\mathcal{N}}(q)) - \chi(\hat{\mathcal{N}}(q)).$$

Maximum parallelization of the algorithm is conceived as follows. Eight sub-fields $O_0, O_1, \ldots, O_7$ are defined in v as follows

$$O_l = v \cup \{(2 \times i + f, 2 \times j + g, 2 \times k + h) | i, j, k = 1, 2, \ldots, f, g, h \in \{0, 1\} \text{ and } 2^2 \times f + 2^1 \times g + 2^0 \times h = l\}$$

such that two points p, $q \in O_l$ are never 26-adjacent. The Euler Characteristic of a 3D digital image can be computed in eight steps and at each step the algorithm uses the following equation

$$\chi(v, 26, 6, \mathcal{B}) = \chi(v, 26, 6, \mathcal{B} - O_i) + \sum_{p \in O_i \cap \mathcal{B}} \chi(\dot{\mathcal{N}}(p)) - \chi(\hat{\mathcal{N}}(p))$$

where $\sum_{p \in O_i \cap \mathcal{B}} \chi(\dot{\mathcal{N}}(p)) - \chi(\hat{\mathcal{N}}(p))$ can be computed in a parallel manner.

4. DISCUSSION

As mentioned before, the existing approaches to computing the Euler Characteristic consider an n-dimensional digital object as a polyhedron composed of n-dimensional homogeneous structures. In those approaches the Euler characteristic of a 3D digital object $\not{p}$ is defined as

$$E(\not{p}) = \sum_{k=0}^{n} (-1)^k \times C_k$$

where C_k is the number of k-dimensional elements in the polyhedral representation of $\not{p}$.

Our approach to computing the Euler Characteristic is based on computing the numbers of black components, tunnels and cavities under a specific connectivity relation. It is possible to extend the approach for different connectivity relation of grid points, say (6, 26), (6, 18) or (18, 6) connectivity. As described in Section 3 a massive parallelization of the algorithm is possible using the concept of sub-fields.

It is easy to note that in the worst case the computation of the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood of a point p due to its deletion needs the configuration of all the 26- neighbors of p. The fastest approach (in worst case sense) needs a look_up_table with 2^{26} entries. Each entry of the look_up_table needs one byte and the complete table needs 64 mbytes. In the worst case our approach is nearly as fast as the fastest approaches (see Case 3 for Class 9) while our average time is much faster (since, our approach overlooks a lot of points for Class i; $0 \leq i \leq 8$ as described in Case 1 and Case 2). Another important aspect of our approach of computing the change in the Euler characteristic in $3 \times 3 \times 3$ neighborhood of a point p due to its deletion is the storage of the look_up_table which is $2^1 + 2^2 + 2^4 + 2^4 + 2^7 + 2^{12} + 2 \times 2^{12}$ bytes i.e. 12 kbytes only.

REFERENCES

1. T. Y. Kong and A. Rosenfeld, Digital topology: introduction and survey, *Comput. Vision Graphics Image Process.* **48**, 357–393 (1989).
2. T. Y. Kong and A. W. Roscoe, A theory of binary digital pictures, *Comput. Graphics Image Process.* **32**, 221–243 (1985).
3. T. Y. Kong, A. W. Roscoe and A. Rosenfeld, Concepts of digital topology, *Topology Applic.* **46**, 219–262 (1992).
4. C. M. Park and A. Rosenfeld, Connectivity and genus in three dimensions, **TR-156**, Computer Science Center, University of Maryland (1971).
5. K. J. Hafford and K. Preston Jr, Three-dimensional skeletonization of elongated solids, *Comput. Vision Graphics Image Process.* **27**, 78–91 (1984).
6. P. K. Saha, B. Chanda and D. Dutta Majumder, Principles and algorithms for 2D and 3D shrinking, **TR/KBCS/2/91**, N.C.K.B.C.S. Library, Indian Statistical Institute, Calcutta, India (1991).
7. P. K. Saha and B. B. Chaudhuri, Detection of 3D simple points for topology preserving transformations with application to thinning, *IEEE Trans. Pattern Anal. Mach. Intell.* **16**, 1028–1032 (1994).
8. P. K. Saha and B. B. Chaudhuri, 3D digital topology under binary transformation with applications, *Comput. Vision Graphics Image Process.* (accepted for publication).
9. H. Bieri and W. Nef, Algorithms for the Euler characteristic and related additive functionals of digital objects, *Comput. Vision Graphic Image Process.* **28**, 166–175 (1984).
10. H. Bieri, Computing the Euler characteristic and related additive functionals of digital objects from their bintree representation, *Comput. Vision Graphics Image Process.* **40**, 115–126 (1987).
11. C. R. Dyer, Computing the Euler number of an image from its quadtree, *Comput. Graphics Image Process.* **13**, 270–276 (1980).
12. K. Voss, Images, objects, and surfaces in Z^n, *Int. J. Pattern Recog. Artif. Intell.* **5**, 797–808 (1991).
13. P. K. Saha, B. B. Chaudhuri, B. Chanda and D. Dutta Majumder, Topology preservation in 3D digital space, *Pattern Recognition* **27**, 295–300 (1994).
14. T. Y. Kong and A. W. Roscoe, Characterization of simply-connected finite polyhedra in 3-space, *Bull. London Math. Soc.* **17**, 575–578 (1985).
15. P. J. Hilton and S. Wylie, *Homology Theory*. Cambridge University Press, Cambridge, U.K. (1960).
16. C. R. F. Maunder, *Algebraic Topology*. Cambridge University Press, Cambridge, U.K. (1980).
17. S. Lobregt, P. W. Verbeek and F. C. A. Groen, Three-dimensional skeletonization: principle and algorithm, *IEEE Trans. Pattern Anal. Mach. Intell.* **2**, 75–77 (1980).
18. M. J. E. Golay, Hexagonal parallel pattern transformations, *IEEE Trans. Comput.* **18**, 733–740 (1969).

About the Author—PUNAM KUMAR SAHA obtained his Bachelor's and Master's degree in computer Science and Engineering from Jadavpur University in 1987 and 1989, respectively. He joined Electronics and Communication Sciences unit, Indian Statistical Institute as a Senior Research Fellow in 1989 where he is presently working as a faculty member. His area of research interests are Fractal Geometry, Three Dimensional Digital Topology and its Application to Image Processing and Computer Vision. He has communicated/published several research papers in reputed journals.

About the Author—PROFESSOR BIDYUT BARAN CHAUDHURI received his B.Sc. (Hons), B.Tech and M.Tech degrees from Calcutta University, India in 1969, 1972 and 1974, respectively and Ph.D. Degree from Indian Institute of Technology, Kanpur in 1980. He joined Indian Statistical Institute in 1978 where he served as the Project Coordinator and Head of National Nodal Center for Knowledge Based Computing. Currently, he is the head of Computer Vision and Pattern Recognition Unit of the institute. His research interests include Pattern Recognition, Image Processing, Computer Vision, Natural Language Processing and Digital Document Processing including OCR. He has published 125 research papers in reputed International Journals and has authored a book entitled *Two Tone Image Processing and Recognition* (Wiley Eastern, 1993). He was awarded Sir J. C. Bose Memorial Award for best engineering science oriented paper published in *JIETE* in 1986 and M. N. Saha Memorial Award for best application oriented papers published in 1989 and 1991. In 1992 he won the prestigious Homi Bhabha Fellowship for working on OCR of the Indian Languages and computer communication for the blind. As a Leverhulme visiting fellow, he worked at Queen's University, U.K. Also, he worked as a visiting faculty member at GSF, Munich and guest scientist at the Technical University of Hannover during 1986–1988 and again in 1990–1991. He is a Senior member of IEEE, member secretary (Indian Section) of International Academy of Sciences, Fellow of National Academy of Sciences (India) and member of many academic and professional bodies. He is a member of the editorial board of the journals such as *Pattern Recognition* and *Vivek*.