ELSEVIER

Ecological Modelling 83 (1995) 295–309

Transmission of Japanese Encephalitis in a 3-population model

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Received 5 January 1994; accepted 15 June 1994

Abstract

The model considered for the spread of Japanese Encephalitis (JE) in a human population of varying size from a reservoir population (pigs, cattle, equines, birds, etc.) through a vector population (particular species of mosquitos) is of SIRS (susceptible–infective–recovered–susceptible) type for the human and reservoir populations and SIS (susceptible–infective–susceptible) type for the vector population. We have considered the logistic differential equation with density-dependent birth rate for the vector population whereas the reservoir population is of constant size. We assume that the human population is regulated by the disease. We also assume that there is a constant recruitment rate of susceptibles into the human population. We perform an equilibrium and stability analysis to find a threshold condition. If the threshold is exceeded, then there is a unique equilibrium with disease present which is locally stable to small perturbations and global stability depends on death rates and the ratio of the equilibrium population sizes of the infected vector and total human populations.

Keywords: Diseases, human; Encephalitis; Multispecies ecosystems

1. Introduction

Japanese Encephalitis (JE) is a vector-borne viral disease which is transmitted from vertebrate reservoir population (pigs, cattle, equines, birds etc.) to susceptible human populations through a particular species of mosquito (culicine species namely, *Culex vishnui*). Man is a dead end of infection, transmission of infection from man to man is impossible. JE is a growing and alarming public health problem in India. In West Bengal of India since 1973, JE has been almost an annual event in the form of an epidemic or a small outbreak engulfing newer and newer rural areas. Every time, the district of Burdwan has been the most affected area in West Bengal, probably forming a hyper-endemic zone. No study so far has been made on the dynamics of JE spread in the endemic area as far as the authors are aware. A regression equation model using a third-order Harmonic Fourier Series having a linear trend has been used by some of the present group (Mukhopadhyay et al., 1993) to simulate the pattern of monthly occurrences of JE.

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SSDI 0304-3800(94)00102-N

The aim of the present study is to investigate the dynamics of this dreaded disease in a 3-population system consisting of vector, reservoir and human populations. Our study is based on the consideration that the vector and human populations are varying in size whereas the size of the reservoir population is fixed and closed. This is because the size of the vector population (mosquito) rapidly changes its size and the human population is also subject to frequent death due to fatality of the disease. On the other hand, the reservoir population which exhibits no disease symptoms are not usually subject to any fatality due to the disease and hence is assumed to be constant in size when the birth rate and death rate are equal (see Mena-Lorca and Hethcote, 1992) and there are no fluctuations of the size due to other reasons such as emigration, immigration etc.

We use similar techniques to Hethcote (1976), Anderson and May (1979), Liu et al. (1986,1987), Castillo-Chavez et al. (1989), Busenberg and Van den Driessche (1990), Greenhalgh (1990), Hethcote and Van den Driessche (1991), Gao and Hethcote (1992), and Mena-Lorca and Hethcote (1992) in constructing as well as analysing the mathematical model. For the global analysis of the model we have also used the methods of Tuljapurkar and Semura (1975), Redheffer (1985), and Rinaldi (1990). In section 2 we have posed the mathematical model of the dynamics of JE. The criteria for the existence of zero (disease-free) and non-zero (disease-present) equilibria have been laid down in section 3. Local and global stability properties of the model system have been investigated in sections 4 and 5 respectively. Finally, the results have been discussed in section 6.

2. Mathematical model

In deriving the model equations, we have first assumed a logistic growth rate for the vector population as considered in a SIRS (susceptible–infective–recovered–susceptible) model by Gao and Hethcote (1992). There is no immune class in the vector population since it acts as a transmitter of virus only. Since the vector population is subject to rapid change, the size is assumed to be varying. There are immune classes in both human and reservoir populations as both get infections through the blood system and consequently immunity develops in these populations, as shown by their immunity titre level. Since there is no fatality in the reservoir population due to the disease, this population has been assumed to be constant by taking the birth and death rates equal. Following Castillo-Chavez et al. (1989), it is assumed that there is a constant rate of recruitment of new susceptibles in the human population and that the number of deaths is proportional to the size in each class. Because of the considerable degree of fatality in man due to the disease, the human population must be varying in size and not constant as for the reservoir population. Infection is not spread by direct contact between reservoir–reservoir, reservoir–man or man–man. It is carried over by mosquitos which play the role of a "transmitter" from infected reservoir population to both susceptible reservoir and human populations. Let $V_1(t)$ and $V_2(t)$ denote the susceptible and infected individuals in vector population; $X_1(t)$, $X_2(t)$ and $X_3(t)$ denote the susceptible, infected and recovered (immune) individuals respectively in human population; $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ denote the susceptible, infected and recovered (immune) individuals respectively in reservoir population. We suggest that the dynamics of the transmission of JE then obey the following equations:

V = Vector

$$\frac{dV_1}{dt} = (\alpha - rV/K)V - \alpha' V_1 - \eta Y_2 V_1/V + \xi V_2 \tag{2.1}$$

$$\frac{dV_2}{dt} = \eta Y_2 V_1/V - (\alpha' + \xi)V_2 \tag{2.2}$$

$$\frac{dV}{dt} = r(1 - V/K)V \tag{2.3}$$

Y = Reservoir

$$\frac{dY_1}{dt} = \mu_1 N_1 - \mu_1 Y_1 - \beta_1 V_2 Y_1 / N_1 + f_1 Y_3 \tag{2.4}$$

$$\frac{dY_2}{dt} = \beta_1 V_2 Y_1 / N_1 - (\gamma_1 + \mu_1) Y_2 \tag{2.5}$$

$$\frac{dY_3}{dt} = \gamma_1 Y_2 - (f_1 + \mu_1) Y_3 \tag{2.6}$$

X, N_2 = Human

$$\frac{dX_1}{dt} = \mu_2 - \mu'_2 X_1 - \beta_2 V_2 X_1 / N_2 + f_2 X_3 \tag{2.7}$$

$$\frac{dX_2}{dt} = \beta_2 V_2 X_1 / N_2 - (\gamma_2 + \varepsilon + \mu'_2) X_2 \tag{2.8}$$

$$\frac{dX_3}{dt} = \gamma_2 X_2 - (f_2 + \delta + \mu'_2) X_3 \tag{2.9}$$

$$\frac{dN_2}{dt} = \mu_2 - \mu'_2 N_2 - \varepsilon X_2 - \delta X_3 \tag{2.10}$$

where

$$X_1(t) + X_2(t) + X_3(t) = N_2(t) \tag{2.11}$$

$$V_1(t) + V_2(t) = V(t) \tag{2.12}$$

$$Y_1(t) + Y_2(t) + Y_3(t) = N_1(\text{constant}) \tag{2.13}$$

and the positive rate constants are denoted as

For vector population (V)

α	= per capita natural birth rate,
α'	= per capita natural death rate,
r	= $\alpha - \alpha'$ = per capita growth rate,
K	= carrying capacity of environment,
ξ	= per capita loss of infectivity of infected vectors,
η	= effective per capita contact rate of infective reservoirs with vector population,

For reservoir population (Y)

μ_1	= per capita natural birth (or death) rate,
γ_1	= per capita recovery rate of infected reservoirs,
f_1	= per capita loss of immunity rate of recovered reservoirs,
β_1	= effective per capita contact rate of infective vectors with reservoir population,

For human population (X,N_2)

μ_2	= the recruitment rate into the susceptible class,
μ'_2	= per capita natural death rate,
ε	= excess per capita death rate of infected individuals,
δ	= excess per capita death rate of recovered individuals,
γ_2	= per capita recovery rate of infected individuals,
f_2	= per capita loss of immunity rate of recovered individuals,
β_2	= effective per capita contact rate of infective vectors with human population.

The parameter η is the average number of adequate contacts per unit time of an infective in the reservoir population with individuals in the vector population and $\beta_1(\beta_2)$ is the average number of adequate contacts per unit time of an infective in the vector population with individuals in the reservoir (human) population.

The model is well-posed in the sense that solutions that are initially non-negative remain so. We consider the feasible region $\Omega = \{(V_1, V_2, Y_2, Y_3, X_1, X_2, X_3) | X_i \geq 0, Y_i \geq 0, V_j \geq 0, i = 1,2,3; j = 1,2; \Sigma V_j(t) = V(t) \leq K, \Sigma Y_i(t) = N_1 \ (= \text{constant}), \Sigma X_i(t) = N_2(t) \leq \mu_2/\mu'_2\}$ and show that it is positively invariant for all $t \geq 0$. Considering separately the boundary points of Ω where $X_i = 0$, $Y_i = 0$ and $V_j = 0$, we see that no solution can exit the feasible region across the boundary. When $V_2 = 0$ and $Y_2 = 0$, then from Eqs. 2.2 and 2.5, $dV_2/dt = 0$, $dY_2/dt = 0$ and from Eqs. 2.6 and 2.8, $dY_3/dt < 0$ if $Y_3 > 0$ and $dX_2/dt < 0$ if $X_2 > 0$ and hence $V_1 = V(t)$, $V_2 = 0$; $Y_2 = 0$, $Y_3 = 0$; $X_1 = N_2(t)$, $X_2 = 0$, $X_3 = 0$ (where $Y_1 = N_1$) be the disease-free solutions in Ω. At $t \to \infty$, these solutions approach the fixed point $(K, 0, 0, 0, \mu_2/\mu'_2, 0, 0)$ in Ω if the initial size of the vector population is positive. Again, within the interior of Ω, the right hand sides of Eqs. 2.1–2.10 are continuously differentiable, guarantying uniqueness of solutions and hence no solution can cross the boundary points of Ω.

3. Threshold and three equilibria

The population size of vector (mosquito) approaches an equilibrium size K (carrying capacity) if $r > 0$ and initial population size, say, $V_0 > 0$. If $r = 0$, then population size $V(t)$ remains at V_0 and if $r < 0$, then solutions $V(t)$ with $V_0 > K$ grow to infinity and solutions with $V_0 < K$ decrease to zero. Because of the rapidly growing behaviour of the vector population and the density-dependent restricted population growth, it is less common to consider $r \leq 0$. The disease will spread in the vector–reservoir system if initially (i.e., at $t = 0$), $dV_2/dt > 0$ and $dY_2/dt > 0$. Let $V_1 \approx V$ and $Y_1 \approx N_1$ at $t = 0$. Hence from Eqs. 2.2 and 2.5, we have $dV_2/dt|_{t=0} > 0$ which implies that $Y_2 > (d'/\eta)V_2$ and $dY_2/dt|_{t=0} > 0$ which implies that $Y_2 < (\beta_1/d)V_2$, where $d' = \alpha' + \xi$ and $d = \gamma_1 + \mu_1$. These two inequalities give $R_0 = R_1R_2 > 1$, where $R_1 = \beta_1/d'$ and $R_2 = \eta/d$.

The parameter $\beta_1(\eta)$ is the average number of adequate contacts per unit time of an infective in the vector (reservoir) population with individuals in the reservoir (vector) population. Since $1/d'$ $(1/d)$ is the death-adjusted average period of infectivity for an individual in the vector (reservoir) population, $R_1 = \beta_1/d'$ $(R_2 = \eta/d)$, which is called the infectious contact number, is the average number of the reservoir (vector) population (both susceptibles and others) contacted by an infective vector (reservoir) individual during its infectious period. Therefore, R_0 is the product of the two infectious contact numbers R_1 and R_2. We shall show in the following theorem that unity is the threshold value of R_0.

Using Eqs. 2.11–2.13, the system 2.1–2.10 can be reduced to

$$\frac{dV_1}{dt} = (\alpha - rV/K)V - \alpha' V_1 - \eta Y_2 V_1/V + \xi V_2 \tag{3.1}$$

$$\frac{dV_2}{dt} = \eta Y_2 V_1/V - d' V_2 \tag{3.2}$$

$$\frac{dY_2}{dt} = \beta_1 V_2 (N_1 - Y_2 - Y_3)/N_1 - dY_2 \tag{3.3}$$

$$\frac{dY_3}{dt} = \gamma_1 Y_2 - fY_3 \tag{3.4}$$

$$\frac{dX_1}{dt} = \mu_2 - \mu'_2 X_1 - \beta_2 V_2 X_1 / N_2 + f_2 X_3 \tag{3.5}$$

$$\frac{dX_2}{dt} = \beta_2 V_2 X_1 / N_2 - b_1 X_2 \tag{3.6}$$

$$\frac{dX_3}{dt} = \gamma_2 X_2 - b_2 X_3 \tag{3.7}$$

where,

$$f = f_1 + \mu_1, \quad b_1 = \gamma_2 + \varepsilon + \mu'_2, \quad b_2 = f_2 + \delta + \mu'_2 \tag{3.8}$$

Theorem 3.1. *There are three possible equilibria for the system* 3.1–3.7.

(i) $R_0 \leq 1$:

There are two equilibria, say, $P_0 = (0, 0, 0, 0, \mu_2/\mu'_2, 0, 0)$ *and* $P'_0 = (K, 0, 0, 0, \mu_2/\mu'_2, 0, 0)$, *which represent disease-free equilibria.*

(ii) $R_0 > 1$:

In addition to the two equilibria defined in (i), *there is a third possible unique equilibrium with disease present, say,* $P_1 = (V_1^*, V_2^*, Y_2^*, Y_3^*, X_1^*, X_2^*, X_3^*)$, *and here the equilibrium value of human population, say,* N_2^*, *satisfies the equation*

$$\psi b_3 N_2^* - [\psi(b_2 + \gamma_2) + b_1 b_2 N_2^*] F(N_2^*) = 0,$$

where

$$\psi = \beta_2 V_2^*, \quad b_3 = \delta\gamma_2 + \varepsilon b_2, \quad F(N_2^*) = \mu_2 - \mu'_2 N_2^*,$$

and $V_2^* = \psi/\beta_2$ *satisfies the following equation*

$$R_0 V_2^* (d' K\zeta + N_1\eta) = K N_1 \eta (R_0 - 1), \quad \textit{where } \zeta = 1 + \gamma_1/f.$$

Proof. We obtain the equilibrium solutions by setting the time derivatives on the right hand sides of Eqs. 3.1–3.7 to zero. We denote the equilibrium values of V_1, V_2, V, Y_2, Y_3, X_1, X_2, X_3, and N_2 by $\hat{V}_1$, $\hat{V}_2$, $\hat{V}$, $\hat{Y}_2$, $\hat{Y}_3$, $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$, and $\hat{N}_2$ (where $\hat{Y}_1 = N_1 - \hat{Y}_2 - \hat{Y}_3$) respectively. From Eq. 2.3, we obtain

$$\frac{dV}{dt} = r(1 - V/K)V.$$

Then $d\hat{V}/dt = 0$ implies either $\hat{V} = 0$ or $\hat{V} = K$.

Case (a): Let $\hat{V} = 0$. Then, it is trivial to show that $\hat{V}_1 = \hat{V}_2 = 0$; $\hat{Y}_2 = \hat{Y}_3 = 0$; $\hat{X}_2 = \hat{X}_3 = 0$, $\hat{X}_1 = \hat{N}_2 = \mu_2/\mu'_2$ (where $\hat{Y}_1 = N_1$).

Case (b): Let $\hat{V} = K$.

From Eq. 3.1,

$$\hat{V}_1 = \left[K\left(K\alpha' + \xi\hat{V}_2\right)\right] / \left[K\alpha' + \eta\hat{Y}_2\right]; \tag{3.9}$$

From Eq. 3.2,

$$\hat{V}_2 = \left[\eta\hat{Y}_2\hat{V}_1\right] / (Kd') \tag{3.10}$$

From Eqs. 3.9 and 3.10,

$$\hat{Y}_2 = d'K\hat{V}_2/\left[\eta\left(K-\hat{V}_2\right)\right] \tag{3.11}$$

From Eq. 3.4,

$$\hat{Y}_3 = (\gamma_1/f)\hat{Y}_2; \tag{3.12}$$

From Eqs. 3.3 and 3.12,

$$\beta_1\hat{V}_2\left(1-\zeta\hat{Y}_2/N_1\right) = d\hat{Y}_2 \tag{3.13}$$

From Eqs. 3.11 and 3.13, we obtain

$$\hat{V}_2\left[KN_1\eta(R_0-1) - R_0\hat{V}_2(d'K\zeta + N_1\eta)\right] = 0 \tag{3.14}$$

Eq. 3.14 implies that either $\hat{V}_2 = 0$ or

$$\hat{V}_2 = \frac{KN_1\eta(R_0-1)}{R_0(d'K\zeta + N_1\eta)}.$$

Since $\hat{V}_2 \geq 0$, the latter value of $\hat{V}_2$ is possible if $R_0 \geq 1$. $\hat{V}_2 = 0$ implies

$$\hat{V}_1 = K, \quad \hat{V}_2 = 0; \quad \hat{Y}_2 = \hat{Y}_3 = 0; \quad \hat{X}_1 = \hat{N}_2 = \mu_2/\mu', \quad \hat{X}_2 = \hat{X}_3 = 0 \quad \left(\text{where } \hat{Y}_1 = N_1\right).$$

From Eq. 3.6,

$$\beta_2\hat{V}_2\hat{X}_1/\hat{N}_2 = b_1\hat{X}_2 \tag{3.15}$$

From Eq. 3.7,

$$\hat{X}_3 = (\gamma_2/b_2)\hat{X}_2, \tag{3.16}$$

From Eqs. 3.5 and 3.15–3.16, we have

$$\hat{X}_1 = \left[\mu_2 b_2 - (b_1b_2 - f_2\gamma_2)\hat{X}_2\right]/(\mu'_2 b_2) \tag{3.17}$$

Using Eqs. 3.16 and 3.17 in Eq. 2.11, we obtain

$$\hat{X}_2 = b_2F\left(\hat{N}_2\right)/b_3 \tag{3.18}$$

From Eqs. 3.15 and 3.18, we have

$$\hat{X}_1 = b_1b_2\hat{N}_2F\left(\hat{N}_2\right)/\left(\beta_2\hat{V}_2b_3\right) \tag{3.19}$$

Again, putting the values of $\hat{X}_1$, $\hat{X}_2$ and $\hat{X}_3$ from Eqs. 3.16, 3.18 and 3.19 in Eq. 2.11, we obtain

$$\beta_2\hat{V}_2b_3\hat{N}_2 - \left\{\beta_2\hat{V}_2(b_2+\gamma_2) + b_1b_2\hat{N}_2\right\}F\left(\hat{N}_2\right) = 0 \tag{3.20}$$

Since $F(\hat{N}_2) = \mu_2 - \mu'_2\hat{N}_2$, Eq. 3.20 is a quadratic equation of $\hat{N}_2$. Since L.H.S. of Eq. 3.20 takes values $\{-\beta_2\hat{V}_2\mu_2(\gamma_2+b_2)\}$ and $\{\beta_2\hat{V}_2b_3\mu_2/\mu'_2\}$ at $\hat{N}_2 = 0$ and $\hat{N}_2 = \mu_2/\mu'_2$ respectively, Eq. 3.20 has a unique positive root, say, $\hat{N}_2 = N_2^*$ when $\hat{V}_2 = V_2^* > 0$. Therefore, we get the unique non-zero equilibrium P_1 (i.e., disease-present) from Eqs. 3.9, 3.11, 3.12, 3.14, 3.16 and 3.18–3.20 when $R_0 > 1$. This completes the proof.

Before going for stability analysis of the system we shall first state some definitions and theorems relating to the stability of matrices (see Berman and Hershkowitz, 1983; Rinaldi, 1990):

$\mathscr{A} = \{A$; there exists a positive diagonal matrix W such that $WA + A^TW$ is positive-definite$\}$ – the diagonally (positive) stable matrices,

$\mathscr{V}$ = {A; there exists a positive diagonal matrix W such that $WA + A^TW$ is negative-definite} – Volterra–Lyapunov (negative) stable matrices,

$\mathscr{P}$ = {A; all the signed principal minors of A, i.e., the quantities

$$M_{i_1,\ldots,i_k} = (-1)^k \det A_{i_1,\ldots,i_k}$$

for any subset $1 \leq i_1 < \cdots < i_k \leq n$ of the integers {1,2, $\cdots$,n}, are positive}

Theorem 3.2. *Let A be a $n \times n$ matrix and A is reducible to a matrix of the form*

$$\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

where the blocks A_1 and A_3 are square. Then $A \in \mathscr{A}$ if $A_1 \in \mathscr{A}$ and $A_3 \in \mathscr{A}$.

Theorem 3.3 (Lyapunov, 1892). *Let A be a $n \times n$ real matrix. Then all the eigenvalues of a matrix A have negative (positive) real parts if and only if there exists a symmetric positive-definite matrix H such that*

$$HA + A^TH$$

is negative (positive) definite. Such a matrix A is said to be negative (positive) stable.

4. Local stability

4.1. Local stability of zero-equilibria

It is trivial to prove that one of the two zero (i.e., disease-free) equilibria, $P_0 = (0, 0, 0, 0, \mu_2/\mu'_2, 0, 0,)$, which always exists, is saddle. We shall now show that another disease-free equilibrium $P'_0 = (K, 0, 0, 0, \mu_2/\mu'_2, 0, 0)$, which also always exists, is locally asymptotically stable (LAS) if $R_0 < 1$ and unstable if $R_0 > 1$:

Using Eqs. 2.11–2.13, we obtain the following Jacobian matrix for the system 3.1–3.7 at P'_0

$$J_0 = \begin{bmatrix} J_{10} & 0 \\ J_{20} & J_{30} \end{bmatrix},$$

where

$$J_{30} = \begin{bmatrix} -\mu'_2 & 0 & f_2 \\ 0 & -b_1 & 0 \\ 0 & \gamma_2 & -b_2 \end{bmatrix},$$

$$J_{20} = \begin{bmatrix} 0 & -\beta_2 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$J_{10} = \begin{bmatrix} -r & d'-r & -\eta & 0 \\ 0 & -d' & \eta & 0 \\ 0 & \beta_1 & -d & 0 \\ 0 & 0 & \gamma_1 & -f \end{bmatrix}$$

We see that all the eigenvalues of the matrix J_{30} are negative whereas the characteristic equation of the matrix J_{10} is

$$(\lambda + r)(\lambda + f)\left[\lambda^2 + (d + d')\lambda + dd'(1 - R_0)\right] = 0$$

Since $r > 0$, all the eigenvalues of the matrix J_{10} are negative real parts if and only if $R_0 < 1$. Applying Theorems 3.2 and 3.3, we can prove that $J_0 \in \mathscr{V}$ if $R_0 < 1$ and $J_0 \notin \mathscr{V}$ if $R_0 > 1$ i.e., the zero (i.e., disease-free) equilibrium which is P_0' is LAS if $R_0 < 1$ and unstable if $R_0 > 1$.

4.2. Local stability of the non-zero equilibrium

Using Eqs. 2.11–2.13, we can obtain the following Jacobian matrix for the system 3.1–3.7 at the non-zero equilibrium point $P_1 = (V_1^*, V_2^*, Y_2^*, Y_3^*, X_1^*, X_2^*, X_3^*)$ as

$$J = \begin{bmatrix} J_1 & 0 \\ J_2 & J_3 \end{bmatrix},$$

where

$$J_3 = \begin{bmatrix} -(\psi' + \mu_2' N_2^*)/N_2^* & b_1 X_2^*/N_2^* & (b_1 X_2^* + f_2 N_2^*)/N_2^* \\ \psi'/N_2^* & -b_1(X_2^* + N_2^*)/N_2^* & -b_1 X_2^*/N_2^* \\ 0 & \gamma_2 & -b_2 \end{bmatrix},$$

(by using Eq. 3.15)

$$J_2 = \begin{bmatrix} 0 & -\beta_2 X_1^*/N_2^* & 0 & 0 \\ 0 & \beta_2 X_1^*/N_2^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} -(r + \eta a_2 a_2') & (d' + \eta a_1 a_2') - r & -\eta a_1 & 0 \\ \eta a_2 a_2' & -(d' + \eta a_1 a_2') & \eta a_1 & 0 \\ 0 & \beta_1(1 - \zeta a_3 a_2') & -(d + \beta_1 a_2 a_3) & -\beta_1 a_2 a_3 \\ 0 & 0 & \gamma_1 & -f \end{bmatrix}$$

(by using Eq. 3.12), where

$$a_i = V_i^*/K, \quad a_j' = Y_j^*/K \quad (i = 1,2; j = 1,\ldots 3,), \quad a_3 = K/N_1 \quad \text{and} \quad \psi' = \psi - b_1 X_2^* \tag{4.21}$$

Then the characteristic equation of the matrix J_3 is

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0,$$

where

$$p_1 = \left[\psi + (b_1 + b_2 + \mu_2')N_2^*\right]/N_2^*,$$

$$p_2 = \left[\psi'(b_1 + b_2) + (b_2 + \gamma_2 + \mu_2')b_1 X_2^* + \{b_1 b_2 + \mu_2'(b_1 + b_2)\}N_2^*\right]/N_2^*,$$

$$p_3 = \left[\{b_3 + \mu_2'(b_2 + \gamma_2)\}\psi' + \mu_2' b_1\{b_2 N_2^* + (b_2 + \gamma_2)X_2^*\}\right]/N_2^*.$$

Let $D_1 = p_1$, $D_2 = p_1 p_2 - p_3$. We shall now show that $D_2 > 0$.

Now,

$$D_2 = \left[q_1 N_2^{*2} + q_2 N_2^* + q_3\right]/N_2^{*2},$$

where,

$$q_1 = (b_1 + b_2)\left[b_1 b_2 + \mu_2'(b_1 + b_2 + \mu_2')\right],$$

$$q_2 = \left[b_1 b_2 + \mu_2'(b_1 + b_2)\right]\psi + \left[b_1^2 + f_2\gamma_2 + (b_1 + b_2)(b_2 + \mu_2')\right]\psi' + \left[\mu_2'^2 + (b_1 + b_2)(b_2 + \gamma_2 + \mu_2')\right]b_1 X_2^*,$$

and

$$q_3 = \left[(b_1 + b_2)\psi' + (b_2 + \gamma_2 + \mu_2')b_1 X_2^*\right]\psi.$$

In the case of the matrix J_1, its characteristic polynomial is $\det(J_1 - \lambda I)$. After elementary row and column operation of the matrix $(J_1 - \lambda I)$ and finally expanding $\det(J_1 - \lambda I)$, we obtain the following characteristic equation:

$$(\lambda + r)\left[\lambda^3 + p_1'\lambda^2 + p_2'\lambda + p_3'\right] = 0,$$

where

$$p_1' = d + d' + f + \beta_1 a_2 a_3 + \eta a_2',$$

$$p_2' = f(d + d') + \eta a_2'(d + f) + \beta_1 a_2 a_3(\gamma_1 + f) + \beta_1 a_2 a_3(d' + \eta a_2'),$$

$$p_3' = df\eta a_2' + \beta_1 a_2 a_3(d' + \eta a_2')(\gamma_1 + f)$$

(using $R_0 a_1(1 - \zeta a_3 a_2') = 1$, which we obtain from Eqs. 3.10, 3.13 and 4.21).

Let $D_1' = p_1'$ and $D_2' = p_1' p_2' - p_3'$.

Now,

$$D_2' = (d + f + \beta_1 a_2 a_3)\left[f(d + d') + \eta a_2'(d + f) + \beta_1 a_2 a_3(\gamma_1 + f) + \beta_1 a_2 a_3(d' + \eta a_2')\right] + dd'f + (d' + \eta a_2')\left[d'f + \eta a_2'(d + f) + \beta_1 a_2 a_3(d' + \eta a_2')\right]$$

Since r, p_i, q_i and p_i' $(i = 1,2,3)$ are greater than zero, D_i and D_i' are greater than zero. Therefore, by applying the Routh–Hurwitz's Theorem (see Wylie and Barrett, 1989), we can say that all the roots of the characteristic equations of the matrices J_1 and J_3 have negative real parts. Again, by applying Theorems 3.2 and 3.3, we see that $J \in \mathscr{V}$. Hence, the non-zero equilibrium P_1 (when it exists, i.e., $R_0 > 1$) is LAS.

5. Global stability of the equilibria

First, we shall state two theorems and then we shall study the global stability property of our system. The inequality $WA > 0$ means that $WA + A^T W$ is positive-definite, where W is a positive diagonal matrix and A^T is the transpose of the matrix A.

Theorem 5.1 (Cross, 1978). *Let A be a 2×2 matrix. Then $A \in \mathscr{V} \Leftrightarrow A \in \mathscr{P}$.*

Theorem 5.2 (Redheffer, 1985). *Let A be a non singular $n \times n$ matrix, where $n \geq 2$, with inverse $A^{-1} = B$ and W a positive diagonal $n \times n$ matrix. Let A^*, B^*, W^* denote the $(n-1) \times (n-1)$ matrices obtained from A, B, W, respectively, deleting the last row and column. Then:*

(i) *if $WA > 0$, we must have $a_{nn} > 0$, $W^*A^* > 0$, and $W^*B^* > 0$;*

(ii) *if $a_{nn} > 0$, $W^*A^* > 0$, and $W^*B^* > 0$, it is possible to choose $W_i > 0$ in such a way that $WA > 0$.*

We shall now prove that the zero-equilibrium, i.e., disease-free equilibrium, which is P_0', is globally asymptotically stable (GAS) if $R_0 \leq 1$. Consider the function

$$L_1(t) = \frac{V_2(t)}{d} + \frac{Y_2(t)}{d'}.$$

Then its derivative is given by

$$\frac{dL_1(t)}{dt} \leq -\left[\frac{d'(1-(d/d')R_1)}{d}V_2 + \frac{d(1-(d'/d)R_2)}{d'}Y_2 + (R_1/N_1 + R_2/V)Y_2V_2\right].$$

Since L_1 is positive-definite and the above inequality shows that dL_1/dt must be negative-definite if $R_1 \leq d'/d$ and $R_2 \leq d/d'$ (which implies $R_0 \leq 1$), then we can say that L_1 is the Lyapunov function when $R_0 \leq 1$. Therefore, $L_1(t) \to 0$ as $t \to +\infty$ when $R_0 \leq 1$. Then, $V_2(t) \to 0$, $Y_2(t) \to 0$ as $t \to +\infty$ and hence from Eqs. 2.4 and 2.6, we see that $Y_3(t) \to 0$ and $Y_1(t) \to N_1$ as $t \to +\infty$. Again from Eq. 2.3, $V(t) \to K$ as $t \to \infty$ (when initial population size of the vector is greater than zero). Therefore, the solutions $(V_1(t), V_2(t), Y_2(t), Y_3(t))$ for the subsystem 3.1–3.4 approach the fixed point $(K, 0, 0, 0)$ in 4-dimensional space as $t \to +\infty$ when $R_0 \leq 1$. Hence, it is obvious to show that all the solutions $(V_1(t), V_2(t), Y_2(t), Y_3(t), X_1(t), X_2(t), X_3(t))$ of the system 3.1–3.7 approach the fixed point $(K, 0, 0, 0, \mu_2/\mu_2', 0, 0)$ which belongs to Ω as $t \to +\infty$ when $R_0 \leq 1$. Therefore, the zero equilibrium (P_0') is GAS when $R_0 = R_1R_2 \leq 1$, whereas another zero equilibrium P_0 is always unstable saddle.

Let us assume that $V = V^*(=K)$ (see Discussion) and make some technical assumptions that $Y(t) = Y_1(t) + Y_2(t)$ and $X(t) = X_1(t) + X_2(t)$. Then, using Eqs. 2.11–2.13, the system 2.1–2.10 can be reduced to

$$\begin{aligned}
\frac{dX}{dt} &= \mu_2 - (f_2 + \mu_2')X - (\gamma_2 + \varepsilon)X_2 + f_2N_2,\\
\frac{dX_2}{dt} &= \beta_2V_2(X - X_2)/N_2 - (\gamma_2 + \varepsilon + \mu_2')X_2,\\
\frac{dN_2}{dt} &= \mu_2 + \delta X - \varepsilon X_2 - (\delta + \mu_2')N_2,\\
\frac{dV_2}{dt} &= \eta'(K - V_2)Y_2 - d'V_2,\\
\frac{dY}{dt} &= f(N_1 - Y) - \gamma_1Y_2,\\
\frac{dY_2}{dt} &= \beta_1'(Y - Y_2)V_2 - dY_2,
\end{aligned} \tag{5.1}$$

where $X_3 = N_2 - X$, $Y_3 = N_1 - Y$, $\eta' = \eta/K$ and $\beta_1' = \beta_1/N_1$. Let

$$\mathscr{D}_0 = \mathscr{D} - \{(\mu_2/\mu_2', 0, \mu_2/\mu_2', 0, N_1, 0)\},$$

where

$$\begin{aligned}
\mathscr{D} = \{&(X, X_2, N_2, V_2, Y, Y_2): X \geq 0, Y \geq 0, X_i \geq 0, Y_i \geq 0\,(i = 2,3), V_2 \geq 0, V_1(t) + V_2(t)\\
&= K, Y(t) + Y_3(t) = N_1, X(t) + X_3(t) = N_2(t) \leq \mu_2/\mu_2'\}
\end{aligned}$$

We now introduce variables u_i $(i = 1,2,\ldots,6)$ in the system 5.1 through the substitutions

$$X = X^*e^{u_1}, \quad X_2 = X_2^*e^{u_2}, \quad N_2 = N_2^*e^{u_3},$$
$$V_2 = V_2^*e^{u_4}, \quad Y = Y^*e^{u_5}, \quad Y_2 = Y_2^*e^{u_6}.$$

The variables u_i obey the following equations:

$$\frac{du_1}{dt} = (X^* e^{u_1})^{-1}\left[-X^*(f_2+\mu_2')(e^{u_1}-1) - X_2^*(\gamma_2+\varepsilon)(e^{u_2}-1) + f_2 N_2^*(e^{u_3}-1)\right],$$

$$\frac{du_2}{dt} = \psi(N_2^* X_2^* e^{u_2+u_3})^{-1}\left[X^*(e^{u_1}-1) - \{X_2^* + (X^*-X_2^*)e^{u_3}\}(e^{u_2}-1) - (X^*-X_2^*)(e^{u_3}-1) + (X^* e^{u_1} - X_2^* e^{u_2})(e^{u_4}-1)\right],$$

$$\frac{du_3}{dt} = (N_2^* e^{u_3})^{-1}\left[\delta X^*(e^{u_1}-1) - \varepsilon X_2^*(e^{u_2}-1) - N_2^*(\delta+\mu_2')(e^{u_3}-1)\right], \tag{5.2}$$

$$\frac{du_4}{dt} = \eta' Y_2^*(V_2^* e^{u_4})^{-1}\left[-K(e^{u_4}-1) + (K - V_2^* e^{u_4})(e^{u_6}-1)\right],$$

$$\frac{du_5}{dt} = -(Y^* e^{u_5})^{-1}\left[fY^*(e^{u_5}-1) + \gamma_1 Y_2^*(e^{u_6}-1)\right],$$

$$\frac{du_6}{dt} = e^{-u_6}\left[d(e^{u_4}-1) + e^{u_4}(d+\beta_1' V_2^*)(e^{u_5}-1) - (d+\beta_1' V_2^* e^{u_4})(e^{u_6}-1)\right].$$

Theorem 5.3. *Let $V = V^* = K$. Let $\mathscr{D}_0'$ is the set $\mathscr{D}_0$ in the new coordinates u_i. Then the origin, $u_i = 0$ $(i = 1, 2, \ldots, 6)$ which belongs to $\mathscr{D}_0'$, for the system 5.2 is globally asymptotically stable (GAS) if*

(i) $\varepsilon \le \delta + \mu_2'$

or

(ii) $\dfrac{V_2^*}{N_2^*} \ge \dfrac{b_1 b_2[\varepsilon - (\delta+\mu_2')]}{\beta_2(\delta+\mu_2')(\gamma_2+b_2)}$ *when* $\varepsilon > \delta + \mu_2'$.

Proof. Let u be the vector of u_i. Consider the function

$$L(u(t)) = W_1 X^*(e^{u_1}-1)^2 + W_2 N_2^* X_2^*(e^{u_2}-1)^2 + W_3 N_2^*(e^{u_3}-1)^2 + W_4 V_2^*(\eta' Y_2^*)^{-1}(e^{u_4}-1)^2 + W_5 Y^*(e^{u_5}-1)^2 + W_6(e^{u_6}-1)^2,$$

where $W_i > 0$ constants.

Observe that $L(u(t))$ is a positive-definite function on the set $\mathscr{D}_0'$. Its derivative is given by

$$\begin{aligned}\frac{dL(u(t))}{dt} &= 2W_1(e^{u_1}-1)\left[-X^*(f_2+\mu_2')(e^{u_1}-1) - X_2^*(\gamma_2+\varepsilon)(e^{u_2}-1) + f_2 N_2^*(e^{u_3}-1)\right]\\
&+ 2W_2\psi e^{-u_3}(e^{u_2}-1)\left[X^*(e^{u_1}-1) - \{X_2^* + (X^*-X_2^*)e^{u_3}\}(e^{u_2}-1) - (X^*-X_2^*)(e^{u_3}-1) + (X^* e^{u_1} - X_2^* e^{u_2})(e^{u_4}-1)\right]\\
&+ 2W_3(e^{u_3}-1)\left[\delta X^*(e^{u_1}-1) - \varepsilon X_2^*(e^{u_2}-1) - N_2^*(\delta+\mu_2')(e^{u_3}-1)\right]\\
&+ 2W_4(e^{u_4}-1)\left[-K(e^{u_4}-1) + (K - V_2^* e^{u_4})(e^{u_6}-1)\right]\\
&- 2W_5(e^{u_5}-1)\left[fY^*(e^{u_5}-1) + \gamma_1 Y_2^*(e^{u_6}-1)\right]\\
&+ 2W_6(e^{u_6}-1)\left[d(e^{u_4}-1) + e^{u_4}(d+\beta_1' V_2^*)(e^{u_5}-1) - (d+\beta_1' V_2^* e^{u_4})(e^{u_6}-1)\right]\end{aligned} \tag{5.3}$$

For simplicity, let us consider $U_i(t) = e^{u_i(t)} - 1$. So, if $u_i(t) = 0$, then $U_i(t) = 0$. Let $dL(u(t))/dt = G(U(t))$, where U is the vector of U_i. Now, if $G(U(t))$ is a negative-definite function, then $L(u(t))$ would

be a Lyapunov function for the system and would ensure global stability about the origin. The Eq. 5.3 can be written in the following matrix form:

$$G(U) = U^T(WA + A^TW)U,$$

where $U = (U_1, U_2, \cdots, U_6)^T$, $W = \text{diag}(W_1, W_2, \cdots, W_6)$, and

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

where A_1 and A_3 are square block matrices and

$$A_1 = \begin{bmatrix} -X^*(f_2 + \mu_2') & -X_2^*(\gamma_2 + \varepsilon) & f_2 N_2^* \\ \dfrac{\psi X^*}{U_3 + 1} & -\dfrac{\psi\{X_2^* + (X^* - X_2^*)(U_3 + 1)\}}{U_3 + 1} & -\dfrac{\psi(X^* - X_2^*)}{U_3 + 1} \\ \delta X^* & -\varepsilon X_2^* & -N_2^*(\delta + \mu_2') \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{\psi\{(X^* - X_2^*) + (X^* U_1 - X_2^* U_2)\}}{U_3 + 1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} -K & 0 & K - V_2^*(U_4 + 1) \\ 0 & -fY^* & -\gamma_1 Y_2^* \\ d & (d + \beta_1 V_2^*)(U_4 + 1) & -\{d + \beta_1' V_2^*(U_4 + 1)\} \end{bmatrix}$$

We shall now show that $P = -A_1 \in \mathscr{A}$ and $Q = -A_3 \in \mathscr{A}$.

Case (a): $P = -A_1$. In this case, we construct the inverse of the matrix P,

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} P_{11} & P_{21} & P_{31} \\ P_{12} & P_{22} & P_{32} \\ P_{13} & P_{23} & P_{33} \end{bmatrix},$$

where P_{ij} are the co-factors of the elements p_{ij} in the determinant of the matrix P and

$$\begin{aligned} \det(p) &= \psi X^* [\mu_2' b_2 N_2^* (X^* - X_2^*)(U_3 + 1) + X_2^* \{\mu_2' N_2^* (\gamma_2 + b_2) + b_3 (X_2^* + X_3^*)\}]/(U_3 + 1) \\ &= \psi X^* [\mu_2' b_2 N_2^* (X^* - X_2^*)(U_3 + 1) + X_2^* (\gamma_2 + b_2)(F(N_2^*) + \mu_2' N_2^*)]/(U_3 + 1), \end{aligned}$$

(by Eqs. 3.16 and 3.18)

$$\begin{aligned} P_{11} &= \psi[N_2^*(\delta + \mu_2')(X^* - X_2^*)(U_3 + 1) + X_2^* \{N_2^*(\delta + \mu_2') - \varepsilon(X^* - X_2^*)\}]/(U_3 + 1) \\ &= \psi[N_2^*(\delta + \mu_2')(X^* - X_2^*)(U_3 + 1) + X_2^* \{N_2^*(\delta + \mu_2') - \varepsilon b_1 b_2 N_2^* F(N_2^*)/(\psi b_3)\}]/(U_3 + 1) \end{aligned}$$

(by Eq. 3.19)

$$\begin{aligned} &= \psi[N_2^*(\delta + \mu_2')(X^* - X_2^*)(U_3 + 1) + F_1 X_2^* \{\psi(\delta + \mu_2')(\gamma_2 + b_2) - b_1 b_2 (\varepsilon - (\delta + \mu_2'))N_2^*\}] \\ &\quad /(U_3 + 1) \end{aligned}$$

(by Eq. 3.20)

$$P_{12} = \psi X^* [N_2^* (\delta + \mu_2') - \delta (X^* - X_2^*)] / (U_3 + 1)$$
$$= \psi X^* [\mu_2' b_3 N_2^* + \delta F(N_2^*)(\gamma_2 + b_2)] / (b_3 (U_3 + 1)),$$

(by Eqs. 3.19 and 3.20)

$$P_{13} = \psi X^* [\delta (X^* - X_2^*)(U_3 + 1) + (\delta - \varepsilon) X_2^*] / (U_3 + 1),$$
$$P_{21} = -N_2^* X_2^* (b_3 + \gamma_2 \mu_2'),$$
$$P_{22} = \mu_2' b_2 X^* N_2^*,$$
$$P_{23} = -b_3 X^* X_2^*,$$
$$P_{31} = \psi [f_2 N_2^* (X^* - X_2^*)(U_3 + 1) + X_2^* \{f_2 N_2^* + (\gamma_2 + \varepsilon)(X^* - X_2^*)\}] / (U_3 + 1),$$
$$P_{32} = -\psi X^* [(f_2 + \mu_2')(X^* - X_2^*) - f_2 N_2^*] / (U_3 + 1),$$
$$P_{33} = \psi X^* [(f_2 + \mu_2')(X^* - X_2^*)(U_3 + 1) + X_2^* (b_1 + f_2)] / (U_3 + 1),$$

and

$$F_1 = F_1(N_2^*) = N_2^* / [\psi(\gamma_2 + b_2) + b_1 b_2 N_2^*].$$

Let P^* and $[(P^{-1})]^*$ be the matrices obtained from P and P^{-1} respectively, deleting the last row and column. By Theorem 5.1, we can easily show that P^* and $[(P)^{-1}]^*$ satisfy the hypotheses of Theorem 5.2 if $P_{11} > 0$ i.e., if either $\varepsilon \leq \delta + \mu_2'$ or,

$$\frac{V_2^*}{N_2^*} \geq \frac{b_1 b_2 [\varepsilon - (\delta + \mu_2')]}{\beta_2 (\delta + \mu_2')(\gamma_2 + b_2)}$$

when $\varepsilon > \delta + \mu_2'$ (these are the sufficient conditions for $P_{11} > 0$). Hence by Theorem 5.2, we have $P \in \mathscr{A}$.

Case (b): $Q = -A_3$. Now, we construct the inverse of the matrix Q,

$$Q^{-1} = \frac{1}{\det(Q)} \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix},$$

where Q_{ij} are similarly defined as P_{ij} and

$$\det(Q) = (U_4 + 1)[d(fY^* V_2^* + K\gamma_1 Y_2^*) + K\beta_1' V_2^* (fY^* + \gamma_1 Y_2^*)],$$
$$Q_{11} = dfY^* + (U_4 + 1)[d\gamma_1 Y_2^* + \beta_1' V_2^* (fY^* + \gamma_1 Y_2^*)],$$
$$Q_{12} = -d\gamma_1 Y_2^*, \quad Q_{13} = dfY^*,$$
$$Q_{21} = -(d + \beta_1' V_2^*)(U_4 + 1)[V_2^* (U_4 + 1) - K],$$
$$Q_{22} = V_2^* (d + K\beta_1')(U_4 + 1),$$
$$Q_{23} = K(d + \beta_1' V_2^*)(U_4 + 1),$$
$$Q_{31} = -fY^* [V_2^* (U_4 + 1) - K],$$
$$Q_{32} = -K\gamma_1 Y_2^*, \quad Q_{33} = KfY^*.$$

In a similar way as in case (a), we can show that $Q \in \mathscr{A}$ without any parametric condition. By applying Theorem 3.2, it is easy to prove that $A \in \mathscr{Y}$ if either $\varepsilon \leq \delta + \mu'_2$ or,

$$\frac{V_2^*}{N_2^*} \geq \frac{b_1 b_2[\varepsilon - (\delta + \mu'_2)]}{\beta_2(\delta + \mu'_2)(\gamma_2 + b_2)}$$

when $\varepsilon > \delta + \mu'_2$.

Therefore, L is negative definite and hence is a Lyapunov function for the system 5.2, ensuring global asymptotic stability of the origin if either of the conditions (i) or (ii) is satisfied. This completes the proof of the theorem.

Hence, by Theorem 5.3, we can say that if $V = V^* = K$ (carrying capacity) and one of the two conditions in Theorem 5.3 is satisfied, then $(X^*, X_2^*, N_2^*, V_2^*, Y^*, Y_2^*)$ is GAS for the system 5.1.

6. Discussion

In this paper we have proposed a mathematical model of the transmission of JE and discussed its dynamics. The spread of JE involves three populations: vector (mosquito), reservoir (pigs, cattle, equines, birds etc.) and man. The peculiarity of the transmission of the disease is that man is the dead end of the infection and transmission of infection does not take place from man to man, man to the reservoir, or man to the vector. Infection is spread from infected reservoir population to susceptible human and reservoir populations through a particular species of mosquito (*Culex vishnui* sp.). Hence modeling transmission of such types of diseases poses some difficulty as these simultaneously involve three populations in an open chain system making it a multidimensional one. Moreover, the rapid change of size of the vector population as well as fatality of the disease makes the population size variable at the vector and human levels so that the analytical study becomes more complex. When $R_0 \leq 1$, i.e., the product of the two infectious contact numbers (R_1 and R_2) is less than or equal to unity, there are the two zero equilibria. The zero equilibrium $P_0 = (0, 0, 0, 0, \mu_2/\mu'_2, 0, 0)$ is always saddle. Biologically this means, this disease-free state with no vector population can never be attained. Another zero equilibrium $P'_0 = (K, 0, 0, 0, \mu_2/\mu'_2, 0, 0)$ is GAS if $R_0 = R_1 R_2 \leq 1$, where $R_1 = \beta_1/d'$ ($R_2 = \eta/d$) is the average number of the reservoir (vector) population (both susceptibles and others) contacted by an infective vector (reservoir) individual during its infectious period. The latter is also a disease-free equilibrium at which the three populations consist of susceptible individuals only. Both these equilibrium points denote disease-free states.

When $R_0 > 1$, there exists three equilibria: two disease-free as given above and one non-zero equilibrium $(V_1^*, V_2^*, Y_2^*, Y_3^*, X_1^*, X_2^*, X_3^*)$ which is the endemic (disease-present) equilibrium. In the case of the disease-free equilibria, one which is saddle, P_0, and the other one, P'_0, is unstable whereas the endemic equilibrium is LAS. Because of the rapidly growing behaviour of the vector population compared to other two populations, we have assumed $V(t) = V^* = K$ (carrying capacity of environment) using the pseudo steady-state hypothesis to study the global stability of the non-zero (i.e., disease-present) equilibrium of our system. This endemic equilibrium is GAS for $\varepsilon \leq \delta + \mu'_2$. On the other hand, if $\varepsilon > \delta + \mu'_2$, the condition for the global stability is that the ratio of the equilibrium population sizes of the infected vector and total human populations exceeds or is equal to the quantity

$$\frac{b_1 b_2[\varepsilon - (\delta + \mu'_2)]}{\beta_2(\delta + \mu'_2)(\gamma_2 + b_2)}.$$

The former condition implies that when the excess per capita death rate due to disease is less than or equal to the total of the per capita natural death rate and excess per capita death rate of recovered individual human, the endemic equilibrium is globally asymptotically stable. The latter condition implies that when the former condition is violated, the global stability of endemic equilibrium depends upon the ratio of the equilibrium population sizes of the infected vector and total human populations. This is quite realistic from biological standpoints. Obviously, if the disease is not fatal, i.e., $\varepsilon = 0$, the former condition of Theorem 5.3 is always satisfied and hence the endemic equilibrium is always globally asymptotically stable. In that case the disease always persists. When $\varepsilon \to \infty$, that is to say, the fatality is very severe, the former condition $\varepsilon \leq \delta + \mu'_2$ is violated and the latter condition of Theorem 5.3 can no more be satisfied except $N_2 \to 0$, and hence we conjecture that the endemic equilibrium is unstable and the only stable equilibrium will be the disease-free equilibrium at which the three populations consist of susceptible individuals only. In that case the disease will not persist. The size of the endemicity (in human population) is directly proportional to $\psi = \beta_2 V_2^*$, that is, the product of the effective contact rate in man and the equilibrium size of the infected vector population.

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