14 August 1995

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 204 (1995) 99–101

Noncommuting spin-1/2 observables and the CHSH inequality

G. Kar [1]

Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B.T. Road, Calcutta 700 035, India

Received 13 April 1995; revised manuscript received 5 June 1995; accepted for publication 20 June 1995
Communicated by P.R. Holland

Abstract

It is shown that a maximally entangled state of two spin-1/2 particles not only gives maximal violation of the CHSH inequality but also gives the largest violation attainable for any pairs of four spin observables that are noncommuting for both systems. Any entangled state implies a violation but it need not be an eigen-state of the relevant Bell operator.

Recently it has been shown that for any entangled state of two quantum systems it is possible to find pairs of observables whose correlations violate the Bell/CHSH inequality [1]. In the special case of two particles of spin J in a singlet state, the maximal violation of the Bell/CHSH inequality allowed by Cirel'son's theorem [2], occurs provided $2J+1$ is even [3,4]. Braunstein et al. [5] showed that for any Bell/CHSH inequality based on noncommuting observables for both systems it is always possible to construct a state which will yield a violation, though not necessarily maximal.

Local realism constrains the statistics of two or more physically separated systems which can be expressed for two systems in terms of the expectation value of some Hermitian operator (the Bell operator, B_{CHSH}) [5] by

$$-2 \leqslant \langle B_{\mathrm{CHSH}} \rangle \leqslant 2. \quad (1)$$

Quantum theory predicts a violation of this inequality if for some state the expectation value exceeds the bound. In operator language the largest violation will be given by the largest eigen-value of this Bell operator. In other words, the states which can produce this largest violation will be eigen-states with this largest eigen-value. The eigen-states which produce violation cannot be product states. In general, they are entangled states of degenerate eigen-vectors of the squared Bell operator B^2_{CHSH} corresponding to the largest eigen-value [5].

In this Letter we shall determine the form of all eigen-states of the Bell operator giving the largest violation attainable for any four spin-1/2 observables that are noncommuting for both systems. They are maximally entangled states (the singlet state is one example) with a relative phase between the two orthogonal vectors in the four-dimensional tensor product Hilbert space.

To show this we consider four spin observables σ_1, σ_2, σ_3 and σ_4 which represent spin measurement along the directions n_1, n_2, n_3 and n_4, respectively. σ_1 and σ_2 act on one particle and σ_3 and σ_4 on the other. The Bell operator for these observables can be written as

$$B_{\mathrm{CHSH}} = \sigma_1\sigma_3 + \sigma_1\sigma_4 + \sigma_2\sigma_3 - \sigma_2\sigma_4. \quad (2)$$

[1] E-mail: sisir@isical.ernet.in.

SSDI 0375-9601(95)00473-4

The square of the Bell operator is given by [6]

$$B^2_{CHSH} = 4(I - [\sigma_1, \sigma_2][\sigma_3, \sigma_4]). \quad (3)$$

Now for any σ_i and σ_j, $[\sigma_i, \sigma_j]$ is given by

$$[\sigma_i, \sigma_j] = -2\mathrm{i} \sin\theta_{ij}\, \sigma_{n_{ij}}, \quad (4)$$

where θ_{ij} is the angle between the unit vectors n_i and n_j and $\sigma_{n_{ij}}$ represents the spin observable corresponding to a spin measurement along the unit vector n_{ij} perpendicular to the plane containing n_i and n_j.

Then (3) can be written as

$$B^2_{CHSH} = 4(I + \sin\theta_{12}\ \sin\theta_{34}\ \sigma_{n_{12}}\sigma_{n_{34}}). \quad (5)$$

The largest eigen-value (λ) of B^2_{CHSH} is given by

$$\lambda = 4(1 + |\sin\theta_{12}\ \sin\theta_{34}|). \quad (6)$$

The corresponding degenerate eigen-vectors are either $\psi_{n_{12}}\psi_{n_{34}}$ and $\psi_{-n_{12}}\psi_{-n_{34}}$ for $\sin\theta_{12}$ and $\sin\theta_{34}$ having the same sign or $\psi_{n_{12}}\psi_{-n_{34}}$ and $\psi_{-n_{12}}\psi_{n_{34}}$ for $\sin\theta_{12}$ and $\sin\theta_{34}$ of opposite sign. Here $\psi_{\pm n_{ij}}$ are eigen-vectors of $\sigma_{\pm n_{ij}}$.

Corresponding to the eigen-value (6) the largest eigen-value (in terms of the absolute value) of B_{CHSH} is

$$\mu = 2(1 + |\sin\theta_{12}\ \sin\theta_{34}|)^{1/2}. \quad (7)$$

This eigen-value corresponds to the largest violation of the Bell/CHSH inequality for the observables concerned. It is obvious from (7) that Cirel'son's bound ($2\sqrt{2}$) is obtained when both θ_{12} and θ_{34} are $\pi/2$.

For simplicity and without loss of generality we assume that the vectors n_i ($i = 1,2,3,4$) lie on the x-y plane and their corresponding azimuthal angles are ϕ_i. We also assume that $\sin(\phi_1 - \phi_2)$ and $\sin(\phi_3 - \phi_4)$ are of the same sign. In that case, the eigen-state of B_{CHSH} giving the largest violation for the observables concerned will be a superposition of eigen-vectors $\psi_z\,\psi_z$ and $\psi_{-z}\,\psi_{-z}$.

Let the eigen-state be

$$\psi = c_1\psi_z\psi_z + c_2\,\mathrm{e}^{\mathrm{i}\phi}\,\psi_{-z}\psi_{-z}, \quad (8)$$

where c_1 and c_2 are real and $c_1^2 + c_2^2 = 1$. With this ψ, the expectation value of B_{CHSH} is given by

$$\langle\psi|B_{CHSH}\psi\rangle = 2c_1c_2[\cos(\phi - \phi_1 - \phi_3) + \cos(\phi - \phi_1 - \phi_4) + \cos(\phi - \phi_2 - \phi_3) - \cos(\phi - \phi_2 - \phi_4)]. \quad (9)$$

Now it is obvious from (9) that the largest eigen-value $2[1 + |\sin(\phi_1 - \phi_2)\sin(\phi_3 - \phi_4)|]^{1/2}$ of B_{CHSH} will be achieved by the above expectation for those ψ for which $|c_1| = |c_2| = 1/\sqrt{2}$ and some ϕ depending on the ϕ_i as c_1c_2 has been factorised out in expression (9).

So we conclude that the entangled states which give the largest possible violation for any four arbitrary spin-1/2 observables that are noncommuting for both systems, must be maximally entangled states.

Let us make this clear from two examples. From the result of Gisin [1] follows that for a normalized state vector ψ of the form

$$\psi_1 = c_1\psi_z\psi_{-z} + c_2\psi_{-z}\psi_z, \quad (10)$$

with $c_1, c_2 \geqslant 0, c_1 \neq c_2$, there exist four spin observables for which the largest possible violation of the Bell/CHSH inequality occurs and this highest violation depends on c_1 and c_2 only [7]. We shall show that although this is the highest possible violation for the particular ψ_1, it is not the maximum possible violation for the four spin observables chosen. So this ψ_1 cannot be an eigen-vector of B_{CHSH} formed by the relevant observables.

If we take the choice of the unit vectors $(a_i)_y = 0$, $(a_i)_x = \sin\theta_i$, $(a_i)_z = \cos\theta_i$ for $i = 1,2,3,4$, where $\theta_1 = \pi/2$, $\theta_2 = 0$,

$$\cos\theta_4 = -\cos\theta_3 = (1 + 4c_1^2c_2^2)^{-1/2},$$

then

$$\langle\psi_1|B_{CHSH}\psi_1\rangle = 2(1 + 4c_1^2c_2^2)^{1/2}. \quad (11)$$

Following (7) the largest eigen-value of B_{CHSH} is given by

$$2[1 + |\sin(\theta_3 - \theta_4)|]^{1/2} = \frac{2(1 + 2c_1c_2)}{(1 + 4c_1^2c_2^2)^{1/2}}. \quad (12)$$

The difference between this largest eigen-value and $\langle\psi_1|B_{CHSH}\psi_1\rangle$ is given by

$$\frac{2(1+2c_1c_2)}{(1+4c_1^2c_2^2)^{1/2}} - 2(1+4c_1^2c_2^2)^{1/2} = \frac{4c_1c_2(1-2c_1c_2)}{(1+4c_1^2c_2^2)^{1/2}}, \qquad (13)$$

which is a positive quantity as $2c_1c_2 < 1$. The largest violation (12) for this choice of spin observables will be given by some maximally entangled state, i.e., $c_1 = c_2 = 1/\sqrt{2}$ and some ϕ depending on θ_3 and θ_4.

The other example is Bell's original example [8]. Here the state is chosen as the singlet state and the σ_i are chosen in a plane (say the x–y plane) with two of them measuring spin in coincident directions. Let $n_1 = n_4$. With these constraints the largest violation (5/2) will be achieved when

$$\phi_1 - \phi_3 = \phi_2 - \phi_3 = \pi/3 = (\phi_1 - \phi_2)/2.$$

Again following (7) the largest eigen-value of B_{CHSH} for the above choice of observables is

$$2[1 + sin^2(\pi/3)]^{1/2} = \sqrt{7} \ \ (> 5/2). \qquad (14)$$

So the singlet state though the maximally entangled state is not the eigen-state of B_{CHSH} for the above choice of observables.

Let us find the exact maximally entangled state whose corresponding eigen-value gives the largest violation for this choice of observables. Let the state be

$$\psi_0 = \frac{1}{\sqrt{2}}(\psi_z\psi_{-z} + e^{i\phi}\,\psi_{-z}\psi_z). \qquad (15)$$

Then

$$\begin{aligned}\langle\psi_0|B_{\mathrm{CHSH}}\psi_0\rangle &= [\cos\phi + 2\cos(\phi - \pi/3) + \cos(\phi + \pi/3)] \\ &= \tfrac{1}{2}(5\cos\phi + \sqrt{3}\sin\phi) = f(\phi) \ \ \text{(say)}. \qquad (16)\end{aligned}$$

It can be easily shown that the maximal value for $f(\phi) = \sqrt{7}$ and this occurs at $\phi = \tan^{-1}(\sqrt{3}/5)$. So the eigen-state giving the largest violation ($\sqrt{7}$) is given by the maximally entangled state of the form given by Eq. (15) with the above value of ϕ.

References

[1] N. Gisin, Phys. Lett. A 154 (1991) 201.
[2] B.S. Cirel'son, Lett. Math. Phys. 4 (1980) 93.
[3] N. Gisin and A. Peres, Phys. Lett. A 162 (1992) 15.
[4] S. Popescu and D. Rohrlich, Phys. Lett. A 169 (1992) 411.
[5] S.I. Braunstein, A. Mann and M. Revzen, Phys. Rev. Lett 68 (1992) 3259.
[6] L.J. Landau, Phys. Lett. A 120 (1987) 4.
[7] A. Mann, B.C. Sanders and W.J. Munro, Phys. Rev. A 51 (1995) 989.
[8] J.S. Bell, Physics (NY) 1 (1964) 195.