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REFERENCES

[1] D. A. Reed and R. M. Fujimoto, *Multicomputer Networks: Message-Based Parallel Processing*. Cambridge, MA: The MIT Press, 1987.

[2] W.-J. Hsu, "Fibonacci cubes—A new interconnection topology," *IEEE Trans. Parallel Distrib. Syst.*, vol. 4, pp. 3–12, Jan. 1993.

[3] W. J. Dally and C. L. Seitz, "Deadlock-free message routing in multiprocessor interconnection networks," *IEEE Trans. Comput.*, vol. C-36, pp. 547–553, May 1987.

[4] M. Y. Chan and S.-J. Lee, "Fault-tolerant embedding of complete binary trees in hypercubes," *IEEE Trans. Parallel Distrib. Syst.*, vol. 4, pp. 277–288, Mar. 1993.

[5] H. P. Katseff, "Incomplete hypercubes," *IEEE Trans. Comput.*, vol. 37, pp. 604–608, May 1988.

[6] N.-F. Tzeng, "Structural properties of incomplete hypercube computers," in *Proc. 10th IEEE Int. Conf. Distrib. Computing Syst.*, May 1990, pp. 262–269.

[7] H.-L. Chen and N.-F. Tzeng, "Distributed identification of all maximal incomplete subcubes in a faulty hypercube," in *Proc. 8th Int. Parallel Processing Symp.*, Apr. 1994, pp. 723–728.

[8] S. R. Deshpande and R. M. Jenevein, "Scalability of binary tree on a hypercube," in *Proc. 1986 Int. Conf. Parallel Processing*, Aug. 1986, pp. 661–668.

[9] A. Y. Wu, "Embedding of tree networks into hypercubes," *J. Parallel Distrib. Computing*, vol. 2, pp. 238–249, 1985.

[10] N.-F. Tzeng, P.-J. Chuang, and H.-L. Chen, "Embeddings in incomplete hypercubes," in *Proc. 1990 Int. Conf. Parallel Processing*, vol. III, Aug. 1990, pp. 335–339.

Hierarchical Classification of Permutation Classes in Multistage Interconnection Networks

Nabanita Das, Bhargab B. Bhattacharya, and Jayasree Dattagupta

***Abstract*— This brief contribution explores a new hierarchy among different permutation classes, that has many applications in multistage interconnection networks. The well-known LC (linear-complement) class is shown to be merely a subset of the closure set of the BP (bit-permute) class, known as the BPCL (bit-permute-closure) class; the closure is obtained by applying certain group-transformation rules on the BP-permutations. It indicates that for every permutation P of the LC class, there exists a permutation P^* in the BP class, such that the conflict graphs of P and P^* are isomorphic, for n-stage MIN's. This obviates the practice of treating the LC class as a special case; the existing algorithm for optimal routing of BPC class in an n-stage MIN, can take care of optimal routing of the LC class as well. Finally, the relationships of BPCL with other classes of permutations, e.g., LIE (linear-input-equivalence), BPIE (bit-permute-input-equivalence), BPOE (bit-permute-output-equivalence) are also exposed. Apart from lending better understanding and an integral view of the universe of permutations, these results are found to be useful in accelerating routability in n-stage MIN's as well as in $(2n-1)$-stage Benes and shuffle-exchange networks.**

***Index Terms*— Multistage interconnection networks (MIN), BP(bit-permute) permutations, linear permutations, baseline network, Benes network, conflict graph, optimal routing.**

I. INTRODUCTION

Design and analysis of multistage interconnection networks (MIN) play a crucial role in large scale parallel processing systems. Various topologies of such networks have been reported in the literature for use in SIMD and MIMD computers. An $N \times N$ unique-path, full-access MIN, e.g., the baseline or omega, has $n(= \log_2 N)$ stages and $(N.n/2)$ binary switches [1]–[6]. One fundamental criterion in the selection and design of a MIN is its permutation capability [7], i.e., the ability to establish simultaneous one-to-one and onto communications among different modules, usually represented as a permutation.

Routing an arbitrary permutation P through a unique-path full-access MIN, may require the usage of common links leading to a conflict, and therefore, may not be realizable in a single pass. The problem of *optimal routing* is to determine the minimum number of passes required to realize P. Equivalently, one needs to partition P into minimum number of subsets, such that transmissions included in each subset are conflict-free, and hence routable in a single pass. The conflict information is usually represented by a graph, called the conflict graph $G(V, E)$ [8], that consists of N vertices each representing a transmission of P; two vertices are adjacent if and only if the corresponding transmissions are conflicting, i.e., they demand a common link in the network. The optimal routing problem can then be mapped to the well-known graph-coloring problem [8], which for an arbitrary permutation, is NP-hard [9]. A heuristic algorithm of complexity $O(N^3)$ was reported in [10] to tackle an arbitrary permutation. However, the BPC (bit-permute-complement) class of permutations, is optimally routable in omega/delta network [8]; the time complexity of the algorithm turns out to be linear in the number of switches. Later, it has been shown that the same algorithm can

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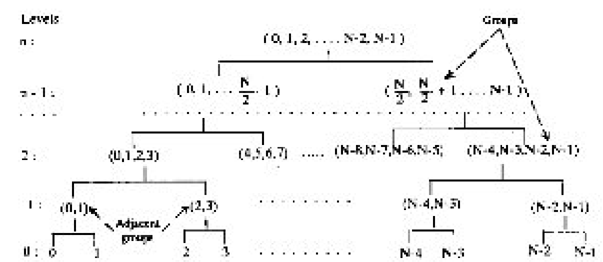

Fig. 1. The input (output) groups at different levels for a baseline network.

be applied to a larger class of permutations, called the BPCL (bit-permute-closure) class [11]. The closure is defined using certain *group transformation* rules on the permutations, that partition them into several equivalence classes: members belonging to the same partition have isomorphic conflict graphs.

Another class of permutations, namely the LC (linear-complement) class is also frequently addressed in the literature [12]–[16]. Self-routing of LC permutations has been studied in $(2n-1)$-stage networks like Benes and shuffle-exchange networks [12], [14]–[16]. Routing techniques have been reported for linear permutations in an omega network, circulating the transmissions twice through it [17]; this is essentially the same as routing through a $2n$-stage shuffle-exchange network. A similar technique was also used much earlier, to route the BPC permutations in omega networks [18].

In this brief contribution, we expose a new hierarchy among different classes of permutations. Needless to say that the BPC class is a subset of the LC class. Our analysis reveals a very interesting property: that the LC class is merely a subset of the BPCL class. Since BPCL is the closure of the BP class under *group-transformation* [11], for every permutation P of the LC class, there exists a permutation P^* in the BP class, such that the conflict graphs of P and P^* are isomorphic; P^* is called the BP-*isomer* of P. Thus every member of LC class can be generated from the BP class, by applying certain restricted forms of group-transformations. This obviates the need of treating the LC class as a special case. The existing algorithm for optimal routing of the BPC-permutations [8] will be applicable for optimal routing of any LC permutation in an n-stage network as well, provided the corresponding BP-isomer is identified. An $O(N^2)$ algorithm for recognizing the BP-isomer, appears in [11], the complexity of which has further been improved to $O(N.n)$ in [19].

Our hierarchical classification also involves other classes of permutations. For instance, the LIE (linear-input-equivalence) permutations are generated from the L permutations by applying input group-interchanges [11]. As LC turns out to be a subset of BPCL, so does the LIE class. It has already been shown that the LIE permutations are self-routable in Benes and $(2n-1)$-stage shuffle-exchange network [16]. This hierarchical classification was also used to develop a fast routing algorithm for the BPCL class of permutations in Benes network [19]. Similar routing algorithm will also exist for a $(2n-1)$-stage shuffle-exchange network.

To summarize, this brief contribution correlates different classes of permutations, namely BP, LC, LIE, BPIE (bit-permute-input-equivalence), BPOE (bit-permute-output-equivalence), and the BPCL class, from the viewpoint of equivalence partitioning defined by group-transformations. Apart from lending better understanding and an integral view of the universe of permutations, these results turn out to be useful in routing LC permutations in n-stage networks, and also in accelerating self-routability in $(2n-1)$-stage Benes and shuffle-exchange networks.

The brief contribution is organized as follows. In Section II, we restate few earlier results on the BPCL class of permutations. Section III exposes the relationship between the LC and the BPCL class, followed by the routing strategy of LC permutations in an n-stage network. The hierarchy among different classes of permutations is discussed in Section IV. Concluding remarks appear in Section V.

II. Group-Transformation Rules and the BPCL Class

As the work reported in this brief contribution is centered around the BPCL class of permutations, we restate in this section, few relevant definitions and earlier results [11] for completeness.

We consider an $N \times N$ baseline network for our study. However, the results will also hold good for other unique-path full-access MIN's, as has been observed in [11].

Definition 1: For an $N \times N$ baseline network, the inputs (outputs) are grouped in different levels as shown in Fig. 1. Any group at level i, $0 \leq i \leq n$, $(n = \log_2 N)$, contains 2^i elements, $\{x, x+1, \cdots, x+2^i-1\}$, where the least element of the group is $x = p.2^i, 0 \leq p < 2^{n-i}$; two groups at level i are said to be adjacent if they have the same parent at level $(i+1)$.

Definition 2: Let $a \leftrightarrow b$ denote: interchange a and b, i.e., replace $a(b)$ by $b(a)$. A group interchange $tX(j:x)$, (where $X = I$ stands for input and $X = O$ refers to output) applied on a permutation P, interchanges the elements of two adjacent groups of inputs (outputs) at level $j, 0 \leq j < n$, following the rule $k \leftrightarrow k+2^j, x \leq k < x+2^j$, where x is the least element of the two groups. This process generates another permutation P', and is denoted by: $tX(j:x)\ [P] \rightarrow P'$.

Example 1: Consider a permutation P: (7 6 2 4 0 3 1 5), (an $N \times N$ permutation P is represented as a sequence of $p(i)$'s, $0 \leq i < N$, such that $p(i)$ is the output corresponding to the input i in P) and the group-interchange $tI(1:4)$, such that $tI(1:4)\ [P] \rightarrow P'$; the interchanging input pairs are: $4 \leftrightarrow 6$ and $5 \leftrightarrow 7$. Hence P': (7 6 2 4 1 5 0 3).

Similarly, $tO(2:0)[P] \rightarrow P''$: (3 2 6 0 4 7 5 1); the output pairs interchanged are $0 \leftrightarrow 4$, $1 \leftrightarrow 5$, $2 \leftrightarrow 6$, $3 \leftrightarrow 7$.

Definition 3: An ordered sequence of group-interchanges on inputs (outputs) is represented as, $SX = \{tX(l_1 : x_1); tX(l_2 : x_2); \cdots ; tX(l_k : x_k)\}$, such that for $i < j$, $l_i \leq l_j$, and if $l_i = l_j$, $x_i < x_j$.

Definition 4: Two sequences of input (output) group-interchanges RX_1 and RX_2 are said to be equivalent, if for every permutation P, $RX_1\ [P] = RX_2\ [P]$.

Remark: For any sequence of input (output) group-interchanges $RX = \{tX(l_1 : x_1); tX(l_2 : x_2); \cdots ; tX(l_k : x_k)\}$, there exists an equivalent *ordered sequence of group-interchanges* on inputs (outputs) $SX = \{tX(w_1 : z_1); tX(w_2 : z_2); \cdots ; tX(w_k : z_k)\}$

Simultaneous application of both input and output group- interchanges in an appropriate order, now leads to the concept of group-transformation.

Definition 5: A group-transformation T is defined as an ordered sequence of group-interchanges on the outputs followed by an ordered sequence of group-interchanges on the inputs (null sequences are also included).

Example 2: A group-transformation $T = \{tO(0:0); tI(1:4)\}$ applied on P transforms it according to the following steps:

$$P : (7 \quad 2 \quad 6 \quad 4 \quad 0 \quad 3 \quad 1 \quad 5)$$

$$\xrightarrow{tO(0:0)} P' : (7 \quad 2 \quad 6 \quad 4 \quad 1 \quad 3 \quad 0 \quad 5)$$

$$\xrightarrow{tI(1:4)} P'' : (7 \quad 2 \quad 6 \quad 4 \quad 0 \quad 5 \quad 1 \quad 3).$$

It has been shown that for any random sequence of input and output group-interchanges, there exists an equivalent group-transformation which imparts the same effect on any permutation. Furthermore, *group-transformation* induces an *equivalence partition* on the set of all permutations.

Definition 6: Given a permutation P, let Closure (P) denote the set of permutations derivable from P by the application of all possible group-transformations. The above set is said to be the closure set of **P**.

Definition 7: The BPCL (bit-permute-closure) set of permutations is defined as follows:

BPCL $= \bigcup_{\forall P \in BP} Closure\,(P)$, where P is a BP permutation defined by the bit-permute-rule,

$x_{n-1}x_{n-2}\cdots x_1x_0 \rightarrow y_{n-1}y_{n-2}\cdots y_1y_0;\ y_i = x_j, 0 \leq i, j < n$ and $y_i \neq y_j$ for $i \neq j$.

The following three important results have been reported in [11].

Result 1: The conflict graphs of all permutations in a closure set are isomorphic.

Result 2: In an $N \times N$ baseline network, for any permutation P, the cardinality of the closure set i.e., $|Closure(P)| \geq 2^{N-1}$.

Result 3: The closure sets of any two BP-permutations are disjoint; hence, $|\text{BPCL}| \geq n! \cdot 2^{N-1}$.

Thus, for each $P_i \in$ BPCL, there exists a unique $P_j \in$ BP, such that the conflict graphs for P_i and P_j are isomorphic; P_j is called the BP-*isomer* or the *seed*-**BP** of P_i. It has been also shown that BPC $\subset$ BPCL.

III. Relationship Between L and BPCL Permutations

In this section, some important results related to LC and BPCL classes are derived.

A. Definitions and Preliminaries

Definition 8: An $N \times N$ permutation P is said to be a linear permutation (L), if there exists an $n \times n$ binary matrix **Q**, such that for every input represented in binary as $(x_{n-1}x_{n-2}\cdots x_1x_0)$, its target output $(y_{n-1}y_{n-2}\cdots y_1y_0)$ is given by the equation: $y = \mathbf{Q}x \cdots (1)$, where x and y represent the column vectors $(x_{n-1}x_{n-2}\cdots x_1x_0)^T$ and $(y_{n-1}y_{n-2}\cdots y_1y_0)^T$ respectively, the superscript T denoting the transpose operation.

By definition, **Q** is nonsingular. All additions are assumed to be modulo 2.

A permutation is a *linear complement permutation* (**LC**) if, $y = \mathbf{Q}x + C \cdots (2)$,
where C is an n-bit column vector.

Example 3: Given

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix};$$

from (1), the linear equations determining the output bits are: $y_0 = x_0 + x_1 + x_2$; $y_1 = x_0 + x_2$; $y_2 = x_0 + x_1$; which defines the L-permutation P: (0 7 5 2 3 4 6 1).

Similarly, to generate a LC permutation P' from **Q**, let $C = (011)^T$. From (2), the linear equations for the output bits are: $y_0 = x_0 + x_1 + x_2$; $y_1 = x_0 + x_2 + 1$; $y_2 = x_0 + x_1 + 1$. Therefore, P': (6 1 3 4 5 2 0 7).

Remark: It is known that the LC class contains the BPC class [13], [14]. If the matrix **Q** in Equation (2) contains exactly one 1 in each row and in each column, then it results a BPC permutation.

B. Linear Permutations and Input Group-Interchanges

The BPCL class of permutations is based on the notion of group-interchanges. To correlate LC and BPCL, we first introduce a canonical nonsingular matrix $M(\mathbf{Q})$, defined below.

The Canonical matrix M(Q):

Definition 9: For two n-bit vectors, $V = (v_0\ v_1 \cdots v_{n-1})^T$ and $V' = (v'_0\ v'_1 \cdots v'_{n-1})^T$, we define an ordering 'less than' ($\langle$), such that, $V \langle V'$, if and only if, $v_i < v'_i$, for some i, $0 \leq i \leq n-1$ and $v_k = v'_k$ for all k, $i < k \leq n-1$.

Example 5: Given $V = (0\ 1\ 1)^T$ and $V' = (1\ 0\ 1)^T$, $V' \langle V$.

Definition 10: Let $\mathbf{Q} = (Q_{j,k})$ be the matrix corresponding to a linear permutation. The column i of **Q** is represented as $Q_{*,i} = (Q_{0,i} Q_{1,i} \cdots Q_{(n-1),i})^T$. From **Q**, we now generate a canonical binary matrix $M(\mathbf{Q})$, such that the column i of $M(\mathbf{Q})$ is denoted by,

$$M(Q)_{*,i} = \min\left\{Q_{*,i} + \sum_{j=0}^{i-1} Q_{*,j}.k_j,\ k_j = 0 \text{ or } 1\right\},$$
$$\text{for } 1 \leq i \leq n-1.$$

where the minimality is dictated by the ordering relation ($\langle$) of Definition 9.

Remark: As **Q** is nonsingular, by construction $M(\mathbf{Q})$ is nonsingular and thereby defines another L permutation.

Example 6: A matrix **Q**, can be transformed to its canonical matrix $M(\mathbf{Q})$, as shown below:

$$\mathbf{Q}: \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \mathbf{Q}': \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\rightarrow M(\mathbf{Q}): \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here, the minimality is attained by making $Q_{*,1} + Q_{*,0} \rightarrow M(Q)_{*,1}$ and $Q_{*,2} + Q_{*,1} + Q_{*,0} \rightarrow M(Q)_{*,2}$.

Lemma 1: For any **Q**, specifying an L permutation, the matrix $M(\mathbf{Q})$ is unique.

Proof: Follows from the definition. □

Definition 11: For any column i in **Q**, $0 \leq i \leq n-1$, let $Q_{j,i} = 1$ *and* $Q_{r,i} = 0$, for all r, $j < r \leq n-1$. Then it will be denoted as $Q_{*,i}(m) = j$. Similar notation will also be used for $M(\mathbf{Q})$.

Lemma 2: In a canonical matrix $M(\mathbf{Q})$, $M(Q)_{*,i}(m) \neq M(Q)_{*,j}(m)$, for $0 \leq i, j \leq n-1$ and $i \neq j$.

Proof: Evident from the construction of $M(\mathbf{Q})$. □

Equivalence of Matrix Operation and Input Group-Interchanges: The following lemma relates certain matrix operations and input group-interchanges.

Lemma 3: Let **Q** define an L-permutation P. The corresponding canonical matrix $M(\mathbf{Q})$ defines another L-permutation, say P^*. Then there exists a sequence of input group-interchanges SI, that transforms P to P^*, i.e., $SI[P] \rightarrow P^*$.

Proof: The L-permutation P is defined by the equation: $y = \mathbf{Q}x$. Let the column $Q_{*,i}$ of **Q** be replaced by $Q_{*,i} + Q_{*,j}$, $0 \leq j < i \leq n-1$; a new nonsingular matrix $\mathbf{Q}'$ is created, that defines another L-permutation P', given by: $y = \mathbf{Q}'x$.

In **Q** and $\mathbf{Q}'$, all columns, excepting the i-th one, are identical. Therefore, all the outputs, excepting those for the inputs with $x_i = 1$, will be the same in P and P'. Since in $\mathbf{Q}'$, $Q'_{*,i} = Q_{*,i} + Q_{*,j}$, the input pairs with $x_i = 1$ and differing in bit x_j are interchanged in P to generate P'. The same transformation can be implemented by the following sequence of input group-interchanges:

$$SI' = \left\{ I(j:x), \quad \middle| x = \sum_{k=j-1}^{n-1} C_k.2^k, C_k = 0 \text{ or } 1, \right.$$
$$\left. \text{for } k \neq i \text{ and } C_i = 1 \right\}.$$

Since, $M(\mathbf{Q})$ is generated from $\mathbf{Q}$ by applying such column transformations only, the lemma follows immediately. □

Example 7: Consider the matrices $\mathbf{Q}$, $\mathbf{Q}'$ and $M(\mathbf{Q})$ of Example 6, which define the L-permutations, P: (0 7 5 2 3 4 6 1), P': (0 7 2 5 3 4 1 6), and P^*: (0 7 2 5 1 6 3 4) respectively. The sequences of input group-interchanges involved in the transformations are: $\{tI(0:2); tI(0:6)\}$ $[P] \rightarrow P'$, and $\{tI(1:4)\}[P'] \rightarrow P^*$.

C. The Canonical Matrix and Output Group-Interchanges

In this subsection, we now study the effect of output group-interchanges on the L-permutation defined by a canonical matrix.

Some Related Properties of $\mathbf{M(Q)}$*:*

Definition 12: In the column i of a canonical matrix $M(\mathbf{Q})$, let $M(Q)_{*,i}(m) = k$. Then the element $M(Q)_{k,i}$ is defined as the E-bit (essential bit) of $M(\mathbf{Q})$, and the remaining non-zero elements are defined as the R-bits (redundant bits) of column i.

Lemma 4: In any $M(\mathbf{Q})$, if $M(Q)_{k,i}$ is the E-bit of column i, then $M(Q)_{k,j} = 0$, for any j, $0 \leq i < j \leq n-1$.

Proof: Let $M(Q)_{k,i}$ be the E-bit of column i and $M(Q)_{k,r} = 1$, where $i < r \leq n-1$. Now $M(Q)_{k,r}$ cannot be the E-bit of column r, by Lemma 2. Since it is 1 in that column, the E-bit must appear in some row s, $s > k$.
Let us now consider the column vector $V = M(Q)_{*,r} + M(Q)_{*,i}$. Clearly $V \{ M(Q)_{*,r}$. It contradicts the definition of $M(\mathbf{Q})$, and therefore proves the lemma. □

Lemma 5: Let p_j denote the number of 1's in row j of $M(\mathbf{Q})$. Then $1 \leq p_j \leq n - j$.

Proof: By Lemma 2, the row j of $M(\mathbf{Q})$ will contain exactly one E-bit of some column c. Lemma 4 implies that all the elements $M(Q)_{j,k} = 0$, for $c < k \leq n-1$. Again, for $0 \leq k < c$, the column k may have $M(Q)_{j,k} = 1$ as an R-bit, if and only if its E-bit appears at some row r, $r > j$. Therefore, at most $(n-1-j)$ columns may have an R-bit in row j. It proves the lemma. □

Lemma 6: Let $\mathbf{Q}^*$ be the nonsingular binary matrix obtained from a canonical matrix $M(\mathbf{Q})$, by setting all R-bits to zero in each column. Let P and P^* be the two L-permutations defined by $M(\mathbf{Q})$ and $\mathbf{Q}^*$ respectively. Then 1) P^* is a BP-permutation, and 2) P^* can be derived from P by applying certain output group-interchanges.

Proof: Consider column i of $M(\mathbf{Q})$; assume that the E-bit is $M(Q)_{k,i}$, and let an R-bit be $M(Q)_{r,i}$, $r < k$. Let us make $M(Q)_{r,i} = 0$, resulting another binary matrix $\mathbf{Q}'$. Since $\mathbf{Q}$ is nonsingular, so will be $\mathbf{Q}'$. Therefore, $\mathbf{Q}'$ generates another L-permutation, say P'.

Let us now examine the relation between P and P'. The linear expressions of all the output bits for P and P' are same, except for the output bit y_r. In the expression of y_r, the coefficient of x_i is 1 for P, whereas it is 0 for P'. Therefore, if an input x with $x_i = 1$, is mapped to an output y in P, then it will be mapped to an output y' in P', such that y and y' differ in bit y_r only. The same change can be accomplished by output group-interchanges in level r, if and only if the affected outputs cover some pairs of adjacent groups in level r.

To find the affected outputs, we put $x_i = 1$, in the expressions of output bits. Each output bit $y_p, p \neq k$, will remain independent, because of the uniqueness of the E-bit in each column $q, q \neq i$. Since the E-bit of the column i, is $M(Q)_{k,i}$, by putting $x_i = 1$, the expression for the output bit y_k contains only some R-bits of other columns j, $j < k$: these R-bits already appear in the expressions for some y_w's, $w > k$. Therefore, the output bit y_k can be expressed as:

$$y_k = \sum_{w=k+1}^{n-1} b_w y_w + 1, \text{ where, } b_w = 0 \text{ or } 1, \text{ for } k < w \leq n-1.$$

As $k > r$, the affected sets of outputs exactly cover some pairs of adjacent groups at level r.

Now, the required sequence of output group-interchanges is:

$$SO = \left\{ tO(r:x), \middle| x = \sum_{j=r+1}^{n-1} C_j . 2^j, \quad C_j = 0 \text{ or } 1, \right.$$
$$\left. \text{for } j \neq k \text{ and } C_k = \sum_{w=k+1}^{n-1} b_w . C_w + 1 \right\},$$

where, b_w is the coefficient of x_w in the expression for y_w.

Therefore, if we set all the R-bits of each column of $M(Q)$ to zero, we obtain a new nonsingular matrix $\mathbf{Q}^*$; this will contain a single 1 in each column and in each row. Hence $\mathbf{Q}^*$ generates a BP-permutation P^*; also a sequence of output group-interchanges is sufficient to generate P^* from P. □

Example 8: The matrix $M(\mathbf{Q})$ of Example 6, generates the L-permutation P : (0 7 2 5 1 6 3 4). Let us transform $M(\mathbf{Q})$ to $\mathbf{Q}^*$ in the following way:

$$M(\mathbf{Q}): \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \mathbf{Q}': \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\rightarrow \mathbf{Q}^*: \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where, $\mathbf{Q}'$ and $\mathbf{Q}^*$ define the permutations P' : (0 6 2 4 1 7 3 5) and P^*: (0 4 2 6 1 5 3 7), respectively. Note that $\{tO(0:4); tO(0:6)\}[P] \rightarrow P'$ and $\{tO(1:4)\}[P'] \rightarrow P^*$; P^* is a BP-permutation.

LC *Permutations as a Subset of* BPCL*:* The main result of this brief contribution is now stated in the following theorem.

Theorem 1: For every L-permutation P, there exists a unique BP-permutation P^*, such that $P \in Closure(P^*)$, or in other words, $L \subseteq BPCL$.

Proof: Follows from the results stated in Lemma 3 and Lemma 6. □

Remark: An LC permutation P', defined by, $y = \mathbf{Q}x + C$, belongs to *Closure*(P), where P is the L permutation, defined by, $y = \mathbf{Q}x$; for each $C(i) = 1$, output group-interchanges at level i, $0 \leq i \leq n-1$, for all possible groups at that level, will transform P' to P. Therefore, it is evident that LC $\subseteq$ BPCL.

Example 9: The L-permutation P : (0 7 5 2 3 4 6 1) is defined by,

$$\mathbf{Q}: \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Assuming $C = (0 \quad 1 \quad 0)^T$, we get the LC permutation P': (2 5 7 0 1 6 4 3). Note that $\{tO(1:0); tO(1:4)\}[P] \rightarrow P'$.

D. Routing of LC Permutations

Since LC $\subseteq$ BPCL, it is clear that the LC permutations will be optimally routable by the same algorithm that routes optimally the BPC permutations [8]. Now, given an arbitrary permutation P, we can decide whether it belongs to the BPCL class, and if so, determine its BP-isomer in $O(N.n)$ time [11], [19]. The generalization of group-transformation rules, as has been described in [11], implies that similar routing will also exist for LC permutations in all equivalent [2] n-stage unique-path full-access MIN's. Thus, the overall complexity of optimal routing of LC-permutations will be $O(N.n)$, i.e., linear in the number of switches.

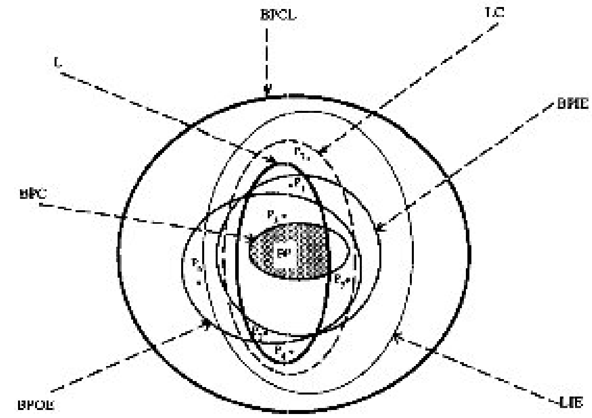

Fig. 2. Hierarchical relationships among various permutation classes.

IV. A New Hierarchy of Permutation Classes

It was known earlier that BPC $\subseteq$ LC. The new result that LC $\subseteq$ BPCL, provides a better understanding of the LC class. We will now introduce other classes of permutations defined by various group-transformations, and then explore the hierarchy among them. This knowledge may lead to some interesting applications, especially in routing through different types of MIN's.

A. Permutation Classes Generated from BP and L Permutations

For an $N \times N$ system, the set of **BP** *permutations* constitutes of $n!$ permutations, where $n = \log_2 N$. The BPCL class is generated by applying all possible group-transformations on the BP class. It has been shown that $|BPCL| \geq n! \cdot 2^{N-1}$ [11]. To have a better idea about the BPCL class, we now consider input group-interchanges and output group-interchanges separately. It has been observed that the number of permutations derivable from P, by the application of all possible ordered sequences of input (output) group-interchanges is 2^{N-1}.

Let $I(P)$ and $O(P)$ denote the set of permutations derivable from P by applying all possible sequences of input and output group-interchanges respectively.

Definition 13: The set of permutations, $BPIE$(BP-input-equivalence), is defined as:
$BPIE = \bigcup_{\forall P \in BP} I(P)$. It is clear that $|BPIE| = n!\,2^{N-1}$.
Similarly, the set $BPOE$(BP*-output-equivalence*) is defined as:
$BPOE = \bigcup_{\forall P \in BP} O(P)$; $|BPOE| = n! \cdot 2^{N-1}$.

Remark: It has been shown in [19] that BPC $\subseteq$ BPIE $\cap$ BPOE.

Lemma 7: $|\text{BPIE} \cap \text{BPOE}| \geq 2^{N-1} - (n! - 1)2^n$.

Proof: As observed earlier, BPC $\subseteq$ BPIE $\cap$ BPOE. For identity permutation P_I, $I(P_I) = O(P_I)$. Now $P_I \in$ BPC. Hence the lemma follows immediately. □

By definition, BP $\subseteq L$. Here, we have established that $L \subseteq$ BPCL, or in other words, all linear permutations can be generated from BP's, by the application of suitable group-transformations.

Remark: $|BP| = n!$, whereas $|L| = 2^{n(n-1)/2} \cdot \prod_{i=0}^{n}(2^i - 1)$. It implies that many linear permutations will have the same BP-isomer.

Definition 14: The LIE(linear-input-equivalence) class of permutations is defined as:
$\text{LIE} = \bigcup_{\forall P \in L} I(P)$.

Remark: Since $L \subseteq$ BPCL, it is evident that LIE $\subseteq$ BPCL.
The LIE and BPIE classes of permutations exhibit a special characteristic in $(2n-1)$-stage networks, like Benes and shuffle-exchange. It has been shown earlier that LC and BPC permutations are self-routable by the least-control routing technique [15]; now any permutation $P' \in$ LIE or BPIE is also self-routable by the same method [16].

B. Hierarchical Relationships Among the Permutation Classes

Fig. 2 shows the hierarchical relations among the different classes of permutations discussed so far. The following observations are immediate.

Obs 1: The BP class is the minimum set of permutations, which by the application of suitable group-transformations, generates all the sets BPIE, BPOE, L, LIE and the BPCL.

Obs 2: The BPCL class of permutations is a superset of all the sets BP, BPIE, BPOE, L and LIE.

Obs 3: The L class of permutations includes members from four different classes, namely 1) BPIE, 2) BPOE, 3) BPIE $\cap$ BPOE and 4) none of 1), 2), 3).

Example 10: The L-permutation P_1: (0 2 3 1 4 6 7 5) is defined by,

$$\mathbf{Q}_1 : \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $\{tI(0:2); tI(0:6)\}[P_1] \rightarrow P^*$: (0 2 1 3 4 6 5 7); since $P^* \in$ BP, $P_1 \in$ BPIE.

Example 11: The L-permutation P_2: (0 3 1 2 4 7 5 6) is defined by,

$$\mathbf{Q}_2 : \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $\{tO(0:2); tO(0:6)\}[P_2] \rightarrow P^*$: (0 2 1 3 4 6 5 7); since $P^* \in$ BP, $P_2 \in$ BPOE.

Example 12: The L-permutation P_3: (0 1 3 2 4 5 7 6) is defined by,

$$\mathbf{Q}_3 : \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $\{tI(0:2); tI(0:6)\}[P_3] \rightarrow P_I$: (0 1 2 3 4 5 6 7); P_I is in BP; also $\{tO(0:2); tO(0:6)\}[P_3] \rightarrow P_I$. Therefore $P_3 \in$ BPIE $\cap$ BPOE.

Example 13: The L-permutation P_4: (0 3 1 2 7 4 6 5) is defined by,

$$\mathbf{Q}_4 : \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that $\{tO(0:2); tO(0:6); tI(0:4); tI(0:6)\}[P_4]$ produces the BP-permutation P: (0 2 1 3 4 6 5 7). Here, both input and output group-interchanges are essential, implying that P_4 is included neither in BPIE, nor in BPOE.

Obs 4: The set L does not cover BPIE $\cap$ BPOE.

Example 14: Consider a permutation P_5: (0 1 5 4 2 3 6 7). Both $\{tI(0:2)\}[P_5]$ and $\{tO(0:4)\}[P_5]$ generate the BP-permutation: (0 1 4 5 2 3 6 7). Therefore $P_5 \in$ BPIE $\cap$ BPOE. But $P_5 \notin L$.

Obs 5: Since BP $\subseteq L$, BPIE $\subseteq$ LIE.

Obs 6: BPOE $\not\subseteq$ LIE.

Example 15: Given P_6: (0 5 1 4 2 6 3 7), note that $\{tO(0:4)\}[P_6]$ generates the BP-permutation: (0 4 1 5 2 6 3 7). Hence $P_6 \in$ BPOE, but $P_6 \notin$ LIE.

Obs 7: LC $\subseteq$ LIE.

Example 16: The L-permutation P': (0 3 1 2 4 7 5 6) is defined by,

$$\mathbf{Q_5}: \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Assuming $C = (0 \quad 0 \quad 1)^2$; we get the LC-permutation P_7: (4 7 5 6 0 3 1 2). Note that $\{tI(2:0)\}[P_7] \rightarrow P'$; i.e., $P_7 \in I(P')$.

We have already observed that BPC $\subseteq$ BPIE $\cap$ BPOE. Moreover, it is easy to see that BP = BPC $\cap$ LC and also BPC $\subseteq$ LC.

These permutations (P_1 through P_7) cited in the above examples, are shown in Fig. 2, indicating their inclusion in different permutation classes.

V. Conclusion

In this brief contribution, we have introduced a new hierarchy among different permutation classes that are of interest to MIN's. We proved that LC $\subseteq$ BPCL. Thus for every LC permutation P, there exists a unique BP-isomer P^*, such that the conflict graphs of P and P^* for an n-stage MIN. are isomorphic. Therefore the optimal routing technique developed earlier for the BPC class of permutations [8], is also applicable to LC permutations. For any permutation $P \in$ BPCL, the BP-isomer of P can be found in $O(N.n)$ time [19]. It has been observed that certain restricted forms of group-transformations are required to generate the LC class from BP. Furthermore, we introduce other classes of permutations and report their mutual relations. Knowledge of this hierarchy helps us in accelerating the routability of different classes of permutations in n-stage unique-path full-access MIN's, as well as in $(2n-1)$-stage MIN's, e.g., Benes and shuffle-exchange networks.

References

[1] T. Feng, "A survey of interconnection networks," *Computer*, vol. 14, pp. 12–27, Dec. 1981.

[2] C. L. Wu and T. Feng, "On a class of multistage interconnection networks," *IEEE Trans. Comput.*, vol. C-29, pp. 694–702, Aug. 1980.

[3] H. J. Siegel, "A model of SIMD machines and a comparison of various interconnection networks," *IEEE Trans. Comput.*, vol. C-28, pp. 907–917, Dec. 1978.

[4] D. K. Pradhan and K. L. Kodandapani, "A uniform representation of single and multistage interconnection networks used in SIMD machines," *IEEE Trans. Comput.*, vol. C-29, pp. 777–790, Sept. 1980.

[5] D. P. Agrawal, "Graph theoretical analysis and design of multistage interconnection networks," *IEEE Trans. Comput.*, vol. C-32, pp. 637–648, July 1983.

[6] S. T. Huang and S. K. Tripathi, "Finite state model and compatibility theory: new analysis tools for permutation networks," *IEEE Trans. Comput.*, vol. C-35, pp. 591–601, July 1986.

[7] T. H. Szymanski and V. C. Hamacher, "On the permutation capability of multistage interconnection networks," *IEEE Trans. Comput.*, vol. C-36, pp. 810–822, July 1987.

[8] C. S. Raghavendra and A. Varma, "Fault-tolerant multiprocessors with redundant path interconnection networks," *IEEE Trans. Comput.*, vol. C-35, pp. 307–316, Apr. 1986.

[9] P. J. Bernherd and D. J. Rosenkrantz, "The complexity of routing through an omega network," in *Proc. 25th Annu. Allerton Conf. Commun., Contr. and Computing*, Sept. 1987.

[10] J. S. Deogun and Z. Fang, "A heuristic algorithm for conflict resolution problem in multistage interconnection networks," in *Proc. 1988 Int. Conf. on Parallel Processing*, Aug. 1988, pp. 475–478.

[11] N. Das, B. B. Bhattacharya, and J. Dattagupta, "Isomorphism of conflict graphs in multistage interconnection networks and its application to optimal routing," *IEEE Trans. Comput.*, vol. 42, pp. 665–667.

[12] M. C. Pease, "The indirect binary n-cube microprocessor array," *IEEE Trans. Comput.*, vol. C-26, pp. 458–473, May 1977.

[13] T. Etzion and A. Lempel, "An efficient algorithm for generating linear transformations in a shuffle-exchange network," *SIAM J. Comput.*, vol. 15, pp. 216–221, Feb. 1986.

[14] S. T. Huang, S. K. Tripathi, N. S. Chen, and Y. C. Tseng, "An efficient routing algorithm for realizing linear permutations on p^t-shuffle-exchange networks," *IEEE Trans. Comput.*, vol. 40, pp. 1292–1298, Nov. 1991.

[15] C. S. Raghavendra and R. V. Boppana, "On self-routing in Benes and shuffle-exchange networks," *IEEE Trans. Comput.*, vol. 40, pp. 1057–1064, Sept. 1991.

[16] N. Das, K. Mukhopadhyaya, and J. Dattagupta, "On self-routable permutations in Benes network," in *Proc. 1991 Int. Conf. Parallel Processing*, Aug. 1991, pp. 1270–1273.

[17] J. Keohane and R. E. Stearns, "Routing linear permutations through the omega network in two passes," in *Proc. 1988 Symp. Frontiers of Massively Parallel Computat.*, 1988, pp. 479–482.

[18] P. C. Yew and D. H. Lawrie, "An easily controlled network for frequently used permutations," *IEEE Trans. Comput.*, vol. C-30, pp. 296–298, Apr. 1981.

[19] N. Das, "Permutation capabilities, routing and fault- tolerance in some multistage interconnection networks," Ph.D. thesis, Dept. of Comput. Sci. and Eng., Jadavpur Univ., Calcutta, India, 1992.