ON SOME TESTS OF SIGNIFICANCE IN SAMPLES FROM BI-POLAR NORMAL DISTRIBUTIONS

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1. Introduction and summary

A $(p+q)$-variate normal distribution is said to be bi-polar if the variates can be divided into two sets of p and q, such that within each set the variances are equal and the correlations are equal, and any variate of the first set has with any variate of the second set the same correlation, called the bi-polar correlation.

This model may be of use in educational problems and Votaw (1950) has considered an application in medicine.

Bi-polarity is a particular case of what Votaw (1948) calls "Compound Symmetry" of normal populations for which he derived the likelihood ratio test and its moments. The distribution of the likelihood criterion to test bi-polarity was obtained in a Gamma series form by Roy (1951).

In the present paper the joint distribution of the maximum likelihood estimates of the elements of the bi-polar covariance matrix has been obtained and a 't' test has been derived for the hypothesis that the bi-polar correlation is zero. The case when the means within each set are equal has been treated separately. The likelihood ratio test for bi-polarity has been found to be unbiased in the case $p=q=2$. A Gamma series expansion has been found for the distribution of the likelihood criterion to test equality of means within each set of variates.

2. Joint distribution of the maximum likelihood estimates of the elements of the bipolar covariance matrix

Let $x_{1i},\ldots,x_{pi}; x_{p+1,i},\ldots,x_{p+q,i}$ $(i = 1, 2, \ldots, n)$ be n independent random observations on the $p+q$ stochastic variables $X_1,\ldots, X_p; X_{p+1},\ldots, X_{p+q}$ distributed normally with means $m_1, m_2, \ldots, m_p; m_{p+1}, \ldots, m_{p+q}$ and covariance matrix

$$\Sigma \equiv \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2' & \Sigma_3 \end{bmatrix}$$

where Σ_1 is a $p\times p$ matrix with all diagonal elements σ_{aa} and other elements $\sigma_{aa'}$, Σ_2 is a $p\times q$ matrix with all elements σ_{ab} and Σ_3 is a $q\times q$ mratix with diagonal elements σ_{bb} and other elements $\sigma_{bb'}$.

The maximum likelihood estimates of the parameters are

$$\hat{m}_j = \bar{x}_j \qquad (j = 1, 2, \ldots, p, p+1, \ldots, p+q),$$

$$\hat{\sigma}_{aa} = \frac{1}{n} S_{aa}, \qquad \hat{\sigma}_{aa'} = \frac{1}{n} S_{aa'}, \qquad \hat{\sigma}_{bb} = \frac{1}{n} S_{bb},$$

$$\hat{\sigma}_{bb'} = \frac{1}{n} S_{bb'}, \qquad \hat{\sigma}_{ab} = \frac{1}{n} S_{ab},$$

$$\left.\begin{aligned}
&\text{where} \quad \bar{x}_j = \frac{1}{n}\sum_{i=1}^{n} x_{ji},\\
&S_{jj'} = \sum_{i=1}^{n}(x_{ji}-\bar{x}_j)(x_{j'i}-\bar{x}_{j'}), \quad j, j' = 1, 2, \ldots, p, p+1, \ldots, p+q,\\
&S_{aa} = \frac{1}{p}\sum_{j=1}^{p} S_{jj}, \qquad S_{aa'} = \frac{1}{\binom{p}{2}}\sum_{j>j'=1}^{p} S_{jj'}, \qquad S_{bb} = \frac{1}{q}\sum_{j=p+1}^{p+q} S_{jj},\\
&S_{bb'} = \frac{1}{\binom{q}{2}}\sum_{j>j'=p+1}^{p+q} S_{jj'}, \quad S_{ab} = \frac{1}{pq}\sum^{p}\sum_{j'=p+1}^{p+q} S_{jj'}.
\end{aligned}\right\} \quad \ldots \quad (2.1)$$

We want to find the joint distribution of S_{aa}, $S_{aa'}$, S_{bb}, $S_{bb'}$, S_{ab}. Let us make the orthogonal transformation

$$(Y_1, \ldots, Y_p \,;\, Y_{p+1}, \ldots, Y_{p+q}) = (X_1, \ldots, X_p \,;\, X_{p+1}, \ldots, X_{p+q}) \begin{bmatrix} A & O \\ O & B \end{bmatrix} \ldots \quad (2.2)$$

where A is a $p \times p$ matrix with first column $\left\{\frac{1}{\sqrt{p}}, \ldots, \frac{1}{\sqrt{p}}\right\}$ chosen so that $A'\Sigma_1 A$ is a diagonal matrix with the first diagonal element $\lambda_1 \equiv \sigma_{aa}+(p-1)\sigma_{aa'}$ and $(p-1)$ other diagonal elements $\lambda_2 \equiv \sigma_{aa}-\sigma_{aa'}$. This is possible since λ_2 is a latent root of Σ_1 of multiplicity $(p-1)$ and $\left\{\frac{1}{\sqrt{p}}, \ldots, \frac{1}{\sqrt{p}}\right\}$ is the latent vector corresponding to the latent root λ_1. B is defined similarly with respect to Σ_2 whose latent roots we shall denote by $\mu_1 \equiv \sigma_{bb}+(q-1)\sigma_{bb'}$ and $\mu_2 \equiv \sigma_{bb}-\sigma_{bb'}$. This will make $A'\Sigma_3 B$ a matrix all elements of which vanish except the leading one which is $\nu = \sqrt{pq}\,\sigma_{ab}$. All the Y's except Y_1 and Y_{p+1} are thus independent; the covariance matrix of Y_1 and Y_{p+1} is $\begin{bmatrix} \lambda_1 & \nu \\ \nu & \mu_1 \end{bmatrix}$; $Y_2, \ldots, Y_p$ have the same variance λ_2 and $Y_{p+1}, Y_{p+2}, \ldots, Y_{p+q}$ have the common variance μ_2. Let us write

$$\bar{y}_j = \frac{1}{n}\sum_{i=1}^{n} y_{ji} \text{ and } L_{jj'} = \sum_{i=1}^{n}(y_{ji}-\bar{y}_j)(y_{j'i}-\bar{y}_{j'}).$$

It is easy to see that

$$\begin{aligned}
L_{11} &= S_{aa}+(p-1)S_{aa'},\\
L_{1,p+1} &= \sqrt{pq}\, S_{ab},\\
L_{p+1,\, p+1} &= S_{bb}+(q-1)S_{bb'},\\
\sum_{j=2}^{p} L_{jj'} &= (p-1)(S_{aa}-S_{aa'}),\\
\sum_{j=p+2}^{p+q} L_{jj} &= (q-1)(S_{bb}-S_{bb'}).
\end{aligned}$$

and since the joint distribution of the $L_{jj'}$'s is

$$\text{const.}\,(L_{11}L_{p+1,\,p+1}-L^2_{1,\,p+1})^{\frac{n-4}{2}}\exp\left\{-\frac{1}{2\Delta}(\mu_1L_{11}-2\nu L_{1,\,p+1}+\lambda_1L_{p+1,\,p+1})\right\}\times$$

$$\times\, dL_{11}dL_{1,\,p+1}dL_{p+1,\,p+1}\times\prod_{j=2}^{p}\exp\left\{-\frac{1}{2}\frac{L_{jj}}{\lambda_2}\right\}(L_{jj})^{\frac{n-1}{2}-1}dL_{jj}\times$$

$$\times\prod_{j=p+2}^{p+q}\exp\left\{-\frac{1}{2}\frac{L_{jj}}{\mu_2}\right\}(L_{jj})^{\frac{n-1}{2}-1}dL_{jj}$$

where $\Delta=\lambda_1\mu_1-\nu^2$, it follows that the joint distribution of S_{aa}, $S_{aa'}$, S_{bb}, $S_{bb'}$, S_{ab} is

$$\text{const.}\;[\{S_{aa}+(p-1)S_{aa'}\}\{S_{bb}+(q-1)S_{bb'}\}-pqS_{ab}^2]^{\frac{n-4}{2}}\times$$

$$\times\exp\left\{-\frac{1}{2\Delta}\Big[\mu_1\{S_{aa}+(p-1)S_{aa'}\}-2\nu\sqrt{pq}S_{ab}+\lambda_1\{S_{bb}+(q-1)S_{bb'}\}\Big]\right\}\times$$

$$\times\exp\left\{-\frac{1}{2\lambda_2}(p-1)(S_{aa}-S_{aa'})\right\}(S_{aa}-S_{aa'})^{\frac{1}{2}(n-1)(p-1)-1}\times$$

$$\times\exp\left\{-\frac{1}{2\mu_2}(q-1)(S_{bb}-S_{bb'})\right\}(S_{bb}-S_{bb'})^{\frac{1}{2}(n-1)(q-1)-1}\times$$

$$\times\, dS_{aa}dS_{aa'}dS_{bb}dS_{bb'}dS_{ab}. \quad \ldots \quad (2.3)$$

3. Joint distribution of the bipolar and intra-class correlation coefficients

Let us define the bipolar correlation as $\rho=\frac{\sigma_{ab}}{\sqrt{\sigma_{aa}\sigma_{bb}}}$, $\rho_1=\frac{\sigma_{aa'}}{\sigma_{aa}}$ and $\rho_2=\frac{\sigma_{bb'}}{\sigma_{bb}}$ and their estimates by $r=\frac{S_{ab}}{\sqrt{S_{aa}S_{bb}}}$, $r_1=\frac{S_{aa'}}{S_{aa}}$ and $r_2=\frac{S_{bb'}}{S_{bb}}$ respectively.

The joint distribution of S_{aa}, S_{bb}, r, r_1 and r_2 is thus

$$\text{const.}\;[\{1+(p-1)r_1\}\{1+(q-1)r_2\}-pqr^2]^{\frac{n-4}{2}}(1-r_1)^{\frac{1}{2}(n-1)(p-1)-1}(1-r_2)^{\frac{1}{2}(n-1)(q-1)-1}$$

$$\times\exp\Big[-\frac{1}{2D}\Big(\frac{S_{aa}}{\sigma_{aa}}\{1+(q-1)\rho_2\}\{1+(p-1)r_1\}-2\sqrt{\frac{S_{aa}S_{bb}}{\sigma_{aa}\sigma_{bb}}}pq\rho r+$$

$$+\frac{S_{bb}}{\sigma_{bb}}\{1+(p-1)\rho_1\}\{1+(q-1)r_2\}\Big)\Big]\times$$

$$\times\exp\left[-\frac{1}{2}\left\{\frac{S_{aa}}{\sigma_{aa}}(p-1)\frac{1-r_1}{1-\rho_1}+\frac{S_{bb}}{\sigma_{bb}}(q-1).\frac{1-r_2}{1-\rho_2}\right\}\right]\times$$

$$\times\,(S_{aa})^{\frac{1}{2}(n-1)p-1}(S_{bb})^{\frac{1}{2}(n-1)q-1}dr\,dr_1\,dr_2\,dS_{aa}dS_{bb}$$

where

$$D=\{1+(p-1)\rho_1\}\{1+(q-1)\rho_2\}-pq\rho^2. \quad \ldots \quad (3.1)$$

Without loss of generality we may take $\sigma_{aa} = \sigma_{bb} = 1$ and write

$$\left.\begin{aligned} \frac{\{1+(q-1)\rho_2\}\{1+(p-1)r_1\}}{2D}+\tfrac{1}{2}(p-1)\frac{1-r_1}{1-\rho_1} &= C_{aa}, \\ \frac{\{1+(p-1)\rho_1\}\{1+(q-1)r_2\}}{2D}+\tfrac{1}{2}(q-1)\frac{1-r_2}{1-\rho_2} &= C_{bb}, \\ \frac{pq\,\rho r}{D} &= B, \\ \nu_1 = \tfrac{1}{2}(n-1)p\,; \quad \nu_2 &= \tfrac{1}{2}(n-1)q, \end{aligned}\right\} \qquad \ldots \quad (3.2)$$

Then expanding $e^{B\sqrt{S_{aa}S_{bb}}} = \sum_{t=0}^{\infty} \frac{B^t}{t!}(S_{aa})^{\frac{1}{2}t}(S_{bb})^{\frac{1}{2}t}$ and integrating over S_{aa} and S_{bb} we derive the joint distribution of r, r_1 and r_2 as

$$\text{const.}\,[\{1+(p-1)r_1\}\{1+(q-1)r_2\}-pqr^2]^{\frac{n-4}{2}}(1-r_1)^{\frac{1}{2}(n-1)(p-1)-1}(1-r_2)^{\frac{1}{2}(n-1)(q-1)-1}\times$$

$$\times \sum_{t=0}^{\infty} \frac{B^t\,\Gamma(\nu_1+\frac{1}{2}t)\Gamma(\nu_2+\frac{1}{2}t)}{t!(C_{aa})^{\nu_1+\frac{1}{2}t}(C_{bb})^{\nu_2+\frac{1}{2}t}}\,dr\,dr_1\,dr_2. \qquad \ldots \quad (3.3)$$

When $\rho=0$ the joint distribution of r, r_1, r_2 comes out as

$$\text{const.}\,[\{1+(p-1)r_1\}\{1+(q-1)r_2\}-pqr^2]^{\frac{n-4}{2}}(1-r_1)^{\frac{1}{2}(n-1)(p-1)-1}(1-r_2)^{\frac{1}{2}(n-1)(q-1)-1}\times$$

$$\times\left[\frac{1+(p-1)r_1}{1+(p-1)\rho_1}+(p-1)\frac{1-r_1}{1-\rho_1}\right]^{-\frac{1}{2}(n-1)p}\left[\frac{1+(q-1)r_2}{1+(q-1)\rho_2}+(q-1)\frac{1-r_2}{1-\rho_2}\right]^{-\frac{1}{2}(n-1)q}dr\,dr_1\,dr_2 \qquad \ldots \quad (3.4)$$

The conditional distribution of r for fixed r_1, r_2 is thus

$$\text{const.}\,[\{1+(p-1)r_1\}\{1+(q-1)r_2\}-pqr^2]^{\frac{n-4}{2}}\,dr \qquad \ldots \quad (3.5)$$

which does not involve any parameter.

It follows therefore that when $\rho=0$ the statistic

$$t = \frac{r}{\sqrt{\frac{\{1+(p-1)r_1\}\{1+(q-1)r_2\}}{pq}-r^2}}\sqrt{n-2} \qquad \ldots \quad (3.6)$$

is distributed as Student's t with $(n-2)$ degrees of freedom, and can be used to test the hypothesis that $\rho=0$. It will be shown in a later section that this statistic provides uniformly most powerful one sided test for the hypothesis that $\rho = 0$.

4. JOINT DISTRIBUTION OF MAXIMUM LIKELIHOOD ESTIMATES OF ELEMENTS OF BIPOLAR COVARIANCE MATRIX WHEN MEANS IN EACH SET ARE EQUAL

When $m_1 = m_2 = \ldots = m_p = m_a$ and $m_{p+1} = m_{p+2} = \ldots = m_{p+q} = m_b$ say, the maximum likelihood estimates of the parameters are

$$\hat{m}_a = \frac{1}{p}\sum_{j=1}^{p} \bar{x}_j, \qquad \hat{m}_b = \frac{1}{q}\sum_{j=p+1}^{p+q} \bar{x}_j,$$

$$\partial_{aa} = \frac{1}{n} S^*_{aa}, \qquad \partial_{aa'} = \frac{1}{n} S^*_{aa'},$$

$$\partial_{bb} = \frac{1}{n} S^*_{bb}, \qquad \partial_{bb'} = \frac{1}{n} S^*_{bb'}, \qquad \partial_{ab} = \frac{1}{n} S^*_{ab},$$

where

$$S^*_{aa} = \frac{1}{p}\sum_{j=1}^{p}\sum_{i=1}^{n}(x_{ji}-\hat{m}_a)^2,$$

$$S^*_{aa'} = \frac{1}{\binom{p}{2}}\sum_{j>j'=1}^{p}\sum_{i=1}^{n}(x_{ji}-\hat{m}_a)(x_{j'i}-\hat{m}_a),$$

$$S^*_{bb} = \frac{1}{q}\sum_{j=p+1}^{p+q}\sum_{i=1}^{n}(x_{ji}-\hat{m}_b)^2,$$

$$S^*_{bb'} = \frac{1}{\binom{q}{2}}\sum_{j>j'=p+1}^{p+q}\sum_{i=1}^{n}(x_{ji}-\hat{m}_b)(x_{j'i}-\hat{m}_b),$$

$$S^*_{ab} = \frac{1}{pq}\sum_{j'=p+1}^{p+q}\sum_{j=1}^{p}\sum_{i=1}^{n}(x_{ji}-\hat{m}_a)(x_{j'i}-\hat{m}_b).$$

To derive the joint distribution of S^*_{aa}, $S^*_{aa'}$, S^*_{bb}, $S^*_{bb'}$, S^*_{ab} we make as before the transformation (2.2) and note that

$$E(Y_j) = 0, \qquad j = 2, \ldots, p,\ p+2, \ldots, p+q,$$

$$S^*_{aa}+(p-1)S^*_{aa'} = L_{11},$$

$$S^*_{bb}+(q-1)S_{bb'} = L_{p+1,\,p+1},$$

$$\sqrt{pq}\, S^*_{ab} = L_{1,\,p+1},$$

$$(p-1)(S^*_{aa}-S^*_{aa'}) = \sum_{j=2}^{p}\sum_{i=1}^{n} y_{ji}^2,$$

$$(q-1)(S^*_{bb}-S^*_{bb'}) = \sum_{j=p+2}^{p+q}\sum_{i=1}^{n} y_{ji}^2.$$

It at once follows that the joint distribution of S^*_{aa}, $S^*_{aa'}$, S^*_{bb}, $S^*_{bb'}$, S^*_{ab} is of the form (2.3) with $\frac{1}{2}n(p-1)-1$ for the index of $(S^*_{aa}-S^*_{aa'})$ and $\frac{1}{2}n(q-1)-1$ for the index of $(S^*_{bb}-S^*_{bb'})$ and therefore the joint distribution of

$$r^* = \frac{S^*_{ab}}{\sqrt{S^*_{aa}S^*_{bb}}}, \quad r^*_1 = \frac{S^*_{aa'}}{S^*_{aa}}, \quad r^*_2 = \frac{S^*_{bb'}}{S^*_{bb}},$$

will be of the form (3.3) with $\frac{1}{2}n(p-1)-1$ as the index of $(1-r_1)$ and $\frac{1}{2}n(q-1)-1$ as the index of $(1-r_2)$ and ν_1 and ν_2 replaced by $\nu'_1 = \frac{1}{2}(np-1)$ and $\nu'_2 = \frac{1}{2}(nq-1)$ respectively.

5. Test of the hypothesis that the bipolar correlation is zero

To test the hypothesis that the bi-polar correlation $\rho=0$ the likelihood-ratio test comes out as

$$L = \left[\frac{1}{1+\frac{t^2}{n-1}}\right]^{\frac{n}{2}} < L_0$$

where t defined by (3.6) is distributed as Student's t with $(n-2)$ degrees of freedom. We shall now prove that this t provides the uniformly most powerful test for one sided alternatives. Let us consider the simple hypothesis $H_0 \equiv H_0\{\rho=0, \sigma^0_{aa}, \sigma^0_{aa'}, \sigma^0_{bb}, \sigma^0_{bb'}, \sigma^0_{ab}\}$ and a simple alternative $H \equiv H\{\rho \neq 0,\ \sigma_{aa},\ \sigma_{aa'},\ \sigma_{bb},\ \sigma_{bb'},\ \sigma_{ab}\}$. Denote by ϕ_0 and ϕ the joint probability density function under H_0 and H respectively. Then since ϕ satisfies the conditions of Ghosh (1948) it follows that all similar regions have the property of detailed similarity and consequently we can construct similar regions and from amongst them choose a most powerful one of size α which is given by (5.1)Λ(5.2) where Λ denotes intersection and (5.1) and (5.2) are defined as follows:

$$\left.\begin{aligned} \theta_{aa} &< S_{aa} < \theta_{aa}+d\theta_{aa}, \\ \theta_{aa'} &< S_{aa'} < \theta_{aa'}+d\theta_{aa'}, \\ \theta_{bb} &< S_{bb} < \theta_{bb}+d\theta_{bb}, \\ \theta_{bb'} &< S_{bb'} < \theta_{bb'}+d\theta_{bb'}, \\ \theta_{ab} &< S_{ab} < \theta_{ab}+d\theta_{ab}, \\ \theta_j &< \bar{x}_j < \theta_j+d\theta_j,\ j=1,2,\ldots,p+q, \end{aligned}\right\} \qquad \ldots \quad (5.1)$$

$$\phi \geqslant \lambda\phi_0 \qquad \ldots \quad (5.2)$$

where λ satisfies

$$\int\limits_{\text{over (5.1)}\Lambda\text{(5.2)}} \phi_0 dv = \alpha \int\limits_{\text{over (5.1)}} \phi_0 dv. \qquad \ldots \quad (5.3)$$

Obviously (5.1)Λ (5.2)$\equiv$(5.1)Λ (5.2′) where (5.2′) is given by

$$S_{ab} \geqslant \lambda' \text{ or } \leqslant \lambda'$$

according as

$$\rho > 0 \text{ or } \rho < 0 \qquad \ldots \quad (5.2')$$

and λ' or λ'' are to be so chosen as to satisfy the condition that the first kind of error is α. But we know from (2.3) that the conditional distribution of S_{ab} when $\rho=0$ is

$$\text{const.}\ [\{S_{aa}+(p-1)S_{aa'}\}\{S_{bb}+(q-1)S_{bb'}\}-pqS_{ab}^2]^{\frac{n-4}{2}}\,dS_{ab}$$

and this suggests that we replace (5.1)Λ (5.2′) by (5.1)Λ (5.2″) where (5.2″) is given by

$$t \equiv \frac{\sqrt{pq}\,S_{ab}\sqrt{(n-2)}}{\sqrt{[\{S_{aa}+(p-1)S_{aa'}\}\{S_{bb}+(q-1)S_{bb'}\}-pqS_{ab}^2]}} > t_\alpha \text{ or } \leqslant -t_\alpha$$

according as $\quad\rho > 0 \quad \text{or} \quad \rho < 0 \qquad \ldots \quad (5.2'')$

where t_α is the upper $100\alpha\%$ point of the t-distribution with $(n-2)$ degrees of freedom. This proves our assertion.

6. Likelihood-ratio test for equality of means within sets

To test the hypothesis that $m_1=m_2=\ldots=m_p$ and $m_{p+1}=\ldots=m_{p+q}$ the likelihood ratio statistic

$$L_1 = \frac{(S_{aa}-S_{aa'})^{p-1}}{(S^*_{aa}-S^*_{aa'})^{p-1}}\cdot\frac{(S_{bb}-S_{bb'})^{q-1}}{(S^*_{bb}-S^*_{bb'})^{q-1}}\cdot\frac{[\{S_{aa}+(p-1)S_{aa'}\}\{S_{bb}+(q-1)S_{bb'}\}-S_{ab}^2pq]}{[\{S^*_{aa}+(p-1)S^*_{aa'}\}\{S^*_{bb}+(q-1)S^*_{bb'}\}-pqS^{*2}_{ab}]}$$

was derived by Votaw (1948) who also proved that when the hypothesis is true

$$E(L_1^t)=\prod_{i=0}^{p-2}\frac{\Gamma\left(\frac{n-1}{2}+\frac{i}{p-1}+t\right)}{\Gamma\left(\frac{n-1}{2}+\frac{i}{p-1}\right)}\;\frac{\Gamma\left(\frac{n}{2}+\frac{i}{p-1}\right)}{\Gamma\left(\frac{n}{2}+\frac{i}{p-1}+t\right)}\prod_{j=0}^{q-2}\frac{\Gamma\left(\frac{n-1}{2}+\frac{j}{q-1}+t\right)}{\Gamma\left(\frac{n-1}{2}+\frac{j}{q-1}\right)}\;\frac{\Gamma\left(\frac{n}{2}+\frac{j}{q-1}\right)}{\Gamma\left(\frac{n}{2}+\frac{j}{q-1}+t\right)}.$$

Now, it is known, that (Roy, 1951) for any statistic l, $0 < l < 1$ for which the tth moment is of the form

$$E(l^t) = \prod\frac{\Gamma(\nu+c_i)}{\Gamma(\nu+b_i)}\cdot\frac{\Gamma(\nu+b_i+t)}{\Gamma(\nu+c_i+t)} \qquad c_i > b_i$$

where ν is of the order of the sample size, the probability density function of $V = -m\log_e l$ can be expanded in an asymptotic series of the form

$$\frac{1}{\Gamma(r)}e^{-V}.\,V^{r-1}+\frac{d_2}{m^2}\left\{\frac{1}{\Gamma(r+2)}e^{-V}V^{r+1}-\frac{1}{\Gamma(r)}e^{-V}V^{r-1}\right\}+\ldots \qquad \ldots \quad (6.1)$$

with $\quad r = \Sigma(c_i-b_i), \quad r_t = \Sigma(c_i^t-b_i^t), \qquad t = 2, 3, \ldots .$

$$\lambda = \frac{r-r_2}{2r}, \qquad m = \nu-\lambda,$$

$$d_2 = \frac{1}{6}\left\{3\lambda^2r+3\lambda r_2+r_3-r\right\},\ldots \text{ etc.}$$

The probability density function of $V = -m \log_e L_1$ can therefore be expanded in the form (6.1) with

$$\left.\begin{aligned} r &= \tfrac{1}{2}(p+q-2), \\ \lambda &= \frac{p+q+2}{4(p+q-2)}, \\ m &= \frac{n}{2}-\lambda, \\ d_2 &= \frac{1}{192}(p+q-2)-\frac{1}{4(p+q-2)}+\frac{1}{24}\left(\frac{1}{p-1}+\frac{1}{q-1}\right). \end{aligned}\right\} \quad \ldots \quad (6.2)$$

7. UNBIASEDNESS OF THE TEST FOR BIPOLARITY WHEN $p = q = 2$

Let X_1, X_2, X_3, X_4 be distributed normally with means m_j and covariance matrix $(\sigma_{jj'})$, $j, j' = 1, 2, 3, 4$. The hypothesis H' of bipolarity is that $\sigma_{11} = \sigma_{22}$ $\sigma_{33} = \sigma_{44}$, $\sigma_{13} = \sigma_{14} = \sigma_{23} = \sigma_{24}$. Make the transformation

$$Y_1 = \frac{X_1+X_2}{\sqrt{2}},\; Y_2 = \frac{X_3+X_4}{\sqrt{2}},\; Y_3 = \frac{X_1-X_2}{\sqrt{2}},\; Y_4 = \frac{X_3-X_4}{\sqrt{2}}$$

and denote the means by m_j' and the covariance matrix by $(\sigma'_{jj'}))j, j' = 1, 2, 3, 4$. Then the hypothesis may be restated as H': $\sigma'_{13} = \sigma'_{14} = \sigma'_{23} = \sigma'_{24} = \sigma'_{34} = 0$ or in words that the four new variates can be divided into the three independent groups (Y_1, Y_2), Y_3 and Y_4. The likelihood ratio statistic for testing bipolarity of the X's is the same as Wilks' (1935) statistic for testing which has been shown to be unbiased by Narain (1950) for all values of m_j' and $\sigma'_{11}, \sigma_{22}, \sigma'_{12}, \sigma'_{33}$ σ'_{44}.

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