ANOTHER LOOK AT THE MOMENT BOUNDS ON RELIABILITY

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Abstract

In this paper a unified derivation of the upper and lower bounds (in terms of the mean) of an IFR, DFR, IFRA, DFRA, NBU or NWU reliability function is presented. The method of proof provides a simpler alternative to the various proofs known in the literature. Moreover, this method can be used to generalize the existing results in two ways, as demonstrated here. First, the bounds for the reliability function are obtained in terms of *any* finite moment in all these cases. Subsequently we provide a moment bound on a reliability function which ages faster or slower than a known one in some sense. The existing literature does not offer any of these generalizations, except for a few results which are available in an unnecessarily complicated form.

CONVEXITY; STAR-SHAPEDNESS; SUPERADDITIVITY; RELIABILITY BOUNDS; HAZARD FUNCTION; IFR; DFR; IFRA; DFRA; NBU; NWU

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60K10

1. Introduction

Moment bounds on the reliability of a unit have been a topic of great interest in the reliability literature. The problem typically consists of computing the sharpest possible upper and lower bound on the reliability, assuming that the lifetime distribution belongs to a common family (e.g. IFR, IFRA, NBU) with a known moment. Various results in this area are available in Barlow and Proschan (1975). Other results may be found in Marshall and Proschan (1972), Korzeniowski and Opawski (1976) and Klefsjö (1982). The results are proved using different techniques. Often the sharpness of the bound has to be proved separately.

In this paper, the upper and lower bounds on the reliability of IFR, IFRA, NBU, DFR, DFRA and NWU classes of life distributions are rederived in a unified way. The advantages of the new approach are as follows:

1. All the proofs involve essentially the same type of arguments.
2. The scope of the known bounds can be broadened in some cases.
3. Sharpness of the bounds follows automatically from the derivation.
4. The results for the known first moment generalize naturally to other positive moments. Many of these generalizations are not available in the literature.

Received 12 January 1993; revision received 20 August 1993.

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5. The derivation generalizes to the case of relative ageing with respect to a known distribution (other than exponential).

6. The derivation enhances understanding of the limiting cases by means of the geometry of hazard functions.

7. Results for the negatively ageing classes follow by the argument of symmetry, although the results are not symmetric.

Suppose F is a life distribution on $[0, \infty)$ with mean μ. Instead of finding a lower bound on the reliability $\bar{F}(=1-F)$, we pose the problem in terms of the hazard function Λ_F defined as $-\log \bar{F}$. We assume $\bar{F}(0-)=1$. Thus Λ_F is a non-decreasing function with $\Lambda_F(0-)=0$ and taking values in $[0, \infty) \cup \{\infty\}$. It may be recalled that F is IFR, IFRA or NBU if and only if Λ_F is convex, star-shaped or superadditive, respectively. A similar set of characterizations can be made for the DFR, DFRA and NWU distributions. The properties of the hazard function are used here in deriving the bounds.

We illustrate the procedure by deriving the lower and upper bounds of a NBU distribution with known first moment. In Section 3 a general procedure for the other classes is suggested. Generalizations are provided in the subsequent sections.

2. Moment bounds on the NBU reliability

2.1. *Lower bound in terms of mean.* Let F be an NBU distribution with mean μ. Thus Λ_F is superadditive, that is, $\Lambda_F(x+y) \geqq \Lambda_F(x) + \Lambda_F(y)$ for each $x, y \geqq 0$. Define the class $\mathscr{C}$ as

$$\mathscr{C} = \left\{\Lambda_F : \Lambda_F \text{ is a superadditive hazard function; } \int_0^\infty \exp(-\Lambda_F(x))dx = \mu\right\}.$$

The objective here is to find $\sup_{\Lambda \in \mathscr{C}} \Lambda(x)$ for every x. As a matter of fact, we only need to consider the case $x < \mu$, since the trivial bound in the other case is attained by the degenerate distribution having mass at μ.

Theorem 2.1. If $\mathscr{C}$ is defined as above, then for $x < \mu$

$$\sup_{\Lambda \in \mathscr{C}} \Lambda(x) = \log \frac{\mu}{\mu - x}.$$

Proof. Pick a member Λ of $\mathscr{C}$. For a fixed $x < \mu$, construct a new hazard function Λ_1 as

$$\Lambda_1(y) = n\alpha \quad \text{if } nx \leqq y < (n+1)x, \qquad n = 0, 1, 2, \cdots,$$

where $\alpha = \Lambda(x)$. Superadditivity of Λ guarantees that it uniformly dominates Λ_1, so that $\int_0^\infty \exp(-\Lambda_1(y))dy \geqq \mu$. On the other hand, Λ_1 is itself a superadditive function. If α is increased monotonically from $\Lambda(x)$ to a very large number, the value of the above integral reduces monotonically and continuously to some number close to x, which is less than μ. Therefore there is some α in $[\Lambda(x), \infty)$ which makes the integral exactly equal to μ. Let Λ_2 be the corresponding variation of Λ_1. It is clear that the latter is a member of $\mathscr{C}$ satisfying $\Lambda_2(x) \geqq \Lambda(x)$. The stated result is obtained by solving for α from $\int_0^\infty \exp(-\Lambda_2(y))dy = \mu$.

It follows from the above theorem that the lower bound on $\bar{F}(x)$ in the range $0 \leqq x < \mu$, when F is NBU, is $1 - x/\mu$. This result was first proved by Marshall and Proschan (1972). In fact the same bound holds for the NBUE class.

2.2. *Upper bound in terms of mean.* The formulation of this problem is identical to the one discussed above. Here we want to find $\inf_{\Lambda \in \mathscr{C}} \Lambda(x)$ where $\mathscr{C}$ is defined as before. In this case the trivial upper bound in the range $x < \mu$ is attained by the degenerate distribution with point mass at μ.

Theorem 2.2. If $\mathscr{C}$ is defined as above, then for $x \geqq \mu$

$$\inf_{\Lambda \in \mathscr{C}} \Lambda(x) = \alpha x,$$

where α satisfes $\int_0^x e^{-\alpha y} dy = \mu$.

The following two lemmas are useful in proving the above theorem. They establish certain interesting properties of the superadditive hazard function.

Lemma 2.1. If Λ is a superadditive hazard function and $\Lambda(a)$ is finite, then there is a subinterval $(0, b)$, of $(0, a)$ such that $\Lambda(y) \leqq \Lambda(a)y/a$ for all $y \in (0, b)$.

Proof. Suppose, for contradiction, that for each $b > 0$ there is a $y \in (0, b)$ such that $\Lambda(y) > \Lambda(a)y/a$. It will be shown that the graphs of the functions $\Lambda(y)$ and $\Lambda(a)y/a$ interesect each other in every subinterval of $(0, a)$.

Let $I = [x_1 - \varepsilon, x_1 + \varepsilon]$ be such a subinterval. Choose $N_1 > x_1/\varepsilon$. Then by hypothesis there is a point x_2 in the interval $(0, x_1/N_1)$ such that $\Lambda(x_2) > \Lambda(a)x_2/a$. Let N_2 be the unique number ($> N_1$) satisfying $x_1/(N_2 + 1) < x_2 \leqq x_1/N_2$. Then the point $y = N_2 x_2$ lying in I satisfes $\Lambda(y) > \Lambda(a)y/a$. Now choose $N_3 > (a - x_1)/\varepsilon$. Let x_3 be a point in $(0, (a - x_1)/N_3)$ such that $\Lambda(x_3) > \Lambda(a)x_3/a$. Suppose N_4 satisfies $(a - x_1)/(N_4 + 1) < x_3 \leqq (a - x_1)/N_4$. Clearly the point $y = a - N_4 x_3$ lies in I and satisfies $\Lambda(y) < \Lambda(a)y/a$. Since Λ is non-decreasing and right-continuous, a limiting argument shows that it is identical to $\Lambda(a)y/a$ over the interval $(0, a)$, which is a contradiction.

Thus we find that a superadditive hazard function behaves somewhat like a star-shaped hazard function in a neighborhood of the origin.

Lemma 2.2. Suppose Λ is a superadditive hazard function and $\Lambda(a)$ is finite. Then for any positive integer n

$$\int_0^{a/2^n} [\exp(-\Lambda(y)) + \exp(-\Lambda(a)/2^{n-1} + \Lambda(y))]dy$$
$$\geqq (1 + \exp(-\Lambda(a)/2^n)) \int_0^{a/2^{n+1}} [\exp(-\Lambda(y)) + \exp(-\Lambda(a)/2^n + \Lambda(y))]dy.$$

Proof. Break up the range of integration of the left-hand side into two parts. The second part can be written as

$$\int_0^{a/2^{n+1}} [\exp(-\Lambda(a/2^n - y)) + \exp(-\Lambda(a)/2^{n-1} + \Lambda(a/2^n - y))]dy.$$

The integrand is of the form $P + R^2/P$ where $P = \exp(-\Lambda(a/2^n - y))$ and $R = \exp(-\Lambda(a)/2^n)$. Using the superadditivity of Λ we can write $0 \leqq R \leqq Q \leqq P \leqq 1$, where $Q = \exp(-\Lambda(a)/2^n + \Lambda(y))$. It follows that $P + R^2/P \geqq Q + R^2/Q$. A rearrangement of the terms of the resulting expression gives the required inequality.

We now return to the proof of the stated theorem.

Proof of Theorem 2.2. Let $\Lambda \in \mathscr{C}$. For a given $x \geqq \mu$ construct

$$\Lambda_1(y) = \begin{cases} \alpha y, & \text{if } y < x \\ \infty, & \text{if } y \geqq x, \end{cases}$$

where $\alpha = \Lambda(x)/x$. Clearly Λ_1 is a superadditive hazard function. We shall show that

$$\int_0^x \exp(-\Lambda(y))dy \geqq \int_0^\infty \exp(-\Lambda_1(y))dy. \tag{1}$$

Indeed, the left-hand side can be written as

$$\begin{aligned} \text{LHS} &= \int_0^{x/2} \exp(-\Lambda(y))dy + \int_0^{x/2} \exp(-\Lambda(x-y))dy \\ &\geqq \int_0^{x/2} [\exp(-\Lambda(y)) + \exp(-\Lambda(x) + \Lambda(y))]dy. \end{aligned}$$

By repeated application of Lemma 2.2 we have for all positive integer n

$$\begin{aligned} \text{LHS} &\geqq (1 + \exp(-\Lambda(x)/2)) \cdots (1 + \exp(-\Lambda(x)/2^n)) \\ &\quad \times \int_0^{x/2^{n+1}} [\exp(-\Lambda(y)) + \exp(-\Lambda(x)/2^n + \Lambda(y))]dy \\ &= (1 + \exp(-\Lambda(x)/2)) \cdots (1 + \exp(-\Lambda(x)/2^n))\exp(-\Lambda(x)/2^{n+1}) \\ &\quad \times \int_0^{x/2^{n+1}} 2\cosh(\Lambda(x)/2^{n+1} - \Lambda(y))dy. \end{aligned}$$

By virtue of Lemma 2.1, there is a finite n such that Λ_1 completely dominates Λ in the interval $(0, x/2^{n+1}]$. Thus we have for each $y \in (0, x/2^{n+1}]$

$$\frac{\Lambda(x)}{2^{n+1}} - \Lambda(y) \geqq \frac{\Lambda(x)}{2^{n+1}} - \Lambda_1(y) \geqq 0.$$

Since cosh is an increasing function and $\Lambda(x) = \Lambda_1(x-)$, we can write

$$\begin{aligned} \text{LHS} &\geqq (1 + \exp(-\Lambda_1(x-)/2)) \cdots (1 + \exp(-\Lambda_1(x-)/2^n))\exp(-\Lambda_1(x-)/2^{n+1}) \\ &\quad \times \int_0^{x/2^{n+1}} 2\cosh(\Lambda_1(x-)/2^{n+1} - \Lambda_1(y))dy. \end{aligned}$$

The last expression is equal to the right-hand side of (1), since all the inequalities in the above derivation become equalities when Λ is replaced by Λ_1. This proves (1). Consequently $\int_0^\infty \exp(-\Lambda_1(y))dy \leqq \mu$. If we now reduce α from the value $\Lambda(x)/x$ to 0, the integral increases continuously and monotonically to x, which is no less than μ. There must be a unique α in $(0, \Lambda(x)/x]$ for which the integral equals μ. The resulting function (call it Λ_2) is a member of $\mathscr{C}$ satisfying $\Lambda_2(x) \leqq \Lambda(x)$. The result stated in the theorem follows.

The upper bound on any NBU reliability function with mean μ is then $e^{-\alpha x}$ where α is as in Theorem 2.2. Although this result was first proved by Korzeniowski and Opawski (1976), the proof given here is considerably simpler and more intuitive. The additional advantage of this proof is that it can be generalized to the case of any known and finite moment (see Section 4). It can also be generalized to the case of relative ageing with respect to a known distribution and a known moment (see Section 5.2). Note that the upper bound coincides with the IFRA upper bound (Barlow and Proschan (1975)). The fact that it actually applies to the NBU class is not very well known.

3. A general derivation for ageing classes

3.1. *Positively ageing classes*. Following the proofs of the above two theorems, we can outline a strategy for deriving bounds for the positively ageing classes. The degenerate distribution with all the mass at μ provides the trivial parts of the upper and lower bounds in each case. The non-trivial parts ($x < \mu$ for lower bound and $x \geqq \mu$ for upper bound) are obtained as follows.

Step I. Define $\mathscr{C}_1$ as the class of hazard functions having a special property (such as convexity, star-shapedness or superadditivity). Let $\mathscr{C}_2$ be the class of hazard functions Λ satisfying the constraint $\int_0^\infty \exp(-\Lambda(y))dy = \mu$. Now pose the problem of minimizing (maximizing) $\bar{F}(x)$ for any given x as that of maximizing (minimizing) $\Lambda(x)$ while Λ must belong to $\mathscr{C}_1 \cap \mathscr{C}_2$.

Step II. Fix x. For a typical member Λ of $\mathscr{C}_1 \cap \mathscr{C}_2$, find a piecewise linear function Λ_1 that satisfies the following properties:

(i) $\Lambda_1(x) = \Lambda(x)$ and $\int_0^\infty \exp(-\Lambda_1(y))dy \geqq (\leqq)\mu$.

(ii) Λ_1 belongs to a subclass $\mathscr{C}_3$ of $\mathscr{C}_1$ which is indexed by one or two shape parameter(s).

(iii) The integral $\int_0^\infty \exp(-\Lambda_1(y))dy$ as well as the value $\Lambda_1(x)$ is a monotone and continuous function of one of these shape parameters (call it α).

Step III. Modify Λ_1 by altering α in such a direction that the integral decreases (increases) as a result while $\Lambda_1(x_0)$ increases (decreases). Argue that there exists a value of α such that $\int_0^\infty \exp(-\Lambda_2(y))dy = \mu$ where Λ_2 is the corresponding modification of Λ_1.

Step IV. It is obvious that $\Lambda_2(x) \geqq (\leqq)\Lambda_1(x) = \Lambda(x)$ where $\Lambda_2 \in \mathscr{C}_3 \cap \mathscr{C}_2 \subset \mathscr{C}_1 \cap \mathscr{C}_2$. Therefore it is enough to search for the maximizer (minimizer) from $\mathscr{C}_3 \cap \mathscr{C}_2$. If this set is a singleton (which happens if α is the *only* parameter characterizing $\mathscr{C}_3$), then $\Lambda_2(x)$ is the solution. Otherwise one has to maximize (minimize) $\Lambda_2(x)$ with respect to the other parameter to obtain the desired result.

Step II is the only one that needs innovation. Table 1 gives a list of appropriate choices for $\mathscr{C}_3$ for a given $\mathscr{C}_1$ and x. Verifying the properties (i)–(iii) is also very easy except for the case of minimizing superadditive hazard functions, which was addressed in Theorem 2.2.

TABLE 1
Choice of $\mathscr{C}_3$ in various cases.

	Objective	
Class of hazard functions ($\mathscr{C}_1$)	Minimize $\Lambda(x)$	Maximize $\Lambda(x)$
Convex	$\mathscr{C}_3 = \{\Lambda_\alpha : 0 \leqq \alpha < \infty\}$ $\Lambda_\alpha(y) = \begin{cases} \alpha y & \text{if } y < x \\ \infty & \text{if } y \geqq x \end{cases}$	$\mathscr{C}_3 = \{\Lambda_{\alpha\beta} : 0 \leqq \alpha < \infty; 0 \leqq \beta \leqq x\}$ $\Lambda_{\alpha,\beta}(y) = \begin{cases} 0 & \text{if } y < \beta \\ \alpha((y-\beta)/(x-\beta)) & \text{if } y \geqq \beta \end{cases}$
Star-shaped	$\mathscr{C}_3 = \{\Lambda_\alpha : 0 \leqq \alpha < \infty\}$ $\Lambda_\alpha(y) = \begin{cases} \alpha y & \text{if } y < x \\ \infty & \text{if } y \geqq x \end{cases}$	$\mathscr{C}_3 = \{\Lambda_\alpha : 0 \leqq \alpha < \infty\}$ $\Lambda_\alpha(y) = \begin{cases} 0 & \text{if } y < x \\ \alpha y & \text{if } y \geqq x \end{cases}$
Superadditive	$\mathscr{C}_3 = \{\Lambda_\alpha : 0 \leqq \alpha < \infty\}$ $\Lambda_\alpha(y) = \begin{cases} \alpha y & \text{if } y < x \\ \infty & \text{if } y \geqq x \end{cases}$	$\mathscr{C}_3 = \{\Lambda_\alpha : 0 \leqq \alpha < \infty\}$ $\Lambda_\alpha(y) = n\alpha$ if $nx \leq y < (n+1)x$

Remark 1. It is easy to see that the minimizer among the superadditive hazard functions happens to be convex (and hence star-shaped). Therefore no separate derivation is needed for the upper bound of the reliability function in the IFR or the IFRA case. If, however, a direct proof is sought, verification of property (i) would be much easier compared to the corresponding step in Theorem 2.2, since Λ_1 would uniformly dominate Λ.

Remark 2. From the construction it is clear that all the upper and lower bounds obtained this way are sharp.

3.2. *Negatively ageing distributions*. On principle, the derivation of upper and lower moment bounds for the reliability functions of negatively ageing classes of distributions (such as DFR, DFRA and NWU) should be similar to those for the positively ageing classes. However, a simple observation makes the search for $\mathscr{C}_3$ in each case unnecessary. By the symmetry of the problem, finding the infimum of a concave ('inverse' star-shaped/subadditive) hazard function is equivalent to finding the supremum of a convex (star-shaped/superadditive) hazard function. Thus the graph of a typical member of $\mathscr{C}_3$ in the negatively ageing case should be that obtained by interchanging the coordinates of a typical graph in the corresponding positively ageing case.

When the objective is to *minimize* $\Lambda(x)$, the above method gives the class of appropriate piecewise linear functions. However, this does not work when we seek to *maximize* $\Lambda(x)$, since interchanging the coordinates would produce a hazard function of the form

$$\Lambda_\alpha(y) = \begin{cases} \alpha y, & \text{if } y < x \\ \alpha x, & \text{if } y \geqq x, \end{cases}$$

for which the mean integral $\int_0^\infty \exp(-\Lambda_\alpha(y))dy$ diverges. However, the following argument can be used to construct $\mathscr{C}_3 \cap \mathscr{C}_2$ directly. For given positive numbers x and M, we can always find $\alpha \in (1/\mu, \infty)$ such that $\alpha x > M$. Subsequently we can find a β in the interval $(0, \alpha)$ such that the concave hazard function

$$\Lambda_2(y) = \begin{cases} \alpha x, & \text{if } y < x \\ \beta(y - x) + \alpha x, & \text{if } y \geqq x \end{cases}$$

satisfies the constraint $\int_0^\infty \exp(-\Lambda_2(y))dy = \mu$. But $\Lambda_2(x)$ is greater than the specified number M, which can be made as large as we want. The function Λ_2 is also inverse star-shaped and subadditive. Therefore the supremum of $\Lambda(x)$ is infinity in each of the cases of interest.

4. Bounds based on any finite moment

If the rth moment μ_r of a life-distribution is known (r may be other than 1 but positive), then the integral constraint becomes $\int_0^\infty \exp(-\Lambda(y))ry^{r-1}dy = \mu_r$. The theorems of Section 2 and the arguments given in Section 3 go through in this case with appropriate modifications. Therefore Table 1 is applicable here. After straightforward algebra, the upper and lower bounds on $\bar{F}$ can be verified to be as in Table 2.

Remark 1. The lower bound on $\bar{F}(x)$ in the IFRA case given in Table 2 is simpler than that given in Barlow and Proschan (1975).

Remark 2. When $r \geqq 1$, the IFR lower bound easily simplifies to

$$\bar{F}(x) \geqq \begin{cases} e^{-\alpha x}, & \text{if } x < \mu_r^{1/r} \\ 0, & \text{if } x \geqq \mu_r^{1/r} \end{cases} \qquad \text{where } \alpha = [\Gamma(r+1)/\mu_r]^{1/r}.$$

Barlow and Proschan (1975) provided only this simpler special case.

Remark 3. None of the bounds given in Table 2 (with the exception of the IFR and IFRA lower bounds) is documented anywhere.

5. Bounds in terms of another distribution

5.1. *Usual notion of relative ageing*. Suppose F and G are life distributions where F is unknown and G is known. The distribution F is said to be more IFR, IFRA or NBU than G depending on whether $G^{-1} \circ F$ (with a suitable definition for G^{-1} when necessary) is convex, star-shaped or superadditive (see Barlow and Proschan (1975)). It

TABLE 2
Bounds on $\bar{F}(x)$ (based on rth moment) in various cases

Class	Upper bound	Lower bound
IFR	$\bar{F}(x) \le \begin{cases} 1, & \text{if } x < \mu_r^{1/r} \\ \delta_x, & \text{if } x \ge \mu_r^{1/r} \end{cases}$	$\bar{F}(x) \ge \begin{cases} \inf_{0\le\beta\le x} e^{-\alpha}, & \text{if } x < \mu_r^{1/r} \\ 0, & \text{if } x \ge \mu_r^{1/r} \end{cases}$
	$\left[\text{where } \int_0^1 r y^{r-1} \delta_x^y \, dy = \frac{\mu_r}{x^r}\right]$	$\left[\text{where } \int_0^\infty \left(\beta + \frac{x-\beta}{\alpha} z\right)^r e^{-z} dz = \mu_r\right]$
IFRA	$\bar{F}(x) \leqq \begin{cases} 1, & \text{if } x < \mu_r^{1/r} \\ \delta_x, & \text{if } x \geqq \mu_r^{1/r} \end{cases}$	$\bar{F}(x) \geqq \begin{cases} \delta_x, & \text{if } x < \mu_r^{1/r} \\ 0, & \text{if } x \geqq \mu_r^{1/r} \end{cases}$
	$\left[\text{where } \int_0^1 r y^{r-1} \delta_x^y \, dy = \frac{\mu_r}{x^r}\right]$	$\left[\text{where } 1 + \int_1^\infty r y^{r-1} \delta_x^y \, dy = \frac{\mu_r}{x^r}\right]$
NBU	$\bar{F}(x) \le \begin{cases} 1, & \text{if } x < \mu_r^{1/r} \\ \delta_x, & \text{if } x \ge \mu_r^{1/r} \end{cases}$	$\bar{F}(x) \geqq \begin{cases} \delta_x, & \text{if } x < \mu_r^{1/r} \\ 0, & \text{if } x \ge \mu_r^{1/r} \end{cases}$
	$\left[\text{where } \int_0^1 r y^{r-1} \delta_x^y \, dy = \frac{\mu_r}{x^r}\right]$	$\left[\text{where } \sum_{j=0}^{\infty} \delta_x^j [(j+1)^r - j^r] = \frac{\mu_r}{x^r}\right]$
DFR	$\bar{F}(x) \leqq \begin{cases} \exp(-rx/x_0), & \text{if } x < x_0 \\ (x_0/x)^r e^{-r}, & \text{if } x \geqq x_0 \end{cases}$	$\bar{F}(x) \ge 0$
	$[\text{where } x_0 = r[\mu_r/\Gamma(r+1)]^{1/r}]$	
DFRA	$\bar{F}(x) \leqq \delta_x$	$\bar{F}(x) \geqq 0$
	$\left[\text{where } \delta_x + \int_1^\infty r y^{r-1} \delta_x^y \, dy = \frac{\mu_r}{x^r}\right]$	
NWU	$\bar{F}(x) \leqq \delta_x$	$\bar{F}(x) \geqq 0$
	$\left[\text{where } \sum_{j=1}^{\infty} \delta_x^j [j^r - (j-1)^r] = \frac{\mu_r}{x^r}\right]$	

is useful to find a bound on $\bar{F}$ when one of its moments is known and F has such a relation with G.

We can assume without loss of generality that F shares the rth moment with G. If G has a different rth moment, we can replace $G(x)$ by $G(ax)$ for a suitable choice of a to make the moments equal.

The basic approach is very similar to those in the previous two sections. The constraint becomes

$$\int_0^\infty \bar{G}(\Lambda(x)) r x^{r-1} dx = \mu_r,$$

where Λ is a convex, star-shaped or superadditive hazard function such that $\Lambda_F = \Lambda_G \circ \Lambda$. Since we only used the non-increasing property of the function e^{-x} in most of the results of Section 3, they go through when it is replaced by $\bar{G}(x)$. The only result that cannot be extended is Theorem 2.2. A tractable solution for this problem may not exist, except for a few special forms of G. The results in the other cases are listed in Table 3. The cases of F being less positively ageing than G (in one of the above senses) are also included.

TABLE 3
Bounds on $\bar{F}(x)$ (based on rth moment and $\bar{G}(x)$) in various cases

$G^{-1} \circ F$	Upper bound	Lower bound
IFR	$\bar{F}(x) \leqq \begin{cases} 1, & \text{if } x < \mu_r^{1/r} \\ \bar{G}(\gamma_x), & \text{if } x \geqq \mu_r^{1/r} \end{cases}$	$\bar{F}(x) \geqq \begin{cases} \inf_{0 \leqq \theta \leqq x} \bar{G}(\alpha), & \text{if } x < \mu_r^{1/r} \\ 0, & \text{if } x \geqq \mu_r^{1/r} \end{cases}$
	$\left[\text{where } \int_0^1 ry^{r-1}\bar{G}(\gamma_x y)dy = \frac{\mu_r}{x^r}\right]$	$\left[\text{where } \int_0^x \left(\beta + \frac{x-\beta}{\alpha} z\right)^r dG(z) = \mu_r\right]$
IFRA	$\bar{F}(x) \leqq \begin{cases} 1, & \text{if } x < \mu_r^{1/r} \\ \bar{G}(\gamma_x), & \text{if } x \geqq \mu_r^{1/r} \end{cases}$	$\bar{F}(x) \geqq \begin{cases} \bar{G}(\gamma_x), & \text{if } x < \mu_r^{1/r} \\ 0, & \text{if } x \geqq \mu_r^{1/r} \end{cases}$
	$\left[\text{where } \int_0^1 ry^{r-1}\bar{G}(\gamma_x y)dy = \frac{\mu_r}{x^r}\right]$	$\left[\text{where } 1 + \int_1^\infty ry^{r-1}\bar{G}(\gamma_x y)dy = \frac{\mu_r}{x^r}\right]$
NBU	$\bar{F}(x) \leqq$?	$\bar{F}(x) \geqq \begin{cases} \bar{G}(\gamma_x), & \text{if } x < \mu_r^{1/r} \\ 0, & \text{if } x \geqq \mu_r^{1/r} \end{cases}$
		$\left[\text{where } \sum_{j=0}^{\infty} \bar{G}(j\gamma_x)[(j+1)^r - j^r] = \frac{\mu_r}{x^r}\right]$
DFR	$\bar{F}(x) \leqq \sup_{\beta \geqq 0} \exp(-\alpha x - \beta)$	$\bar{F}(x) \geqq 0$
	$\left[\text{where } \int_0^\infty ry^{r-1}\bar{G}(\alpha y + \beta)dy = \mu_r\right]$	
DFRA	$\bar{F}(x) \leqq \bar{G}(\gamma_x)$	$\bar{F}(x) \geqq 0$
	$\left[\text{where } \bar{G}(\gamma_x) + \int_1^\infty ry^{r-1}\bar{G}(\gamma_x y)dy = \frac{\mu_r}{x^r}\right]$	
NWU	$\bar{F}(x) \leqq \bar{G}(\gamma_x)$	$\bar{F}(x) \geqq 0$
	$\left[\text{where } \sum_{j=1}^{\infty} \bar{G}(j\gamma_x)[j^r - (j-1)^r] = \frac{\mu_r}{x^r}\right]$	

When F is more IFR than G and they share the rth moment ($r \geqq 1$), it is easy to see that the bound given above reduces to

$$\bar{F}(x) \geqq \begin{cases} \bar{G}(x), & \text{if } x < \mu_r^{1/r} \\ 0, & \text{otherwise,} \end{cases}$$

as given in Barlow and Proschan (1975).

5.2. *Alternative notion of relative ageing.* Since $G^{-1} \circ F = \Lambda_G^{-1} \circ \Lambda_F$, the above notions of relative ageing can be characterized by the convexity, star-shapedness or superadditivity of $\Lambda_G^{-1} \circ \Lambda_F$. Kalashnikov and Rachev (1986) introduced a new type of relative ageing via the convexity of $\Lambda_F \circ \Lambda_G^{-1}$. One can also attribute star-shapedness or superadditivity to this composite function. The interesting feature of this series of partial ordering is that each has a meaningful implication. These are as follows:

1. If the hazard rates λ_F and λ_G exist and $\lambda_G > 0$, then $\Lambda_F \circ \Lambda_G^{-1}$ is convex if and only if λ_F/λ_G is increasing.
2. If $\Lambda_G > 0$, then $\Lambda_F \circ \Lambda_G^{-1}$ is star-shaped if and only if Λ_F/Λ_G is increasing.
3. If $\bar{F}, \bar{G} > 0$, then $\Lambda_F \circ \Lambda_G^{-1}$ is superadditive if and only if

$$\bar{F}^{-1}\left(\frac{\bar{F}(x+t)}{\bar{F}(t)}\right) \geq \bar{G}^{-1}\left(\frac{\bar{G}(x+t)}{\bar{G}(t)}\right).$$

The quantity on the left-hand side may be interpreted as the *rescaled quantile of the life distribution of a component which is t units old*. In other words,

$$\boldsymbol{P}\left[X > \bar{F}^{-1}\left(\frac{\bar{F}(x+t)}{\bar{F}(t)}\right)\right] = \boldsymbol{P}[X_t > x],$$

X_t being the remaining life of a component which is t units old, and $X = X_0$.

Suppose the ageing of F is either faster or slower than that of a known distribution G in the above sense and that the rth moment of F is known to be μ_r. Then the moment constraint can be shown to be equivalent to

$$\int_0^\infty e^{-\Lambda(y)} r[\Lambda_G^{-1}(y)]^{r-1} \, d\Lambda_G^{-1}(y) = \mu_r,$$

where Λ is convex, star-shaped, superadditive, concave, 'inverse' star-shaped or subadditive. Here $\Lambda_F = \Lambda \circ \Lambda_G$. Thus it is enough to find a bound on Λ. This problem is very similar to that of Section 4. Table 1 is relevant here. An elaborate derivation is unnecessary since no further insight can be gained. It should be noted that even the 'NBU upper bound' generalizes in this case, since Lemma 2.2 continues to hold when the integrals are with respect to any positive measure.

6. Concluding remarks

An extremely simplified form of the technique presented in this paper was used by Barlow and Proschan (1965) to derive the lower bound on the reliability of an IFR distribution. In their later volume (1975) they gave a different (statistical) proof. A variation of the method suggested here may be used to derive reliability bounds based on a known quantile or a known moment *and* a quantile. It is expected that the method will also be useful for classroom teaching. As a matter of fact, the lemmas and theorems of Section 2, the procedure of Section 3 and the functions described in Table 1 are easily understood by means of diagrams drawn on a blackboard. Since the hazard functions of NBUE, DMRL and HNBUE classes are not known to have any special geometrically appealing property, the method may not be useful in deriving bounds for these classes.

References

BARLOW, R. E. AND PROSCHAN, F. (1965) *Mathematical Theory of Reliability*. Holt Reinhart and Winston, New York.

BARLOW, R. E. AND PROSCHAN, F. (1975) *Statistical Theory of Reliability and Life Testing: Probability Models*. Holt Reinhart and Winston, New York.

HAINES, A. AND SINGPURWALA, N. D. (1974) Some contributions to the stochastic characterization of wear. In *Reliability and Biometry*, ed. F. Proschan and R. J. Serfling, pp. 47–80. SIAM, Philadelphia.

KALASHNIKOV, V. V. AND RACHEV, S. T. (1986) Characterization of queuing models and their stability. In *Probability Theory and Mathematical Statistics*, ed. Yu. K. Prohorov et al. 2, pp. 37–53. VNU Science Press.

KLEFSJÖ, B. (1982) The HNBUE and HNWUE classes of life distributions. *Naval Res. Logist. Quart.* **29**, 331–344.

KORZENIOWSKI, A. AND OPAWSKI, A. (1976) Bounds for reliability in the NBU, NWU, NBUE and NWUE classes. *Zast. Mat. Appl. Math.* **15**, 1–5.

MARSHALL, A. W. AND PROSCHAN, F. (1972) Classes of distributions applicable in replacement with renewal theory implications. *Proc. 6th Berkeley Sym. Math. Statist. Prob.* 395–415.