MEASURABLE SETS IN PRODUCT SPACES AND THEIR PARAMETRIZATIONS

By

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CONTENTS

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INTRODUCTION AND SUMMARY

In recent years, several authors (Wesley [45], Bourgain [5], Cenzer and Mauldin [9], Ioffe [13], Mauldin [23] and Srivastava and Sarbadhikari [31]) have established a number of results on the measurable parametrizations of Borel sets in the product of two Polish spaces. These results are not only interesting in their own right but also find applications in the integration of correspondences, mathematical economics etc. The prototype of these parametrization results is the following result of Lusin : If the vertical sections of a Borel set are all countably infinite, then it can be written as a countable union of disjoint Borel graphs. Recent work in the area has taken off from this point and concentrated for the most part on parametrizing Borel sets whose vertical sections are uncountable. Here, of course, the results have to be formulated with some care, for a classical result states that such Borel sets need not even contain a Borel graph. Roughly speaking, the results assert that Borel sets with uncountable vertical sections can be parametrized by a function of two variables, which is a Borel isomorphism in one variable and jointly measurable with respect to some σ-field on the product. Beginning with Ioffe [47], people have also looked for parametrizing functions that have pleasant topological properties in one variable, instead of being Borel isomorphisms.

In many cases, however, while the sets being parametrized belong to product σ-fields, the parametrizing function obtained is measurable only with respect to large σ-fields on the product that cannot be related to reasonable σ-fields generated by rectangles. In this thesis we look at both kinds of parametrizations and show that in very general situations parametrizations can be obtained that are measurable with respect to natural product σ-fields, which as is well-known are of a much simpler descriptive character.

As the above suggests we are in a sense concerned with product structure. In this spirit we next consider the question of " approximating " complex sets in the product by sets in product σ-fields. In this thesis we do this for C-sets in the product of two Polish spaces. We show that given any such set one can find a set in the σ-field generated by rectangles with one side a C-set and the other a Borel set which approximates it (in measure or category) uniformly over all sections. Such a formulation unifies and simplifies many results about C-sets, as then many questions about them reduce to ones about sets in product σ-fields.

A feature about this thesis worth mentioning is that even in establishing boldface results (and most of this thesis concerns itself with such results) we have had to use results from the effective theory in what appears to us to be an essential manner.

The organisation of the thesis and the main results are as follows. We fix the basic definitions and notation in Section 0.

In Section 1, we prove the following abstract version of Lusin's theorem : Let X be Polish, and $(T, \underline{\underline{M}})$ a measurable space, and suppose B in $\underline{\underline{M}} \otimes \underline{\underline{B}}_X$ has countable vertical sections. Then, when $\underline{\underline{M}}$ is closed under operation $(\underline{\underline{A}})$, B is a countable union of $\underline{\underline{M}}$-measurable graphs. This answers, in part, a question of Ioffe [13].

We then prove, in Section 2, some representation theorems for G_δ-valued multifunctions that correspond to uniform versions of a well-known theorem of Mazurkiewicz. A typical result is : Let T, X be Polish with X zero-dimensional, and $F : T \longrightarrow X$ a Borel measurable G_δ-valued multifunction such that $F(t)$ is both dense and boundary in $cl(F(t))$ for each t. Then there is a Borel measurable map $f : T \times \Sigma \longrightarrow X$ such that $f(t,.)$ is a homeomorphism onto $F(t)$ for each t. These results are used in the sequel.

In Section 3, we first obtain a parametrized version of the Von Neumann selection theorem, namely : Let T, X be Polish and B a Borel set in $T \times X$ with uncountable vertical sections. Let $\underline{\underline{B}}(\underline{\underline{A}}(T))$ stand for the analytic σ-field on T. Then there is a $\underline{\underline{B}}(\underline{\underline{A}}(T)) \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}$-measurable map $f : T \times (\Sigma \dot{\cup} N) \longrightarrow X$ such that $f(t,.)$ is a one-one, continuous map onto B^t for each $t \in T$.

This answers a question of Cenzer and Mauldin [9] and improves earlier results in several different directions chiefly in that f above is measurable with respect to a product structure, and in that the selections for B induced by f are the best possible. We then prove an uncountable counterpart of the theorem in Section 1, that is, we show that if $\underline{\underline{M}}$ is closed under operation $(\underline{\underline{A}})$, then any $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$ with uncountable vertical sections can be parametrized by an $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \cup N}$-measurable map with the above properties. This improves and gives a definable version of a result of Ioffe [13] and Bourgain [5], who showed that the above holds when $\underline{\underline{M}}$ is a complete σ-field. We then show, in Section 4, that many of our results can be improved (if the sets B above are assumed to have condensed sections) to obtain uniform versions of the fact that a set is condensed and Borel iff it is a one-one, continuous image of Σ. Our results are seen to be optimal in many different ways.

Having so far considered sets in product σ-fields, we then take up the question of approximating definable sets in the product $T \times X$, of two Polish spaces, by sets in product σ-fields. Let $\underline{\underline{S}}(Z)$ denote the class of C-sets in Z, for Polish Z. In Section 5, we show that if $A \in \underline{\underline{S}}(T \times X)$, then there are B and C in $\underline{\underline{S}}(T) \otimes \underline{\underline{B}}_X$ such that $B \subseteq A \subseteq C$ and $C^t - B^t$ is meager for each t. Many questions about A then reduce to ones about the simpler sets B and C. We study the consequences of this theorem and show that it yields as immediate corrolaries many results about C-sets available in the literature, such as a recent selection theorem of Burgess [6] and some computations of Vaught [42]. We also prove a similar statement for measure.

In Section 6, we establish a parametrization theorem of Sarbadhikari and Srivastava [31] in the set up of Kuratowski and Ryll-Nardzewski. More precisely, we have : Let X be Polish, T a non-empty set and $\underline{\underline{M}}$ a field on T. Let $F : T \to X$ be an $\underline{\underline{M}}_{\sigma}$-measurable multifunction with $Gr(F) \in (\underline{\underline{M}} \times \underline{\underline{U}})_{\sigma\delta}$, $\underline{\underline{U}}$ being the topology of X. Then there is a map $f : \Sigma \to X$ such that $f(t,.)$ is continuous, open, and onto $F(t)$ for $T \in T$, and $f(.,\sigma)$ is an $\underline{\underline{M}}_{\sigma}$-measurable selection of F for $\sigma \in \Sigma$. One consequence of this is that various known selection theorems (such as Srivastava [36], Burgess [7]) hold in completely abstract situations. We then give two counterexamples. The first answers a question raised in the above paper [31] and shows that such continuous, open Borel measurable representations may lead to analytic, non-Borel sets. In the second we show that many known selection theorems do not extend to analytic sets.

A refinement of the method in the second example above settles the following question of J.R. Steel [41] in the negative : If A is an analytic set with σ-compact vertical sections, then does A contain an analytic set with compact sections having the same projection to the first coordinate ? This we do in Section 7 .

In Section 8 we show that an effectivization of an article of Mauldin [23] yields a criterion for Borel parametrizations akin to the Δ-uniformization criterion.

In Section 9 we prove an effective analogue of the theorem in [31] on continuous, open representations of G_δ-valued multifunctions. This is from a joint article with R. Barua [2] where such representations are applied to obtain basis theorems.

0. Definitions and notation : In this section we fix the basic definitions and notation used in this thesis.

We denote by ω the set of natural numbers and reserve $\mathbb{N}$ for the set of positive integers. Seq will denote the set of finite sequences of natural numbers. We will, when convenient, identify Seq with the set of Godel numbers of finite sequences of natural numbers, particularly in parts where we take recourse to effective theoretic methods. What we mean will, in each case, be clear from the context.

For $k \in \omega$, S_k will be the set of all elements of Seq of length k. For $s \in$ Seq, $|s|$ or $lh(s)$ will denote the length of s and if $i < |s|$ is a natural number, s_i will denote the $(i+1)^{st}$ coordinate of s. Thus, we may look upon $s \in S_k$ as the sequence $(s_0, s_1, \ldots, s_{k-1})$, and write $s = \langle s_0, s_1, \ldots, s_{k-1} \rangle$. For $i \leq |s|$, $s \restriction i$ denotes $\langle s_0, s_1, \ldots, s_{i-1} \rangle$ and for $s, t \in$ Seq, we use $s\hat{\ }t$ to denote the catenation of s and t. If $n \in \omega$, sn denotes $s\hat{\ }\langle n \rangle$. For $s, t \in$ Seq, we write $s \prec t$ if t extends s.

$\mathbb{R}$ denotes the real line and Σ the space ω^ω, where the latter is given the product of discrete topologies on ω. It is well known that Σ is homeomorphic to the space of irrationals. For $\sigma \in \Sigma$, $\sigma \restriction i$ will denote $\langle \sigma(0), \sigma(1), \ldots, \sigma(i-1) \rangle$; here if $i = 0$, $\sigma \restriction i$ will correspond to the empty sequence, e. If $s \in$ Seq, the set $\{\sigma \in \Sigma : \sigma(i) = s_i \text{ for } i < |s|\}$ will be denoted by $\Sigma(s)$. These sets form a base for the topology on Σ. Finally,

$\Sigma \uplus N$ denotes the discrete (topological) union of Σ and N, N being given the discrete topology.

Let T and X be non-empty sets. A <u>multifunction</u> $F : T \longrightarrow X$ is a function whose domain is T and whose values are subsets of X. If nothing explicit is stated the values of F will be non-empty. For $E \subseteq X$, we denote by $F^{-1}(E)$ the set $\{t \in T : F(t) \cap E \neq \emptyset\}$. By $Gr(F)$ we mean $\{(t,x) \in T \times X : x \in F(t)\}$, and call it the <u>graph of</u> F. A function f $T \longrightarrow X$ is called a <u>selector</u> for F if $f(t) \in F(t)$, $t \in T$. We also call f a <u>selection</u> or <u>uniformization</u> for $Gr(F)$.

If $\mathcal{A}$ and $\mathcal{B}$ are σ-fields on T and X respectively, then $\mathcal{A} \otimes \mathcal{B}$ will denote the <u>product σ-field</u> on $T \times X$; and if $\mathcal{M}$ and $\mathcal{N}$ are families of subsets of T and X respectively, then $\mathcal{M} \times \mathcal{N}$ stands for $\{A \times B : A \in \mathcal{M}$ and $B \in \mathcal{N}\}$. The <u>power set</u> of T will be denoted by $\mathcal{P}(T)$, and for $\mathcal{A} \subseteq \mathcal{P}(T)$, $\mathcal{A}^c$ will be $\{A \subseteq T : A^c \in \mathcal{A}\}$ and the smallest countably additive (resp. countably multiplicative) family of subsets of T containing $\mathcal{A}$ will be denoted by $\mathcal{A}_\sigma$ (resp. $\mathcal{A}_\delta$). If $W \subseteq T \times X$ then the vertical (at t) and horizontal (at x) <u>sections</u> of W will be denoted by W^t and W_x respectively.

Suppose X and Y are separable metric spaces. A multifunction $F : T \longrightarrow X$ is called $\mathcal{A}$-measurable if $F^{-1}(V) \in \mathcal{A}$ for every open $V \subseteq X$. Similarly, a point map $f : T \longrightarrow X$ is $\mathcal{A}$-measurable if $f^{-1}(V) \in \mathcal{A}$ for open V in X. A map $f : T \times Y \longrightarrow X$

if called an $\underline{\underline{A}}$-measurable <u>Caratheodory map</u> if for each $t \in T$, the map $f(t,.) : Y \longrightarrow X$ is continuous and the map $f(.,y) : T \longrightarrow X$ is $\underline{\underline{A}}$-measurable for each $y \in Y$. A Caratheodory map is <u>open</u> (resp. <u>closed</u>) if, for each t, $f(t,U)$ is relatively open (resp. closed) in the range of $f(t,.)$ for open $U \subseteq Y$. Observe that if $\underline{\underline{A}}$ is a σ-field on T, then any $\underline{\underline{A}}$-measurable Caratheodory map is $\underline{\underline{A}} \otimes \underline{\underline{B}}_Y$-measurable, where $\underline{\underline{B}}_Y$ is the Borel σ-field of Y.

Let $B \subseteq T \times X$. A map $g : T \times Y \longrightarrow X$ is said to <u>parametrize</u> B if $g(t,.)$ maps Y onto B^t for each $t \in T$. If T and Y are Polish, g is Borel measurable, and $g(t,.)$ is one-one for each $t \in T$, then g is called a Borel parametrization. If g is a Caratheodory map parametrizing B, then B is said to have a <u>Caratheodory representation</u>. We shall also say "g induces B" for "g parametrizes B". One can analogously speak of maps g on $T \times Y$ into X inducing a multifunction $F : T \longrightarrow X$ (instead of $Gr(F)$). Observe that if F is induced by an $\underline{\underline{A}}$-measurable Caratheodory map then F is $\underline{\underline{A}}$-measurable.

For X separable metric, $\underline{\underline{B}}_X$ will always denote the Borel σ-field of X. When X is Polish, the analytic σ-field on X, that is, the smallest σ-field on X containing the analytic subsets of X, will be denoted by $\underline{\underline{B}}(\underline{\underline{A}}(X))$. If $Y \subseteq X$, $\underline{\underline{B}}(\underline{\underline{A}}(Y))$ stands for the trace of $\underline{\underline{B}}(\underline{\underline{A}}(X))$ on Y.

We will say that a σ-field $\underline{\underline{A}}$ on T is closed under operation $(\underline{\underline{A}})$ if the result of operation $(\underline{\underline{A}})$ performed on a system of sets $\{A_{n_1 n_2 \ldots n_k}\}$ with each $A_{n_1 \ldots n_k} \in \underline{\underline{A}}$, belongs to $\underline{\underline{A}}$. Well-known examples of σ-fields closed under operation $(\underline{\underline{A}})$ are (for Polish X), any σ-field complete with respect to a σ-finite measure on $\underline{\underline{B}}_X$, the class of universally measurable subsets of X, various definable subclasses thereof, such as Selivanovskii's class of C-sets, Kolmogorov's R-sets, and Blackwell's BP-sets.

Suppose $\underline{\underline{M}}$ is a countably generated σ-field on T generated by $\{M_n\}_{n \geq 1}$. By the characteristic function of the sequence $\{M_n\}$ we mean the function $f : T \to [0,1]$ given by

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{3^n} \cdot I_{M_n}, \quad \text{where}$$

I_{M_n} is the indicator function of the set M_n. Let $S = f(T)$. Then, as is well known, f is a bimeasurable function (that is, a measurable function that carries measurable sets to measurable sets) between $\underline{\underline{M}}$ and $\underline{\underline{B}}_S$.

For $E \subseteq X$, cl(E) will denote the closure of E and $\delta(E)$, the diameter of E.

We will in some places take recourse to effective theoretic methods. All our terminology and notation from the effective theory (subject to the ones fixed above) is from Moschovakis [25]. Here, we only fix the following: when Γ is a point class $\underset{\sim}{\Gamma}$ denotes the boldface class corresponding to Γ.

All else is either widely known or available in Kuratowski [15].

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1. <u>Parametrizing sets with countable sections</u> : Lusin's theorem on sets with countable sections is well-known, namely, that if B is a Borel subset of the product of two Polish spaces, with countable vertical sections, then B is a countable union of Borel graphs [19]. This raises the following general question : Let $(T, \underset{=}{M})$ be a measurable space, and X a Polish space. Suppose $B \in \underset{=}{M} \otimes \underset{=}{B}_X$ and B has countable vertical sections. Then, is B a countable union of $\underset{=}{M}$-measurable functions ? We shall see below that this is indeed true if $\underset{=}{M}$ is a σ-field closed under operation $(\underset{=}{A})$. Our proof is simple modulo certain methods from the effective theory. It is of some interest that the above abstract problem couched in purely boldface terms seems to require the methods of the effective theory - any classical proof must surely be very hard. This method will also play a crucial role in the sequel when we parametrize sets with uncountable sections.

Before getting down to the proofs we might add that in [13], Ioffe has raised a similar question : Let B be the graph of an $\underset{=}{M}$-measurable, closed and countable valued multifunction on T into X. Then, is B the union of countably many $\underset{=}{M}$-measurable selections ? However, in his specific situation Ioffe only needs that this be valid when $\underset{=}{M}$ is closed under operation $(\underset{=}{A})$, and as mentioned above, we obtain this partial answer to his question.

1.1 <u>Lemma</u> : Let $P \subseteq \omega^\omega \times \omega^\omega$ be in Δ^1_1. Then there are countably many Π^1_1-recursive partial functions $f_0, f_1, \ldots,$ each defined on ω^ω into ω^ω, such that whenever P^x, the vertical section of P at x is countable,

$$(\forall y)(P(x,y) \longleftrightarrow (\exists n)(f_n(x)\downarrow \text{ and } f_n(x)=y))$$

Proof : Fix a Π^1_1-recursive partial function $d : \omega^\omega \times \omega \longrightarrow \omega^\omega$ which parametrizes points in $\Delta^1_1(x)$, x running through ω^ω [25, 4D.2]. Define, for $n \geq 0$, $f_n : \omega^\omega \longrightarrow \omega^\omega$ by :

$$f_n(x)\downarrow \ \& \ f_n(x)=y \longleftrightarrow d(x,n)\downarrow \text{ and } P(x,d(x,n)) \text{ and } y=d(x,n).$$

It follows that the f_n's so defined are Π^1_1-recursive partial functions. Now, by the effective perfect set theorem [25, 4F.1], any countable $\Delta^1_1(x)$ set contains only $\Delta^1_1(x)$ points. Consequently, for any x, if P^x, a set in $\Delta^1_1(x)$, is countable, we have,

$$(\forall y)(P(x,y) \longrightarrow (\exists n)(d(x,n)\downarrow \text{ and } d(x,n)=y)).$$

The conclusion now follows.

It is clear that the above can be relativized to any $z \in \omega^\omega$. The following is then easy.

1.2 Lemma : Let S be separable metric and X Polish. Suppose B is a Borel subset of $S \times X$ such that B^t is nonempty and countable for each $t \in S$. Then there is a sequence $\{f_n\}_{n\geq 1}$ of $\underline{\underline{B}}(\underline{\underline{A}}(S))$-measurable functions such that

$$B = \bigcup_{n\geq 1} Gr(f_n).$$

Proof : Let T be a separable metric completion of S. As B is Borel in $S \times X$, there is $C \subseteq T \times X$, absolute Borel such that $C \cap (S \times X) = B$. Furthermore as T and X can be Borel isomorphically imbedded in ω^ω, we may without loss of generality assume that $T = X = \omega^\omega$. As C is Borel, C is in $\Delta^1_1(z)$ for some $z \in \omega^\omega$. A straightforward application of Lemma 1.1 (relativized to z), together with the facts that the domain of a $\Pi^1_1(z)$-recursive partial function is a set in $\Pi^1_1(z)$ and that the inverse image of a basic clopen set under such a map is again a set in $\Pi^1_1(z)$ yields maps $\{g_n : n \geq 1\}$ with the required measurability covering B. The restriction of each g_n to S is a partial $\underline{\underline{B}}(\underline{\underline{A}}(S))$-measurable map defined on a set in $\underline{\underline{B}}(\underline{\underline{A}}(S))$.

Define now $g : S \longrightarrow X$ by :

$$g(t) = g_n(t) \text{ if } n \text{ is the least integer } n \text{ such that } g_n(t) \text{ is defined.}$$

Then g is $\underline{\underline{B}}(\underline{\underline{A}}(S))$-measurable and defined on the whole of S. Finally put

$$\begin{aligned} f_n(t) &= g_n(t) \quad \text{if } g_n(t) \text{ is defined} \\ &= g(t) \quad \text{otherwise.} \end{aligned}$$

These do the job.

1.3 Theorem : Let $(T, \underline{\underline{M}})$ be a measurable space, $\underline{\underline{M}}$ being a σ-field closed under operation $(\underline{\underline{A}})$. Let X be Polish and suppose $B \subseteq T \times X$, $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$ and B has countable and nonempty vertical sections. Then there is a sequence $\{f_n\}_{n\geq 1}$ of $\underline{\underline{M}}$-measurable functions on T into X such that for each $t \in T$,

$$B^t = \{f_n(t) : n \geq 1\} .$$

<u>Proof</u> : As $B \in \underline{\underline{M}} \, \textcircled{x} \, \underline{\underline{B}}_X$, there are countably many rectangles $\{M_n \times U_n\}_{n\geq 1}$, with $M_n \in \underline{\underline{M}}$ and U_n open in X such that $B \in \sigma(\{M_n \times U_n\}_{n\geq 1})$. Let $\underline{\underline{M}}_o = \sigma(\{M_n\}_{n\geq 1})$. Then $\underline{\underline{M}}_o$ is a countably generated sub σ-field of $\underline{\underline{M}}$. Let $g : T \longrightarrow [0,1]$ be the characteristic function of the sequence $\{M_n\}_{n\geq 1}$ and suppose $g(T) = S \subseteq [0,1]$. Then $g^{-1}(\underline{\underline{B}}_S) = \underline{\underline{M}}_o \subseteq \underline{\underline{M}}$.

Define $\emptyset : T \times X \longrightarrow S \times X$ by : $\emptyset(t,x) = (g(t),x)$. It is easy to see that $\emptyset$ is bimeasurable between $(T \times X, \underline{\underline{M}}_o \textcircled{x} \underline{\underline{B}}_X)$ and $(S \times X, \underline{\underline{B}}_S \textcircled{x} \underline{\underline{B}}_X)$. Set $C = \emptyset(B)$, so $C \in \underline{\underline{B}}_S \textcircled{x} \underline{\underline{B}}_X$, i.e., C is a Borel set in $S \times X$ and C has countable and non-empty vertical sections. An application of 1.2 now yields a sequence $\{g_n\}_{n\geq 1}$ of $\underline{\underline{B}}(\underline{\underline{A}}(S))$-measurable maps on S into X covering C. As $\underline{\underline{M}}$ is closed under operation $(\underline{\underline{A}})$, we have $g^{-1}(\underline{\underline{B}}(\underline{\underline{A}}(S))) \subseteq \underline{\underline{M}}$. It follows that if we now define f_n on T by $f_n(t) = g_n(g(t))$, then the f_n's have the desired properties.

What happens for arbitrary $\underline{\underline{M}}$? We have not been able to settle the issue. Notice however that the general question is equivalent to asking the question about the Borel σ-field on a coanalytic set. For, arguing as in 1.3, it is easily seen that we might without loss of generality assume that $B \in \underline{\underline{B}}_S \textcircled{x} \underline{\underline{B}}_X$

where $S \subseteq [0,1]$. Now find $B_1 \in \underline{\underline{B}}_{[0,1]} \otimes \underline{\underline{B}}_X$ such that $B_1 \cap (S \times X) = B$. Then $R = \{r \in [0,1] : B^r$ is countable and nonempty$\}$ is a coanalytic set containing S. It is enough now to solve the problem for $B_1 \cap (R \times X)$.

A related question is the following : suppose $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$ and B^t is a singleton for each t. Then is B the graph of an $\underline{\underline{M}}$-measurable function ? This is true when $\underline{\underline{M}}$ is closed under operation $(\underline{\underline{A}})$. Again, the general question is equivalent to studying the behaviour on coanalytic subsets of $[0,1]$. The answer could well be undecidable in ZFC. However both theorems are valid on an analytic set equipped with the Borel σ-field. In the first case, this is just Lusin's theorem for Borel sets with countable sections and the second is an immediate consequence of the first separation theorem for analytic sets.

2. Homeomorphic Caratheodory representations for G_δ-valued multifunctions : We will now begin our study of parametrizing measurable sets with uncountable sections. In this section, we establish a Caratheodory representation theorem for G_δ-valued multifunctions that will be used in the sequel. The study of measurable G_δ-valued multifunctions was initiated by Srivastava who in addition to proving the basic selection theorem [36], obtained several parametrization results [31,37], some in collaboration with H. Sarbadhikari [31]. We will need and prove a parametrization theorem of a special type, namely one in which the induced sectional maps are homeomorphisms. The basic result corresponds to a " uniform " version of Mazurkiewicz's theorem that a dense G_δ-subset of a zero-dimensional Polish space whose complement is also dense is a homeomorph of the space of irrationals [15,p.441]. We use this technique later in the section to obtain one-one Caratheodory representations for measurable G_δ-valued multifunctions taking dense-in-itself values in a zero-dimensional space. Our method is, as usual, to carry out the appropriate proof for the " single-section " case uniformly.

For the rest of this section T and X will be Polish spaces, X being, moreover, zero-dimensional. We fix a countably generated sub σ-field $\underline{\underline{A}}$ of the Borel σ-field $\underline{\underline{B}}_T$ on T. We also fix an $\underline{\underline{A}}$-measurable G_δ-valued multifunction $F : T \longrightarrow X$. The graph of F will be denoted by G, and we will assume furthermore that $G \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$. Apart from the assumption on the dimension of X, this is the set-up in Srivastava [36] where it is shown that $\underline{\underline{A}}$-measurable selections exist for such multifunctions.

We begin by stating three well-known lemmas. The first two are proved in [36] and the third is a result of Blackwell [3]. 2.1 is basic to most parametrization results on G_δ-valued multifunctions and is a corollary to a beautiful result of Saint-Raymond [29].

2.1 <u>Lemma</u> : Let T, X be Polish spaces and let $\underline{\underline{A}}$ be a countably generated sub σ-field of $\underline{\underline{B}}_T$. Suppose $B \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$ and B^t is a G_δ in X for each $t \in T$. Then there exist sets $B_n \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$ such that $B_n \supseteq B_{n+1}$, B_n^t is open in X for each $t \in T$ and $n \geq 0$, and $B = \bigcap_{n \geq 0} B_n$.

2.2 <u>Lemma</u> : Let $(T, \underline{\underline{A}})$ and $(X, \underline{\underline{B}})$ be measurable spaces. Let $\underline{\underline{A}}$ be atomic and $B \in \underline{\underline{A}} \otimes \underline{\underline{B}}$. Then $B^t = B^{t'}$ holds for t and t' belonging to the same atom of $\underline{\underline{A}}$.

2.3 <u>Lemma</u> : Let T be Polish and $\underline{\underline{A}}$ a countably generated sub σ-field of $\underline{\underline{B}}_T$. Then $A \in \underline{\underline{A}}$ iff $A \in \underline{\underline{B}}_T$ and A is a union of atoms of $\underline{\underline{A}}$.

Invoking 2.1 we write :

$$G = \bigcap_{n \geq 0} G_n$$

where $G_n \supseteq G_{n+1}$, $G_n \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$, and G_n^t is open for each $n \geq 0$ and $t \in T$. Also $\{r_0, r_1, \ldots\}$ will be a fixed dense subset of X,

d a fixed complete metric for X, and $\{V(n)\}_{n \geq 0}$ a fixed clopen base for X consisting of nonempty sets. Denote the projection function on $T \times X$ to T by π_T. We will now prove several lemmas.

2.4 <u>Lemma</u> : Let T, X be as above. Assume further, that X is compact. Suppose $B \subseteq T \times X$, $B \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$, and B^t is open for $t \in T$. Then $\{t \in T : B^t \text{ is not closed}\} \in \underline{\underline{A}}$.

<u>Proof</u> : It follows from 2.3 and a well-known result of Kunugui and Novikov [16] that $\pi_T(T \times X) - B) \in \underline{\underline{A}}$ and that $(T \times X) - B$ is the graph of an $\underline{\underline{A}} \cap \pi_T(T \times X) - B)$ - measurable, compact valued multifunction H on $\pi_T((T \times X) - B)$. Now use the fact that B^t is open and X is compact to verify that

$$\{t \in T : B^t \text{ is not closed}\} = \{t \in \pi_T((T \times X) - B) : (\forall n)(\exists m)\ (r_m \in B^t \text{ and } d(r_m, H(t)) < 1/(n+1))\}.$$

As H is $\underline{\underline{A}}$-measurable, it follows that $d(r_m, H(t))$ is an $\underline{\underline{A}}$-measurable function of t, and consequently that the set on the left belongs to $\underline{\underline{A}}$.

The proof given above does not use the fact that X is zero-dimensional.

2.5 <u>Lemma</u> : Let T, X be as in 2.4. Suppose G^t is not closed for each $t \in T$. Let $\varepsilon > 0$. Then there exist $\underline{\underline{A}}$-measurable maps $p(\cdot, n) : T \longrightarrow \omega$ for each $n \geq 0$ such that

(i) $\delta(V(p(t,n))) < \varepsilon$, $t \in T$, $n \geq 0$.

(ii) $V(p(t,n)) - \bigcup_{m<n} V(p(t,m)) \neq \emptyset$.

(iii) $G^t \subseteq \bigcup_{n\geq 0} V(p(t,n)) \subseteq G_0^t$.

<u>Proof</u> : Let $T_n = \{t \in T : G_n^t \text{ is not closed}\}$, $n \geq 0$.

By 2.4 each $T_n \in \underline{\underline{A}}$. As $G^t = \bigcap_{n\geq 0} G_n^t$ and G^t is not closed for each $t \in T$, we see that for each t there is $n \geq 0$ such that G_n^t is not closed. Thus $\bigcup_{n\geq 0} T_n = T$. Disjointify the T_n's : set $W_0 = T_0$, $W_n = T_n - \bigcup_{i<n} T_i$, $n \geq 1$. Then $\bigcup_{n\geq 0} W_n = \bigcup_{n\geq 0} T_n$, $W_n \subseteq T_n$, $W_n \in \underline{\underline{A}}$ and the W_n's are pairwise disjoint. Let $C = \bigcup_{n\geq 0} ((W_n \times X) \cap G_n)$. Then $C \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$, $G \subseteq C \subseteq G_0$, and C has open, non-closed vertical sections.

Define, for $n \geq 0$,

$$R_n = \{t \in T : V(n) \subseteq C^t\}, \text{ if } \delta(V(n)) < \varepsilon$$
$$= \emptyset \qquad\qquad , \text{ otherwise.}$$

Notice that if $\delta(V(n)) < \varepsilon$, then R_n is the complement of the projection of a Borel subset of $T \times X$ with compact vertical sections. It follows that for each $n \geq 0$, $R_n \in \underline{\underline{B}}_T$. Moreover, R_n is a union of $\underline{\underline{A}}$-atoms. It follows from 2.3 that $R_n \in \underline{\underline{A}}$.
Now define, by induction on n,

$$p(t,0) = m \text{ , if } m \text{ is the first natural number } \ell \text{ such that } t \in R_\ell .$$

$p(t,n+1) = m$, if m is the first natural number $\ell > p(t,n)$ such that $V(\ell) - \bigcup_{i \leq n} V(p(t,i)) \neq \emptyset$ and $t \in R_\ell$.

Observe that, as G^t is not closed for every t, the $p(t,n)$'s are well-defined on the whole of T. For, if not, there would be t and n such that $\bigcup_{m \leq n} V(p(t,m)) = G^t$, which is impossible as the $V(n)$'s are clopen sets. One easily checks that the $p(t,n)$'s are $\underline{\underline{A}}$-measurable functions such that $\bigcup_{n \geq 0} V(p(t,n)) = G^t$. They, therefore, satisfy the conditions in the lemma.

2.6 <u>Lemma</u> : Let the hypotheses of 2.5 be in force. Let $\varepsilon > 0$. Then there exist $\underline{\underline{A}}$-measurable maps $p(\cdot\, , n) : T \longrightarrow \omega$ for each $n \geq 0$ such that

(i) $\delta(V(p(t,n))) < \varepsilon$, $t \in T$, $n \geq 0$.

(ii) $(V(p(t,n)) - \bigcup_{m < n} V(p(t,m))) \cap G^t \neq \emptyset$.

(iii) $G^t \subseteq \bigcup_{n \geq 0} V(p(t,n)) \subseteq G_0^t$.

<u>Proof</u> . Apply Lemma 2.5 to get maps $q_0'(t,n)$, $n \geq 0$ such that $\delta(V(q_0'(t,n))) < \varepsilon$ for $t \in T$, $n \geq 0$; $V(q_0'(t,n)) - \bigcup_{m < n} V(q_0'(t,m)) \neq \emptyset$ and $G^t \subseteq \bigcup_{n \geq 0} V(q_0'(t,n)) \subseteq G_0^t$.

Let $T_n = \{t \in T : (V(q_0'(t,n)) - \bigcup_{m < n} V(q_0'(t,m))) \cap G^t \neq \emptyset\}$, $n \geq 0$.

As G is the graph of an $\underline{\underline{A}}$-measurable multifunction and the functions $q_o'(t,n)$ are $\underline{\underline{A}}$-measurable it follows that $T_n \in \underline{\underline{A}}$ for $n \geq 0$. For the purposes of this proof we will make a temporary definition by putting $V(\infty) = \emptyset$. Define, by induction on n, functions $q_o(t,n)$ as follows :

$$
\begin{aligned}
\text{Put } q_o(t,n) &= q_o'(t,k) \text{ , where } k \text{ is the } (n+1)^{st} \text{ integer } \ell \\
&\qquad \text{such that } t \in T_\ell \text{, if there is one such} \\
&= \infty \text{ , otherwise .}
\end{aligned}
$$

Then $q_o(t,n)$, $n \geq 0$, are $\underline{\underline{A}}$-measurable extended natural number-valued maps. Moreover,

$$G^t \subseteq \bigcup_{n \geq 0} V(q_o(t,n)) \subseteq G_o^t .$$

For each $k \geq 0$, apply the above argument to ~~T, X, G~~ $T, X, G, \{G_n, n \geq k\}$. Then, for each $k \geq 0$, we will obtain $\underline{\underline{A}}$-measurable extended natural number-valued maps $q_k(t,n)$, $n \geq 0$, with the following properties :

(a) $\delta(V(q_k(t,n))) < \varepsilon$ for $t \in T$, $n \geq 0$.

(b) $G^t \subseteq \bigcup_{n \geq 0} V(q_k(t,n)) \subseteq G_k^t$.

(c) $q_k(t,n) < \infty \Longrightarrow (V(q_k(t,n)) - \bigcup_{m < n} V(q_k(t,m))) \cap G^t \neq \emptyset$.

Define $C_k \subseteq T \times X$ by :

$$(t,x) \in C_k \Longleftrightarrow x \in \bigcup_{n \geq 0} V(q_k(t,n)).$$

As the $q_k(t,n)$'s are $\underline{\underline{A}}$-measurable maps, $C_k \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$ for $k \geq 0$.

Moreover, C_k^t is open, $k \geq 0$, $t \in T$, and $\bigcap_{k \geq 0} C_k = G$. Now, by assumption, G^t is not closed for each t. Consequently, for each $t \in T$ there is $k \geq 0$ such that C_k^t is not closed. Put $S_n' = \{t \in T : C_n^t \text{ is not closed}\}$, $n \geq 0$. By 2.4, $S_n' \in \underline{\underline{A}}$ for $n \geq 0$. Disjointify the S_n''s : set $R_0 = S_0'$, $R_n = S_n' - \bigcup_{i<n} S_i'$, $n \geq 1$. Then $\bigcup R_n = T$, the R_n's are pairwise disjoint, C_n^t is not closed for $t \in R_n$, and $R_n \in \underline{\underline{A}}$ for $n \geq 0$. Define $C \subseteq T \times X$ by:

$$C = \bigcup_{n \geq 0} ((R_n \times X) \cap C_n) .$$

Then $C \in \underline{\underline{A}} \ (\overline{\underline{x}}) \ \underline{\underline{B}}_X$, $G \subseteq C \subseteq G_0$ and C has open, non-closed sections. For $t \in R_k$, $C^t = C_k^t$. As C^t is not closed, $C_k^t = \bigcup_{n \geq 0} V(q_k(t,n))$ is not closed. Since the $V(n)$'s are from a clopen base, it follows that on R_k, $q_k(t,n) < \infty$ for every $n \geq 0$. To complete the proof we now need only put,

$$p(t,n) = q_k(t,n) \ , \quad \text{if} \quad t \in R_k .$$

2.7 <u>Lemma</u> : Let T and X be as in Lemma 2.4. Assume that for each clopen V in X such that $G^t \cap V \neq \emptyset$, we have $G^t \cap V$ is not closed. Then there is a system $\{p(t,s) : s \in Seq\}$ of $\underline{\underline{A}}$-measurable maps on T into ω such that

(i) $s \in S_k$, $k \geq 1 \Longrightarrow \delta(V(p(t,s))) < 1/k$ for $t \in T$.

(ii) For $s \in S_k$,

$$(V(p(t,s)) - \bigcup\{V(p(t,(s \restriction (k-1))i)) : i < s_{k-1}\}) \cap G^t \neq \emptyset .$$

(iii) For $s \in S_k$, $n \geq 0$,

$$V(p(t,sn)) \subseteq V(p(t,s)) - \bigcup\{V(p(t,(s\restriction(k-1))i)) : i < s_{k-1}\}.$$

(iv) $G^t \subseteq \bigcup\{V(p(t,s)) : s \in S_k\} \subseteq G^t_{k-1}$, $t \in T$, $k \geq 1$.

Proof : We will define such a system by induction on $|s|$.
Put $p(t,e)$ = least n such that $V(n) = X$ (we assume here for convenience that X is in $\{V(n)\}_{n \geq 0}$). For $|s| = 1$, we obtain maps $p(t,n)$ as in Lemma 2.6 with $\varepsilon = 1$. Suppose $p(t,s)$ has been defined for $s \in S_k$. Now fix $s \in S_k$. We have to define $p(t,sn)$ for $n \geq 0$. Let $j = s_{k-1}$. For $d \in S_{j+1}$, define

$$T(d) = \{t \in T : p(t,(s\restriction(k-1))0) = d_0, \ldots\ldots \ldots\ldots\ldots p(t,(s\restriction(k-1))s_{k-1}) = d_j\}.$$

By induction hypothesis, all the maps specified within brackets above are $\underline{\underline{A}}$-measurable. It follows that $T(d) \in \underline{\underline{A}}$ for each $d \in S_j$, $\bigcup\{T(d) : d \in S_{j+1}\} = T$, and the $T(d)$'s are pairwise disjoint. Apply Lemma 2.6 with $T(d)$ playing the role of T, $V(d_j) - \bigcup\{V(d_i) : i < j\}$ playing the role of X, $G \cap (T(d) \times (V(d_j) - \bigcup\{V(d_i) : i < j\}))$ playing the role of G, $G_n \cap (T(d) \times (V(d_j) - \bigcup\{V(d_i) : i < j\}))$ playing the role of G_m for $m \geq k$ (we ignore here $G_0, G_1, \ldots, G_{k-1}$), and $\varepsilon = 1/(k+1)$. Note that for each $t \in T$ the hypothesis of the Lemma implies that the " new " G^t is not closed in the " new " X. One therefore obtains $\underline{\underline{A}}$-measurable maps $q_d(t,n)$ defined on $T(d)$ satisfying the conditions in Lemma 2.6 specialized to the above set-up, where the base for the "new" X consists precisely of those $V(n)$ s.t. $V(n) \subseteq V(d_j) - \bigcup\{V(d_i) : i < j\}$, with their original labelling unchanged.

If one now defines $p(t,sn)$ by

$$p(t,sn) = q_d(t,n) \quad \text{on} \quad T(d)$$

it is easy to verify that all the conclusions of the lemma are satisfied.

We will now drop the technical assumption on the compactness of X assumed since Lemma 2.4.

2.8 Lemma : Let all the assumptions of Lemma 2.7 be in force except the one on the compactness of X. Then there is a Caratheodory map $f : T \times \Sigma \longrightarrow X$ such that

(i) $f(t,\cdot)$ is a homeomorphism of Σ onto $F(t)$, $t \in T$.

(ii) $f(\cdot,\sigma)$ is an $\underline{\underline{A}}$-measurable selector for F, $\sigma \in \Sigma$.

Proof : By taking a zero-dimensional compactification of X we see that we may assume X to be compact without loss of generality. Let $\{p(t,s) : s \in Seq\}$ be then the maps obtained from 2.7. The result easily follows by defining $f(t,\sigma)$ to be the unique element of $\bigcap_{n \geq 1} V(p(t,\sigma | n))$. This definition is legitimate and condition (i) is easily seen to be true. Condition (ii) is established by the identity :

$$f(t,\sigma) \in W \longleftrightarrow (\exists n)(V(p(t,\sigma | n)) \subseteq W)$$

for each open W in X. As $p(t,\sigma | n)$ is $\underline{\underline{A}}$-measurable, the right hand side is an $\underline{\underline{A}}$-measurable condition.

We will now state the promised uniform version of the Mazurkiewicz theorem.

2.9 Theorem : Let T be Polish, X zero-dimensional and Polish, and $\underline{\underline{A}}$ a countably-generated sub σ-field of $\underline{\underline{B}}_T$. Let $F : T \to X$ be an $\underline{\underline{A}}$-measurable G_δ-valued multifunction with $G_r(F) \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$. Assume further that for each $t, cl(F(t)) - F(t)$ is dense in $cl(F(t))$. Then there is a Caratheodory map $f : T \times \Sigma \to X$ satisfying

(i) For each $t \in T$, $f(t,.)$ is a homeomorphism of Σ and $F(t)$.

(ii) For each $\sigma \in \Sigma$, $f(.,\sigma)$ is an $\underline{\underline{A}}$-measurable selector for F.

Proof : Observe that $F(t) \cap W$ is not closed in any clopen set W for which $F(t) \cap W \neq \emptyset$. Lemma 2.8 now applies to yield the theorem.

2.10 Theorem : Let T be a Polish space and X a zero-dimensional Polish space. Let $\underline{\underline{A}}$ be a countably generated sub σ-field of $\underline{\underline{B}}_T$. Suppose $F : T \to X$ is an $\underline{\underline{A}}$-measurable multifunction taking dense-in-itself G_δ-values. Assume further that $Gr(F) \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$. Then there is a Caratheodory map $f : T \times (\Sigma \dot{\cup} N) \to X$ such that

(i) For each $t \in T$, $f(t,.)$ is a $1-1$, continuous map of $\Sigma \dot{\cup} N$ onto $F(t)$ such that $f(t,.)|_\Sigma$ is a homeomorphism.

(ii) For each $\sigma \in \Sigma$, $f(.,\sigma)$ is an $\underline{\underline{A}}$-measurable selector for F.

Proof : As F is an $\underline{\underline{A}}$-measurable G_δ-valued multifunction, by a theorem of Srivastava [36] it admits a sequence of disjoint $\underline{\underline{A}}$-measurable selectors, say $f_1(t), f_2(t), \ldots\ldots$ such that $\{f_n(t)\}_{n \geq 1}$ is dense in $F(t)$ for $t \in T$ [Observe that we can take the f_i's to have disjoint graphs since each $F(t)$ is infinite]. Define a new multifunction $H : T \longrightarrow X$ by :

$$H(t) = F(t) - \{f_n(t)\}_{n \geq 1} .$$

Clearly $cl(H(t)) - H(t) = cl(F(t)) - H(t)$ is dense in $cl(H(t))$. Theorem 2.9 therefore applies to H yielding a homeomorphic Caratheodory representation $h(t,\sigma)$ for H as in 2.9. Now put $f(t,\sigma) = h(t,\sigma)$ if $\sigma \in \Sigma$, and $f(t,n) = f_n(t)$ if $n \in \mathbb{N}$. This f satisfies the conclusions of the theorem.

An easy consequence of Theorem 2.10 is its validity in any Polish space X which becomes zero-dimensional on the removal of countably many points. In particular Theorem 2.10 holds for multifunctions taking values in $\mathbb{R}$. Thus we have

2.11 <u>Corollary</u> : Let T be a Polish space and $\underline{\underline{A}}$ a countably generated sub σ-field of the Borel σ-field on T. Suppose F is an $\underline{\underline{A}}$-measurable multifunction on T into $\mathbb{R}$ taking dense-in-itself G_δ-values and such that, moreover, $Gr(F) \in \underline{\underline{A}} \otimes \underline{\underline{B}}_{\mathbb{R}}$. Then the conclusion of Theorem 2.10 holds.

2.12 <u>Remark</u> : As the maps $f(t,.)$ in Theorem 2.10 are 1-1, continuous maps, they are, a fortiori, Borel isomorphisms. We have therefore obtained the " Borel parametrization " result of Srivastava and Sarbadhikari [31] for such multifunctions when they take values in (topologically) one-dimensional Polish spaces. However, our proof is more " effective " as it does not go into the Cantor-Bernstein kind of argument employed there. Also, as observed by them, we cannot drop the assumption that $F(t)$ is dense-in-itself for each t, as then F need not even admit a Borel parametrization.

2.13 Remark : Let A be an analytic space and X a zero-dimensional Polish space. Suppose $F : A \longrightarrow X$ is a $\underline{\underline{B}}_A$-measurable $\mathcal{G}_\delta$-valued multifunction such that $Gr(F) \in \underline{\underline{B}}_A \otimes \underline{\underline{B}}_X$. Fix a Polish space T and a continuous map f on T onto A. Put $\underline{\underline{A}} = f^{-1}(\underline{\underline{B}}_A)$. Then $\underline{\underline{A}}$ is a countably generated sub σ-field of $\underline{\underline{B}}_T$. Define $H : T \longrightarrow X$ by $H(t) = F(f(t))$. Then H is an $\underline{\underline{A}}$-measurable multifunction with $Gr(H) \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$. One easily sees now that by assuming various conditions on the values of F, which clearly carry over to H, one can prove theorems for $\underline{\underline{B}}_A$-measurable multifunctions on an analytic space A, corresponding to all the theorems we have proved so far. The technique is analogous to the one employed in [36].

Finally, before moving on to the next section, we add that in a later section we will obtain a one-one Caratheodory representation for the multifunction F in Corollary 2.11 in which the parametrizing space $\Sigma \dot{\cup} \mathbb{N}$ is replaced by Σ.

3. Parametrizing sets with uncountable sections   In this section we use the results obtained in the previous section to solve some questions on one-one Caratheodory representations. Our concern here is the existence, for suitable multifunctions, of representations expressing them as unions of " continuously " indexed families of selections. The reader might see Ioffe [12] for a discussion of this notion with reference to optimal control theory.

Probably the first appearance of a " parametrization " result is in a paper of R.A. Purves [27], who proved the converse of Lusin's theorem on sets with countable sections by showing that if f is a bimeasurable map on one Polish space into another (bimeasurable with respect to their Borel σ-fields), then f is countable to one outside a countable subset of the range. The proof essentially boiled down to looking at the graph of f turned around and Borel parametrizing a suitable portion of it.

In the kind of parametrization problem we are interested in, the first step was taken by Wesley [45] who proved, in connection with certain questions in Mathematical Economics, the following " uncountable " version of Lusin's theorem :

Theorem : Let B be a Borel subset of $I \times I$ with uncountable vertical sections. Then there is an $\underline{\underline{L}}(I \times I)$-measurable map f on $I \times I$ onto B such that :

(i) for each $t \in I$, $f(t,.)$ is a Borel isomorphism of I onto $\{t\} \times B^t$, and

(ii) for each $s \in I$, $\pi_2 \circ f(.,s)$ is an $\underline{\underline{L}}(I)$-measurable selection for B.

(Here, $\underline{\underline{L}}(I)$ and $\underline{\underline{L}}(I \times I)$ denote the Lebesgue measurable subsets of I and $I \times I$ respectively).

While Wesley's proof required forcing techniques, Cenzer and Mauldin in a subsequent paper [9] obtained a more "descriptive" result without any recourse to metamathematical methods, in answer to a question arising in work of Choksi on group automorphisms [10]. Their result is as follows :

<u>Theorem</u> : Let B be a Borel subset of $I \times I$ with uncountable vertical sections. Then there is an $S(I \times I)$-measurable map f on $I \times I$ onto B such that (a) $f(t,.)$ on I onto $\{t\} \times B^t$ is a Borel isomorphism for each $t \in I$, and (b) f^{-1} is $S(B)$-measurable.

(Recall that $S(X)$ denotes the family of C sets in X, for metric X)

Notice that, in the above, for each s, $\pi_2 \circ f(.,s)$ is an $S(I)$-measurable selection for B. Several questions now arise. Recall that the celebrated theorem of Von Neumann [26] and Yankov [46] states, in particular, that any such B has a $\underline{\underline{B}}(\underline{\underline{A}}(I))$-measurable selection, where $\underline{\underline{B}}(\underline{\underline{A}}(I))$ is the σ-field generated by the analytic subsets of I. Cenzer and Mauldin therefore asked whether B can be so parametrized that the induced individual selections are all $\underline{\underline{B}}(\underline{\underline{A}}(I))$-measurable. Secondly, and probably more importantly, the measurability of the maps f in the above theorems cannot be related to any pleasant " product structure " on $I \times I$ - it is not true, for example, that the families $\underline{\underline{L}}(I \times I)$, and $\underline{\underline{S}}(I \times I)$ are sub σ-fields of $\underline{\underline{L}}(I) \,(\overline{x})\, \underline{\underline{L}}(I)$ (See [28]). As the structure of product σ-fields is in general much simpler it would be desirable to rectify this " defect " by obtaining parametrizations that are measurable with respect to appropriate product σ-fields. Finally, it would be useful to replace the Borel isomorphisms in the above theorems by one-one continuous maps. We show that all this can be done. Indeed, we show that there is a one-one Caratheodory map f on $I \times (\Sigma \biguplus N) \longrightarrow I$ inducing B, with f $\underline{\underline{B}}\,(\underline{\underline{A}}(I))\,(\overline{x})\,\underline{\underline{B}}_{\Sigma \biguplus N}$ - measurable. This is, in a sense, the best parametrization possible, for an arbitrary Borel set with uncountable sections.

Along similar lines, Bourgain [5] and Ioffe [13] improved Wesley's theorem by proving the following :

Theorem : Let $(T, \underline{\underline{M}}, \mu)$ be a complete measure space, with σ-finite μ, and let X be Polish. Suppose $B \in \underline{\underline{M}}\,(\overline{x})\,\underline{\underline{B}}_X$ has uncountable vertical sections. Then, there is a μ-null set E and a map $f : T \times (\Sigma \biguplus N) \longrightarrow X$ such that (a) f is $\underline{\underline{M}}\,(x)\,\underline{\underline{B}}_{\Sigma \biguplus N}$ - measurable, and (b) $f(t,.)$ is a one-one, continuous map on $\Sigma \biguplus N$ onto B^t, for each $t \in T - E$.

Ioffe then asked whether this theorem could be unified with the theorem of Cenzer and Mauldin by replacing the complete σ-field $\underline{\underline{M}}$ by an arbitrary σ-field $\underline{\underline{M}}$ closed under Souslin's operation $(\underline{\underline{A}})$. This, as we shall see below, our methods easily accomplish, thereby yielding an uncountable analogue of Theorem 1.3. In particular, therefore, for $\underline{\underline{M}}$ one can take the class of universally measurable subsets of a Polish space or appropriate definable sub σ-fields thereof. We might add that all the maps we construct have inverses measurable with respect to the product σ-field (in a sense that will be clear on a reading of Theorem 3.11). An immediate consequence of this is that the maps we construct induce only sets in $\underline{\underline{M}} \otimes \underline{\underline{B}}_X$ with uncountable sections. Moreover, if $(T, \underline{\underline{M}})$ is a measurable space and every $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$ with uncountable vertical sections admits an $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \uplus N}$ - measurable Caratheodory representation, then $\underline{\underline{M}}$ is closed under operation $(\underline{\underline{A}})$. Thus the condition that $\underline{\underline{M}}$ is closed under operation $(\underline{\underline{A}})$ is a necessary condition for the existence of such parametrizations suggesting that such σ-fields provide the natural setting for studying parametrizations.

Before we get to the main theorem we prove the specific consequence of the results of the previous section that we will require :

3.1 <u>Lemma</u> : Let $(T, \underline{\underline{M}})$ be a measurable space. Let $F : T \longrightarrow \Sigma$ be an $\underline{\underline{M}}$-measurable multifunction taking nonempty and perfect values.

Then there is a map $k : T \times (\Sigma \biguplus N) \longrightarrow \Sigma$ such that

(i) k is $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \biguplus N}$-measurable,

(ii) for each $t \in T$, $k(t, .)$ is a one-one, continuous map on $\Sigma \biguplus N$ onto $F(t)$, and

(iii) for each $t \in T$, the restriction of $k(t, .)$ to Σ is a homeomorphism.

Proof : As F is a closed-valued measurable multifunction and $\underline{\underline{M}}$ is a σ-field, by well known theorems (See [13]), there are countably many $\underline{\underline{M}}$-measurable selectors for F, say $g_1, g_2, \ldots$ such that for each t, $\{g_n(t)\}_{n \geq 1}$ is dense in $F(t)$. Moreover, as $F(t)$ is perfect for each $t \in T$, $\{g_n(t)\}_{n \geq 1}$ is a dense-in-itself sequence for every $t \in T$.

We will now reduce the problem to a form to which the methods of Section 2 can readily be applied. To begin with notice that as each g_n is $\underline{\underline{M}}$-measurable there is a countably generated sub σ-field $\underline{\underline{M}}_o$ of $\underline{\underline{M}}$ such that every g_n is measurable with respect to $\underline{\underline{M}}_o$. Let $\{M_n\}_{n \geq 1}$ be a generator for $\underline{\underline{M}}_o$ and let $m : T \longrightarrow [0,1]$ be the characteristic function of the sequence $\{M_n\}_{n \geq 1}$.

Put $R = m(T)$. The map m is then a bimeasurable map with respect to $\underline{\underline{M}}_o$ and $\underline{\underline{B}}_R$ (that is, a measurable map that carries measurable sets to measurable sets). Thus, to each $g_n : T \longrightarrow \Sigma$ there corresponds a $\underline{\underline{B}}_R$-measurable map $h_n : R \longrightarrow \Sigma$ such that $g_n(t) = h_n(m(t))$, $t \in T$. Hence

$F(t) = cl(\{h_n(m(t))\}_{n \geq 1})$ for $t \in T$. By a well-known result [15; p. 434], each h_n extends to a $\underline{\underline{B}}_{[0,1]}$ - measurable map $h_n' : [0,1] \longrightarrow \Sigma$. Let

$$S = \{r \in [0,1] : \{h_n'(r)\}_{n \geq 1} \text{ is a dense-in-itself sequence}\}$$
$$= \{r \in [0,1] : (\forall p \geq 1)(\forall n)(\exists k)\,(0 < d(h_p'(r),\, h_k'(r)) < 1/_{(m+1)})\},$$

d being a metric on Σ.

It follows that S is a Borel subset of $[0,1]$. Moreover, as $\{g_n(t)\}_{n \geq 1}$ is dense-in-itself for each t, we have,

$\{h_n(r')\}_{n \geq 1} = \{h_n'(r')\}_{n \geq 1}$ is dense-in-itself for each $r' \in R$. Thus, $R \subseteq S$. Define a multifunction $H : S \longrightarrow \Sigma$ by :

$$H(r) = cl(\{h_1'(r),\, h_2'(r), \ldots\}).$$

Then, as is easily checked,

(i) H is a $\underline{\underline{B}}_S$ - measurable multifunction,

(ii) $Gr(H) \in \underline{\underline{B}}_S \otimes \underline{\underline{B}}_\Sigma$,

(iii) $H(s)$ is perfect (and, therefore, a dense-in-itself G_δ) for each $s \in S$, and

(iv) $H(m(t)) = F(t)$ for each $t \in T$.

Thus, as S is absolute Borel, Theorem 2.10 applies to the multifunction H (this is, for example, ensured by Remark 2.13). We therefore obtain a $\underline{\underline{B}}_S \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}$ - measurable Caratheodory map $k_o : S \times (\Sigma \dot{\cup} N) \longrightarrow \Sigma$ satisfying :

(a) $k_o(r,\cdot)$ is a one-one, continuous map on $\Sigma \bigcup N$ onto $H(r)$ for each $r \in S$, and

(b) $k_o(r,\cdot)$ restricted to Σ is a homeomorphism for $r \in S$.

Now define $k : T \times (\Sigma \bigcup N) \longrightarrow \Sigma$ by :

$$k(t,\sigma) = k_o(m(t),\sigma).$$

As $R \subseteq S$, the map k is well-defined on the whole of T. It is easily seen that this k satisfies the required conditions.

The following has been observed in [9].

3.2 <u>Lemma</u> : Let X, Y be Polish and let $A \subseteq X \times Y$ be an analytic set with closed sections. Then $A' = \{(x,y) : y \in$ the perfect kernel of $A^x\}$ is an analytic set.

<u>Proof</u> : Let $\{V_n\}_{n \geq 0}$ be a base for the topology of Y. Observe that

$$A' = \{(x,y) : (\forall n)(y \in V_n \longrightarrow V_n \bigcap A^x \text{ is uncountable})\}.$$

It is well known that the right hand side of the implication above is an analytic condition [15]. It follows that A' is an analytic set.

3.3 <u>Lemma</u> : Let X and Y be Polish. Suppose $C \subseteq X \times Y$ is a coanalytic set with open vertical sections. Then $C = \bigcup_{n \geq 1} C_n \times V_n$, with C_n coanalytic in X and V_n open in Y.

<u>Proof</u> : Let $\{V_n : n \geq 1\}$ be a base for Y.

Put $C_n = \{x \in X : V_n \subseteq C^x\} = X - \{x \in X : V_n \bigcap (Y - C^x) \neq \emptyset\}$.

As $(X \times Y - C)$ is an analytic set, and the projection of an analytic set is analytic, C_n is easily seen to be coanalytic in view of the last equality. Furthermore, as C^x is open for each x, $C = \bigcup_{n \geq 1} C_n \times V_n$.

<u>WARNING</u> : There is no 3.4. This has been suppressed from an earlier version as the lemma under this heading was found to be unnecessary.

3.5 <u>Lemma</u> : Let $(T, \underline{\underline{M}})$ be a measurable space and let X be Polish. Suppose $f : T \times \Sigma \longrightarrow X$ is an $\underline{\underline{M}} \,\bar{\otimes}\, \underline{\underline{B}}_\Sigma$ -measurable map such that for each $t \in T$, $f(t,.)$ is a homeomorphism on Σ into X. Then the canonical map $\hat{f} : T \times \Sigma \longrightarrow T \times X$ induced by f satisfies : $\hat{f}(M) \in \underline{\underline{M}} \,\bar{\otimes}\, \underline{\underline{B}}_X$ for each $M \in \underline{\underline{M}} \,\bar{\otimes}\, \underline{\underline{B}}_\Sigma$, where $\hat{f}(t,\sigma) = (t, f(t,\sigma))$.

<u>Proof</u> : Plainly, it suffices to show that the range of $\hat{f}$ is in $\underline{\underline{M}} \,\bar{\otimes}\, \underline{\underline{B}}_X$ (for then, as $\hat{f}$ preserves sections, the range of every basic rectangle would be so).

We begin by observing that for each open $W \subseteq X$, $f^{-1}(W)$ is a set in $\underline{\underline{M}} \,\bar{\otimes}\, \underline{\underline{B}}_\Sigma$ with open sections. Let $\{\sigma_n\}_{n \geq 1}$ be a dense subset of Σ. It is easy to see then that for each $(n_1, n_2, \ldots, n_k) \in$ Seq, the sets

$$\{t \in T : (f^{-1}(W))^t \subseteq \Sigma(n_1 n_2, \ldots, n_k)\}$$

$$= \{t \in T : (\forall n \geq 1)((t,\sigma_n) \in f^{-1}(W) \longrightarrow \sigma_n \in \Sigma(n_1 n_2, \ldots, n_k))\} \in \underline{\underline{M}},$$

and $\{t \in T : (f^{-1}(W))^t \neq \emptyset\} = \{t \in T : (\exists n)((t,\sigma_n) \in f^{-1}(W))\} \in \underline{\underline{M}}$.

Fix now for each $k \geq 1$, a countable base $\{V(k,n)\}_{n \geq 1}$ for X, so that for each $n \geq 1$, $\delta(V(k,n)) < 1/k$.

Define, for $k \geq 1$, $G_k \subseteq T \times X$ by :

$$(t,x) \in G_k \longleftrightarrow (\exists n \geq 1)(\exists (n_1, n_2, \ldots, n_k) \in S_k)$$

$$[x \in V(k,n) \text{ and } (f^{-1}(V(k,n)))^t \subseteq \Sigma(n_1 n_2, \ldots, n_k) \text{ and } (f^{-1}(V(k,n)))^t \neq \emptyset]$$

Put $G = \bigcap_{k \geq 1} G_k$. Then $G \in \underline{\underline{M}}\ (\ddot{x})\ \underline{\underline{B}}_X$.

<u>Claim</u> : Range of $\hat{f} = G$.

Fix $t \in T$. Let $(t,x) \in$ Range of $\hat{f}$. Then there is $(n_1, n_2, \ldots, n_k, \ldots) \in \Sigma$ such that $f(t, (n_1, n_2, \ldots)) = x$. As $\{V(k,n)\}_{n \geq 1}$ is a base for X and the inverse of $f(t,.)$ is continuous, it follows that there is $n \geq 1$ such that n and $(n_1, n_2, \ldots, n_k)$ witness the fact that $(t,x) \in G_k$. This holds for any k.

Conversely, suppose $(t,x) \in G_k$, for every $k \geq 1$. Then for each $k \geq 1$, we have a natural number m_k and $(n_1^k, \ldots, n_k^k) \in S_k$ such that $x \in V(k, m_k)$,

$$f(t,.)^{-1}(V(k,m_k)) \subseteq \Sigma\ (n_1^k, \ldots, n_k^k)$$

and $f(t,.)^{-1}(V(k,m_k)) \neq \emptyset$ for each $k \geq 1$.

Consider m_1 and (n_1^1). As $x \in V(k,m_k)$ for every $k \geq 1$ and $\delta(V(k,m_k)) < 1/k$, it follows that there is a $k > 1$ such that $V(k,m_k) \subseteq V(1,m_1)$. Therefore,

$$f(t,.)^{-1}(V(k,m_k)) \subseteq f(t,.)^{-1}(V(1,m_1)).$$

It follows that $\Sigma(n_1^k,\ldots,n_k^k) \cap \Sigma(n_1^1) \neq \emptyset$, and consequently that $(n_1^k,\ldots,n_k^k)$ is an extension of (n_1^1).

Applying the same argument to m_k and $(n_1^k,\ldots,n_k^k)$ and so on, we see that there is a single sequence $(n_1,n_2,\ldots) \in \Sigma$ and an increasing sequence $(k_1,k_2,\ldots)$ of positive integers such that

(i) for each $\ell \geq 1$, $(n_1,n_2,\ldots,n_{k_\ell}) = (n_1^{k_\ell},\ldots,n_{k_\ell}^{k_\ell})$,

(ii) $\emptyset \neq f(t,.)^{-1}(V(k_\ell, m_{k_\ell})) \subseteq \Sigma(n_1,\ldots,n_{k_\ell})$, and

(iii) $x \in V(k_\ell, m_{k_\ell})$

Put $y = f(t,(n_1,n_2,\ldots))$. The continuity of the map $f(t,.)$ now shows that $y=x$, and, therefore, that $(t,x) \in$ Range of $\hat{f}$. This proves the claim thereby completing the proof of Lemma 3.5.

The reader might have noticed that not surprisingly we have made no use of the completeness of X in the above argument.

The preliminaries complete, we will now state the first theorem of this section.

3.6 <u>Theorem</u> : Let T, X be Polish spaces and B a Borel subset of $T \times X$ such that B^t is uncountable for each $t \in T$. Then there is a map $h : T \times (\Sigma \uplus N) \longrightarrow X$ satisfying :

(i) h is $\underline{\underline{B}}(\underline{\underline{A}}(T)) \otimes \underline{\underline{B}}_{\Sigma \uplus N}$-measurable,

(ii) $h(t,.)$ is a one-one, continuous map on $\Sigma \dot{\cup} N$ onto B^t for each $t \in T$, and

(iii) if $\hat{h} : T\times(\Sigma \dot{\cup} N) \longrightarrow T\times X$ is the canonical map induced by h then $\hat{h}$ is $(\underline{\underline{B}}(\underline{\underline{A}}(T)) \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}, \underline{\underline{B}}(\underline{\underline{A}}(T)) \otimes \underline{\underline{B}}_X)$ - measurable and $\hat{h}^{-1}$ is $(\underline{\underline{B}}(\underline{\underline{A}}(T)) \otimes \underline{\underline{B}}_X|_B, \underline{\underline{B}}(\underline{\underline{A}}(T)) \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N})$ -measurable. [Recall $\hat{h}(t,y) = (t, h(t,y))$].

Step I We will first prove the theorem for sets contained in $T\times\Sigma$ with closed vertical sections. So let $B \subseteq T\times\Sigma$ be such that B^t is closed and uncountable for each $t \in T$. Let A be the perfect kernel of B, that is define $A \subseteq T\times\Sigma$ by :

$$A^t = \text{perfect kernel of } B^t.$$

By Lemma 3.2, A is an analytic subset of $T\times\Sigma$ with non-empty and perfect sections. Thus $(T\times\Sigma) - A$ is a coanalytic set with open sections and we may write, in view of Lemma 3.3,

$$T\times\Sigma - A = \bigcup_{n\geq 1} (S_n\times V_n),$$

where S_n is coanalytic in T and V_n is open in Σ.

Consequently, $B - A = \bigcup_{n\geq 1} (B\cap(S_n\times V_n))$

Now, $B\cap(T\times V_n)$ is a Borel set having countable vertical sections on S_n and Lemma 1.2, therefore, shows that each $B\cap(S_n\times V_n)$ is a countable union of graphs of (partial) $\underline{\underline{B}}(\underline{\underline{A}}(T))$-measurable functions each defined on a set in $\underline{\underline{B}}(\underline{\underline{A}}(T))$ [Note that each S_n being coanalytic is in $\underline{\underline{B}}(\underline{\underline{A}}(T))$]. It follows that $B-A$ is a countable union of graphs of such (partial) functions, say, $g_1, g_2, \ldots$.

If we now define a multifunction $F : T \longrightarrow \Sigma$ by :

$$F(t) = A^t,$$

then F is a non-empty, perfect set valued multifunction measurable with respect to $\underline{\underline{B}}(\underline{\underline{A}}(T))$. Lemma 3.1 plainly applies to yield a map $k : T \times (\Sigma \dot{\cup} N) \longrightarrow \Sigma$ such that

(a) k is $\underline{\underline{B}}(\underline{\underline{A}}(T)) \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}$ - measurable,

(b) for each $t \in T$, $k(t,.)$ is a one-one, continuous map on $\Sigma \dot{\cup} N$ onto A^t, and

(c) for each $t \in T$, the restriction of $k(t,.)$ to Σ is a homeomorphism.

Put $k_n(t) = k(t,n)$ for $n \in N$. Then each k_n is a $\underline{\underline{B}}(\underline{\underline{A}}(T))$ - measurable function on T into Σ. Further, for each $t \in T$, $\{g_n(t)\}_{n \geq 1} \cup \{k_n(t)\}_{n \geq 1}$ is infinite (since $\{k_n(t)\}_{n \geq 1}$ is so). It follows that there exist $\underline{\underline{B}}(\underline{\underline{A}}(T))$ - measurable (total) functions $f_n : T \longrightarrow \Sigma$ with disjoint graphs such that

$$\bigcup_{n \geq 1} Gr(f_n) = (\bigcup_{n \geq 1} Gr(g_n)) \cup (\bigcup_{n \geq 1} Gr(k_n)).$$

Finally define $h : T \times (\Sigma \dot{\cup} N) \longrightarrow \Sigma$ by

$$h(t,\sigma) = k(t,\sigma), \quad \text{if } \sigma \in \Sigma$$

$$h(t,n) = f_n(t), \quad \text{if } n \in N.$$

Then, for each $t \in T$, $h(t,.)$ is a map on $\Sigma \biguplus N$ onto B^t. It is easily seen that h satisfies the conditions (i) and (ii) specified in the theorem. The measurability of the map $\hat{h}$ specified in (iii) is an easy consequence of the measurability of h and the product structure of the σ-field sitting on the range of $\hat{h}$. For if $A_n \times W_n$ is a basic rectangle in $\underline{\underline{B}}(\underline{\underline{A}}(T)) \otimes \underline{\underline{B}}_\Sigma$, then $\hat{h}^{-1}(A_n \times W_n) = (A_n \times \Sigma \biguplus N) \cap h^{-1}(W_n)$, as $\hat{h}$ preserves sections. As for the measurability of the map $\hat{h}^{-1}$ specified in the theorem, notice that it suffices to see that $h|_{T\times N}$ and $h|_{T\times\Sigma}$ are both " forward-measurable ". The former follows from the fact that the graph of a $\underline{\underline{B}}(\underline{\underline{A}}(T))$-measurable function is in $\underline{\underline{B}}(\underline{\underline{A}}(T)) \otimes \underline{\underline{B}}_\Sigma$ and the latter is an immediate consequence of Lemma 3.5 as $h|_{T\times\Sigma} = k|_{T\times\Sigma}$, and $k(t,.)$ is a *homeomorphism for each* t.

<u>Step II</u> We will now establish the general case by reducing it to the case treated in Step I, as in Cenzer and Mauldin. So we now have $B \subseteq T \times X$ with uncountable vertical sections as in the theorem. As B is absolute Borel, it is the image under a one-one, continuous map f of a closed subset C of Σ. Denote by W the set $\{(t,\sigma) \in T\times C : \pi_T(f(\sigma)) = t\}$. Then W is a Borel subset of $T\times\Sigma$ with uncountable, closed sections, and the function $g : W \longrightarrow B$ defined by $g(t,\sigma) = f(\sigma)$ is a Borel isomorphism of W onto B such that $\pi_X \circ f$ maps W^t onto B^t in a one-one, continuous manner. By Step I, W can be " parametrized " by a map $e : T\times\Sigma \biguplus N \longrightarrow \Sigma$ satisfying all the conditions specified in the theorem. Define $h : T\times\Sigma \biguplus N \longrightarrow X$ by :

$$h(t,y) = \pi_X \circ f \circ e(t,y) .$$

The map h clearly satisfies (i) and (ii). To see (iii), observe that $h(t,y) = g \circ e\,(t,y)$. As g preserves sections, for any basic rectangle in $\mathbf{B}(\mathbf{A}(T)) \otimes \mathbf{B}_\Sigma$, say $M_n \times W_n$, $g(M_n \times W_n) = g(T \times W_n) \cap (M_n \times X)$. As g is a Borel isomorphism, this last set is in $\mathbf{B}(\mathbf{A}(T)) \otimes \mathbf{B}_X$. The map g therefore carries sets in $\mathbf{B}(\mathbf{A}(T)) \otimes \mathbf{B}_\Sigma$ to sets in $\mathbf{B}(\mathbf{A}(T)) \otimes \mathbf{B}_X$. It follows immediately that $\hat{h}^{-1}$ has the required measurability. Similarly $\hat{h}$ has the required measurability. This completes the proof.

3.7 <u>Remark</u> : As observed at the beginning of this section the Von Neumann selection theorem is the best possible selection theorem for an arbitrary Borel set, in the sense that the analytic σ-field is the smallest natural σ-field with respect to which one can always get a measurable selection. We are not interested here in the descriptive nature of the graph of the selection. Our theorem 3.6 therefore cannot be improved in the general setting in which it is stated. Furthermore, Mauldin [23] has obtained interesting necessary and sufficient conditions for a Borel set in the product to be Borel parametrizable. Theorem 3.6 together with this result of Mauldin, therefore, seems to be a complete solution to the problem of parametrizing Borel sets in the product of two Polish spaces.

We might also add that in [9] Cenzer and Mauldin had also shown that any analytic set with uncountable sections contains $2^{\aleph_0}$ disjoint selections measurable with respect to the analytic σ-field by obtaining a parametrization that goes into the analytic set.

3.8 Remark : Mauldin [loc.cit.] has shown that a Borel set with uncountable vertical sections is Borel parametrizable if and only if it contains a Borel set with compact and perfect vertical sections. The question then arises : when does a Borel set with uncountable sections in the product of two Polish spaces admit a one-one Borel measurable Caratheodory representation ? In particular, do the two notions coincide ?

For Borel sets in $T\times\Sigma$ with closed, uncountable sections the answer is fairly simple. Let $B \subseteq T\times\Sigma$ be a Borel set with closed and uncountable vertical sections, with T Polish. Then the following are equivalent :

(a) there is a Borel measurable map f on $T\times(\Sigma \uplus N)$ into Σ such that $f(t,.)$ is a one-one, continuous map on $\Sigma \uplus N$ onto B^t for $t\in T$.

(b) The perfect kernel of B defined by :

$$(t,x)\in A \Longleftrightarrow x\in \text{ perfect kernel of } B^t$$

is the graph of a Borel measurable multifunction.

(b) $\Longrightarrow$ (a) is an immediate consequence of Lemma 3.1 and the fact that $B - A$ being a Borel set with countable sections is a countable union of Borel graphs. To see (a) $\Rightarrow$ (b) let $\{\sigma_n\}_{n \geq 1}$ be a dense subset of Σ. Then the perfect kernel A of B is

$$\left\{(t,x) : x \in cl \{f(t,\sigma_n) : n \geq 1\}\right\}.$$

As each $f(\cdot,\sigma_n)$ is $\underline{\underline{B}}_T$-measurable, the result follows.

It follows immediately now that there is a set B which is Borel parametrizable but not induced by a one-one Borel measurable Caratheodory map. To see this define $B \subseteq \Sigma \times \Sigma$ by $B = C \cup D$ where C is a Borel set with compact, perfect vertical sections contained in $\Sigma \times \Sigma(1)$ and D is a Borel set with uncountable closed sections contained in $\Sigma \times \Sigma(2)$ that is not Borel uniformizable. The validity of (b) for B would imply that D contains a perfect valued Borel measurable multifunction, so that D would then be Borel uniformizable. On the other hand, according to a result of Mauldin [23], B is Borel parametrizable.

3.9 <u>Remark</u> : As already pointed out the first appearance of a " parametrization " result was in Purves' proof [27] of the converse to Lusin's theorem on countable to one measurable maps. As an interesting aside we will outline an argument for this beautiful result, partly as an illustration of how parametrization problems arise in descriptive set theory. While the proof we will present uses essentially the same ideas as that of Purves it is now possible to bring about considerable simplifications in the technical details.

We will prove the contrapositive of the result, namely that if a measurable map f on one Polish space into another takes on uncountably many values of f uncountably often, then f is not bimeasurable. This will be done in several steps.

<u>Step I</u> : It is easily seen that one may assume that f is a continuous map on a closed subspace C of Σ into Σ. Furthermore, as the set of values of uncountable order of a measurable function is an analytic set [15] and every uncountable analytic set contains a copy of Σ, due to the assumptions of " uncountability " made on f, we may further take it that f is on C onto Σ and every value of f is taken on uncountably often.

<u>Step II</u> : Define now $G \subseteq \Sigma \times C$ by :

$$G = \{(y,x) : f(x) = y\}.$$

As f is now continuous and uncountable to one, G is a Borel subset of $\Sigma \times C$ with closed and uncountable vertical sections. Furthermore, $\pi_C : G \longrightarrow C$ is one-one, and therefore a Borel isomorphism.

<u>Step III</u> : Observe now that it suffices to obtain an uncountable Borel set $T \subseteq \Sigma$ and a Borel isomorphism g on $T \times 2^{\omega}$ into G such that $g(t,.)$ maps 2^{ω} into $\{t\} \times G^t$ for each $t \in T$.

To see this notice that as T is uncountable and Borel, there is a Borel subset D of $T \times 2^{\omega}$ such that $\pi_T(D)$ is an analytic, non-Borel set. Then as g is a Borel isomorphism and preserves sections $g(D)$ is Borel and $\pi_T(g(D)) = \pi_T(D)$ is not Borel. However as π_C is a Borel isomorphism on $G \longrightarrow C$ and $g(D) \subseteq G$, $\pi_C(g(D)) = E$, say, is Borel in C. It is easily seen that $f(E) = \pi_\Sigma(g(D)) = \pi_T(g(D))$. It follows that $f(E)$ is not Borel. Thus, f is not bimeasurable.

Step IV : Towards getting hold of g and T let λ be a non-atomic probability measure on Σ, and let A be the (sectionwise) perfect kernel of G as in Lemma 3.2. Then A is an analytic set with non-empty, perfect sections. Fix a base $\{V_n\}_{n \geq 1}$ for C. Let $A_n = \pi_\Sigma(A \cap (\Sigma \times V_n))$. Each A_n is analytic and therefore λ-measurable. Easy arguments now show that there is a Borel λ-null set N such that $A_n - N$ is Borel for each $n \geq 1$. Let $T = \Sigma - N$ and $B_1 = A \cap (T \times C)$. It follows then that B_1 is the graph of a perfect valued Borel measurable multifunction on T into C. As $B_1 \subseteq G$ it suffices now to define a Borel isomorphism g on $T \times 2^{\omega}$ into B_1 preserving sections.

Step V : Standard arguments now yield the map g on $T \times 2^{\omega}$ into B_1. One has only to carry out " uniformly " the method of getting a copy of the Cantor set inside a perfect set as in Mauldin [23] and Srivastava [31]. For completeness we outline a proof.

Fix a clopen base $\{V_n : n \geq 1\}$ for C such that $V_1 = C$, and a metric d on C compatible with its topology such that $\delta(C) < 1$. We will define a system of positive integer valued functions coded by finite sequences of 0's and 1's, $\{p_s : s \in 2^{<\omega}\}$ satisfying :

(i) $p_s : T \longrightarrow N$ is $\underline{\underline{B}}_T$-measurable, for each s.

(ii) $\delta(V_{p_s(t)}) < \frac{1}{2^{|s|}}$, for each s.

(iii) $|s_1| = |s_2|$ & $s_1 \neq s_2 \Longrightarrow V_{p_{s_1}(t)} \bigcap V_{p_{s_2}(t)} = \emptyset$.

(iv) $s_1 \prec s_2 \Longrightarrow V_{p_{s_2}(t)} \subseteq V_{p_{s_1}(t)}$.

(v) $V_{p_s(t)} \bigcap B_1^t \neq \emptyset$ for each s.

To see that such a system exists put $p_e = 1$.

Suppose p_s has been defined for some $s \in 2^{<\omega}$, satisfying the above properties.

Define $p_{s0}(t)$ = least $m \geq 1$ for which there is $n \geq 1$ such that $\delta(V_m), \delta(V_n) < \frac{1}{2^{|s|+1}}$; $V_m \bigcap V_n = \emptyset$;

$$V_m \subseteq V_{p_s(t)}, \quad V_n \subseteq V_{p_s(t)},$$

and $V_m \bigcap B_1^t \neq \emptyset$, $V_n \bigcap B_1^t \neq \emptyset$.

As B_1^t is perfect and by induction hypothesis, $V_{p_s(t)} \bigcap B_1^t \neq \emptyset$, for each t, there is m such that $p_{s0}(t) = m$. As B_1 is the graph of a measurable multifunction, it is easy to see that p_{s0}

is $\underline{\underline{B}}_T$ - measurable. Now define,

$$p_{s1}(t) = \text{least } n \text{ which witnesses the fact that } p_{so}(t) = n, \text{ if } p_{so}(t) = n.$$

p_{so}, p_{s1} so defined satisfy the required conditions.

Now put $g(t,(i_1,i_2,\ldots)) = \left(t, \bigcap V_{p_{\langle i_1,i_2,\ldots,i_{n-1}\rangle}(t)}\right)$, for $(i_1,i_2,\ldots) \in 2^\omega$. This does the job.

The reader might have noticed that we could have invoked Theorem 3.6 to " parametrize " G and then " cut it down " to get a suitable Borel parametrization into G or used Lemma 3.1 in Step V. We have refrained from doing so only to clarify that one does not require all that has gone into these results.

We turn now to a solution of Ioffe's question on whether the theorem of Cenzer and Mauldin can be unified with the theorem of Ioffe and Bourgain referred to in the introduction to this section, by replacing the complete σ-field therein by an arbitrary σ-field closed under operation $(\underline{\underline{A}})$. We establish an abstract theorem which amounts to showing that Theorem 3.6 holds even when T is an arbitrary subset of a Polish space.

If $\underline{\underline{N}}$ is a σ-field on a set T, then $\underline{\underline{A}}(\underline{\underline{N}})$ will denote the family of all subsets of T obtained as the result of operation $(\underline{\underline{A}})$ performed on a system of sets from $\underline{\underline{N}}$.

Our abstract theorem can then be formulated as follows :

3.10 Theorem : Let $(T, \underline{\underline{N}})$ be a measurable space. Let X be Polish and suppose $B \in \underline{\underline{N}} \otimes \underline{\underline{B}}_X$ has uncountable vertical sections. Let $\underline{\underline{M}} = \sigma(\underline{\underline{A}}(\underline{\underline{N}}))$. Then there is a map $h : T \times (\Sigma \dot{\cup} N) \longrightarrow X$ satisfying

(i) h is $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}$ - measurable,

(ii) $h(t,.)$ is a one-one, continuous map of $\Sigma \dot{\cup} N$ onto B^t for each $t \in T$, and

(iii) if $\hat{h} : T \times (\Sigma \dot{\cup} N) \longrightarrow T \times X$ is the canonical map induced by h, then $\hat{h}$ is $(\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}, \underline{\underline{M}} \otimes \underline{\underline{B}}_X)$-measurable and $\hat{h}^{-1}$ is $(\underline{\underline{M}} \otimes \underline{\underline{B}}_X|_B, \underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N})$ - measurable.

Proof : As the proof is similar to that of Theorem 3.6, we will only outline the proof. Notice that as $B \in \underline{\underline{N}} \otimes \underline{\underline{B}}_X$, there are countably many rectangles $\{T_n \times V_n\}_{n \geq 1}$, with $T_n \in \underline{\underline{N}}$ and V_n open in X such that

$$B \in \sigma\text{-field generated by } \{T_n \times V_n\}_{n \geq 1} .$$

Let $\underline{\underline{N}}_o = \sigma(\{T_n\}_{n \geq 1})$ and let $m : (T, \underline{\underline{N}}_o) \longrightarrow [0,1]$ be the characteristic function of the sequence $\{T_n\}$. Put $m(T) = R$ and let $W \subseteq R \times X$ be defined by

$$W = \{(m(t),x) : (t,x) \in B\} .$$

By the properties of m, $W \in \underline{\underline{B}}_R \otimes \underline{\underline{B}}_X$ and W^r is uncountable for each $r \in R$. Also observe that

$$m^{-1}(\underline{\underline{B}}(\underline{\underline{A}}(R))) \subseteq \sigma(\underline{\underline{A}}(\underline{\underline{N}}_o)) \subseteq \sigma(\underline{\underline{A}}(\underline{\underline{N}})) = \underline{\underline{M}}$$

In this argument we will conduct several " transfers " and at each step claim that it suffices to parametrize the set obtained at that stage by a map with the appropriate measurability properties. These are easy enough to see due to the product structures of the σ-fields we are considering and the fact that all the maps we construct preserve sections as in the proof of Theorem 3.6. At the first step we claim that it is enough now to parametrize W by a map h satisfying all the conditions specified in Theorem 3.6 with T replaced by R and B by W. To see this one must bear in mind the above observations since, for example, the composition of analytic σ-field measurable maps need not be measurable with respect to the analytic σ-field.

To see now the existence of the required parametrization for W one has only to follow the various steps in the proof of Theorem 3.6.

As $W \in \underline{\underline{B}}_R \otimes \underline{\underline{B}}_X$, there is absolute Borel $W' \in \underline{\underline{B}}_{[0,1]} \otimes \underline{\underline{B}}_X$ such that $W' \cap (R \times X) = W$. Now find $B' \subseteq [0,1] \times \Sigma$, $B' \in \underline{\underline{B}}_{[0,1]} \otimes \underline{\underline{B}}_\Sigma$ with closed vertical sections and a Borel isomorphism g of B' onto W' as in the proof of Theorem 3.6. Notice that B'^t is uncountable whenever $(W')^t$ is so. Thus, as before, the problem of parametrizing W' on R reduces to that of parametrizing B on R.

Once again define $A \subseteq [0,1] \times \Sigma$ by :

$$A = \{(t,x) : x \in \text{ perfect kernel of } B'^t\}.$$

Arguments identical to the ones used in Theorem 3.6 now show that $B' - A$ can be written as a countable union of graphs of $\underline{\underline{B}}(\underline{\underline{A}}([0,1]))$-measurable functions, and consequently that $(B' - A) \cap (R \times \Sigma)$ is a countable union of graphs of $\underline{\underline{B}}(\underline{\underline{A}}(R))$ - measurable functions. On the other hand A being an analytic set with perfect sections, Lemma 3.1 applies to yield a suitable parametrization of A on $\pi_{[0,1]}(A)$. As B'^t is uncountable for each $t \in R$, we have $R \subseteq \pi_{[0,1]}(A)$. Thus A and therefore B' can be parametrized on R. This completes the proof.

Now if $\underline{\underline{M}}$ is a σ-field closed under operation $(\underline{\underline{A}})$ then $\sigma(\underline{\underline{A}}(\underline{\underline{M}})) \subseteq \underline{\underline{M}}$. The following is therefore immediate and answers Ioffe's question.

3.11 <u>Theorem</u> : Let $(T, \underline{\underline{M}})$ be a measurable space, $\underline{\underline{M}}$ being a σ-field closed under operation $(\underline{\underline{A}})$. Let X be Polish and suppose $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$ has uncountable vertical sections. Then there is a map $h : T \times (\Sigma \dot{\cup} N) \longrightarrow X$ satisfying :

(i) h is $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}$ - measurable,

(ii) $h(t,.)$ is a one-one, continuous map of $\Sigma \dot{\cup} N$ onto B^t for each $t \in T$, and

(iii) if $\hat{h} : T \times (\Sigma \dot{\cup} N) \longrightarrow T \times X$ is the canonical map induced by h, then $\hat{h}$ is $(\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}, \underline{\underline{M}} \otimes \underline{\underline{B}}_X)$ -measurable, and $\hat{h}^{-1}$ is $(\underline{\underline{M}} \otimes \underline{\underline{B}}_X|_B, \underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N})$-measurable.

Not surprisingly maps h with the above properties induce only sets in $\underline{\underline{M}} \otimes \underline{\underline{B}}_X$ with uncountable vertical sections. This simple consequence we state as the next theorem.

3.12 Theorem : Let $(T, \underline{\underline{M}})$ be a measurable space, $\underline{\underline{M}}$ being a σ-field closed under operation $(\underline{\underline{A}})$. Let X be Polish. Suppose $B \subseteq T \times X$. Then the following are equivalent.

(a) $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$, and B has uncountable vertical sections.

(b) There is a map $h : T \times (\Sigma \dot{\cup} N) \longrightarrow X$ satisfying properties (i), (ii), and (iii) of Theorem 3.11.

Proof : (a) $\Longrightarrow$ (b) is just Theorem 3.11. On the other hand, (b) $\Longrightarrow$ (a) holds for any σ-field $\underline{\underline{M}}$. One only needs the fact that h preserves sections. To see this, fix a countably generated sub σ-field $\underline{\underline{M}}_0$ of $\underline{\underline{M}}$ such that h is $\underline{\underline{M}}_0 \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}$-measurable. Then $\hat{h}$ is $(\underline{\underline{M}}_0 \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}, \underline{\underline{M}}_0 \otimes \underline{\underline{B}}_X)$-measurable. Now get a countably generated sub σ-field $\underline{\underline{M}}_1$ of $\underline{\underline{M}}$ such that $\hat{h}^{-1}$ is $(\underline{\underline{M}}_1 \otimes \underline{\underline{B}}_X|_B, \underline{\underline{M}}_0 \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N})$-measurable. Put $\underline{\underline{N}} = \underline{\underline{M}}_0 \vee \underline{\underline{M}}_1$. Then h satisfies (i), (ii) and (iii) of Theorem 3.11 with $\underline{\underline{M}}$ replaced by $\underline{\underline{N}}$. We will check the measurability of the map $\hat{h}^{-1}$. The other arguments are similar. So fix $P \in \underline{\underline{N}}$ and V open in $\Sigma \dot{\cup} N$. Then $\hat{h}(P \times V) = (P \times X) \cap \hat{h}(T \times V)$, as h preserves sections. As $T \times V \in \underline{\underline{M}}_0 \otimes \underline{\underline{B}}_{\Sigma \dot{\cup} N}$, we have

$\hat{h}(T \times V) \in \underline{\underline{M}}_1 \otimes \underline{\underline{B}}_X|_B$. It follows that $\hat{h}(P \times V)$ is in $\underline{\underline{N}} \otimes \underline{\underline{B}}_X|_B$ (as P is in $\underline{\underline{N}}$ and $\underline{\underline{M}}_1 \subseteq \underline{\underline{N}}$).

Let f be the characteristic function of a generator for $\underline{\underline{N}}$, and put $Y = f(T) \subseteq \mathbb{R}$. Define the [redacted] map $g : Y \times \Sigma \biguplus N \longrightarrow X$ by :

$$g(y, z) = h(t, z) \quad \text{if} \quad t \in f^{-1}(y).$$

Then g is well-defined and if $C \subseteq Y \times X$ is defined by $C = \emptyset(B)$ where $\emptyset : T \times X \longrightarrow Y \times X$ is given by $\emptyset(t,x) = (f(t),x)$, then by the bimeasurability of f, g satisfies (i), (ii) and (iii) with $(T, \underline{\underline{N}})$ replaced by $(Y, \underline{\underline{B}}_Y)$ and B replaced by C. Finally, by (iii), g is a Borel isomorphism of $Y \times (\Sigma \biguplus N)$ and C, preserving sections. By a well-known extension theorem [15], extend g to a Borel isomorphism g_1 on $\mathbb{R} \times (\Sigma \biguplus N)$ into $\mathbb{R} \times X$ and look at

$$D = \left\{ (r,z) : \pi_1(g_1(r,z)) = r \right\}.$$

Then D is absolute Borel and $B = \emptyset^{-1}(g_1(D) \bigcap (Y \times X))$. It follows that $B \in \underline{\underline{N}} \otimes \underline{\underline{B}}_X \subseteq \underline{\underline{M}} \otimes \underline{\underline{B}}_X$. This completes the proof.

We conclude by observing that our results are also optimal in the sense that the condition in Theorem 3.12 that $\underline{\underline{M}}$ be closed under operation $(\underline{A})$ is necessary if one requires that the map h be measurable with respect to the product σ-field.

3.13 <u>Proposition</u> : Let $(T, \underline{\underline{M}})$ be a measurable space. Suppose for every $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_\Sigma$ with B^t uncountable for each $t \in T$, there is an $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot\cup N}$ - measurable map $h : T \times \Sigma \dot\cup N \longrightarrow \Sigma$ such that $h(t,.)$ is continuous and onto B^t, $t \in T$. Then $\underline{\underline{M}}$ is closed under operation $(\underline{\underline{A}})$.

Proof : Let $\{A_{n_1 n_2 \ldots n_k}\}$ be a system of sets with $A_{n_1 n_2 \ldots n_k} \in \underline{\underline{M}}$ for each $n_1, n_2, \ldots, n_k$. Then, as is well-known, there is $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_\Sigma$ such that $\pi_T(B) = \underline{\underline{A}}(A_{n_1 n_2 \ldots n_k})$. For, one need only take $B = \bigcap_{k \geq 1} \bigcup_{S_k} (A_{n_1 n_2 \ldots n_k} \times \Sigma(n_1 n_2, \ldots, n_k))$. As Σ is homeomorphic to $\Sigma(0)$, we may without loss of generality, assume $B \subset T \times \Sigma(0)$. Now put $C = B \cup (T \times \Sigma(1))$. Then $C \in \underline{\underline{M}} \otimes \underline{\underline{B}}_\Sigma$ and C^t is uncountable for each t. Let h be an $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot\cup N}$ - measurable map on $T \times (\Sigma \dot\cup N) \longrightarrow \Sigma$ inducing C as in the theorem. Let $D = \{(t,\sigma) : h(t,\sigma) \in B^t\}$. As h preserves sections, $\pi_T(D) = \pi_T(B)$. Also, B^t is clopen in C^t, $t \in T$. Thus, as $h(t,.)$ is continuous, D^t is open for each t. Also as h is $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot\cup N}$ - measurable, $D \in \underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma \dot\cup N}$. Fix now a dense set $\{r_n : n \geq 1\}$ in $\Sigma \dot\cup N$. Then

$$\pi_T(B) = \pi_T(D) = \{t \in T : (t, r_n) \in D \text{ for some } n \geq 1\} \in \underline{\underline{M}}.$$

Thus $\underline{\underline{A}}(\{A_{n_1 n_2 \ldots n_k}\}) \in \underline{\underline{M}}$, and the proof is complete.

4. We have so far considered the parametrization by appropriate functions of sets B in a product space $T \times X$ when (a) each B^t is countable, and (b) each B^t is uncountable. To motivate the results of this section we recall the following result of Sierpinski [33]: A subset of $\mathbb{R}$ is condensed and Borel if and only if it is a one-one continuous image of Σ. One might therefore ask for an analogous parametrization of B when (c) each B^t is condensed. Specifically, we improve some of our earlier results by showing that the dense-in-itself G_δ-valued multifunction in Corollary 2.11 is induced by an $\underline{\underline{A}} \otimes \underline{\underline{B}}_\Sigma$-measurable one-one Caratheodory map* defined on $T \times \Sigma$ and also that if in Theorem 3.11 it is assumed that the set B has condensed sections, then B is induced by a one-one Caratheodory map* defined on $T \times \Sigma$ with all the measurability properties specified in Theorem 3.11. These answer in part questions raised in [44]. We will adapt Sierpinski's proof so that, on the one hand, it goes through for any Polish space and, on the other, can be carried over " uniformly ".

<u>In what follows X denotes a fixed Polish space. We also fix the following notation : $(T, \underline{\underline{M}})$ will stand for a measurable space, $h : T \times \Sigma \longrightarrow X$ a fixed $\underline{\underline{M}} \otimes \underline{\underline{B}}_\Sigma$-measurable, one-one Caratheodory map*, and $s_0 : T \longrightarrow X$ and $\underline{\underline{M}}$-measurable map such that</u>

(i) <u>$s_0(t) \notin \text{Range}\,(h(t,.))$ for $t \in T$, and</u>

(ii) <u>$s_0(t) \in \text{cl}\,(\text{Range}\,(h(t,.)))$ for $t \in T$.</u>

Footnote : * A Caratheodory map $h : T \times \Sigma \to X$ is <u>one-one</u> if for each $t \in T$, the map $h(t,\cdot)$ is one-one.

Put $C = \{(t,x) \in T \times X : (\exists \sigma \in \Sigma)(h(t,\sigma) = x)\}$ and $B = C \cup Gr(s_o)$. Finally, let d be a fixed metric on X.

We will break up the proof into several lemmas. Furthermore, we will state them so that they apply to both the situations when $\underline{M}$ is a σ-field closed under operation $(\underline{A})$ and also when T is Polish and $\underline{M}$ a countably generated sub σ-field of its Borel σ-field.

Before plunging into the technicalities it would be best to state roughly what we intend to do. In 2.11 and 3.11 we have a one-one Caratheodory map f on $T \times (\Sigma \dot{\cup} N)$ parametrizing a given set G. Let f' be the restriction of f to $T \times \Sigma$. The first step is then to redefine f' so that it takes each $\Sigma(\langle n \rangle)$ to a dense subset of G^t. As G^t is condensed, $f(t,n)$ is a limit point of G^t and therefore of $f'(t,\Sigma(\langle n \rangle))$ for each n, for the redefined f'. Each $f(\cdot,n)$ will then correspond to a function s_o with the properties listed above where for h we take f'_n, the restriction of f' to $\Sigma(\langle n \rangle)$. Fix n. It is easy to see then that to complete the proof it suffices to redefine h to get h' defined on $T \times \Sigma(\langle n \rangle)$ inducing B, and then to put the h''s and B's together.

In what follows, unless otherwise stated, $\underline{M}$ is any σ-field.

4.1 <u>Lemma</u> : Suppose $r : T \longrightarrow X$ is $\underline{M}$-measurable. Then $\{(t,x) : d(x,r(t)) < \varepsilon\}$ is a set in $\underline{M} \otimes \underline{B}_X$ with non-empty and open vertical sections for each $\varepsilon > 0$.

Proof : This is clear.

4.2 Lemma : Suppose $P \subseteq T \times \Sigma$, $P \in \underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma}$, and P^t is non-empty and open for each $t \in T$. Assume that for each open $V \subseteq \Sigma$, $\{t \in T : V \subseteq P^t\} \in \underline{\underline{M}}$. Then there is a map $\varphi : T \times \Sigma \to \Sigma$ such that

(i) φ is $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma}$-measurable,

(ii) for each $t \in T$, $\varphi(t,.)$ is a homeomorphism of Σ and P^t.

Proof : Fix an enumeration $u_1, u_2, \ldots\ldots$ of Seq.

Define

$$T_n = \{t \in T : \Sigma(u_n) \subseteq P^t \text{ and } \Sigma(s) \nsubseteq P^t \text{ for every } s \prec u_n\}.$$

Then $T_n \in \underline{\underline{M}}$ (by our assumptions on $\underline{\underline{M}}$).

Define $\underline{\underline{M}}$-measurable maps $\{p(t,n),\ n \geq 1\}$ taking values in Seq $\cup \{\infty\}$ as follows (we put $\Sigma(\infty) = \emptyset$) :

$p(t,1) = u_m$ if m is the first integer ℓ such that $t \in T_\ell$

and for $n \geq 1$,

$p(t,n+1) = u_m$ if m is the first integer $\ell > p(t,n)$ such that $t \in T_\ell$, if there is one such

$= \infty$ otherwise .

As each $T_\ell \in \underline{\underline{M}}$, the maps $p(t,n)$ are $\underline{\underline{M}}$-measurable. Further, they satisfy :

(a) $\{\Sigma(p(t,n)) : n \geq 1\}$ is a discrete family in the topology of its union, and

(b) $P^t = \bigcup_{n \geq 1} \Sigma(p(t,n))$, $t \in T$.

Let $R_o = \{t \in T : p(t,n) \neq \infty$ for every $n \geq 1\}$.

$R_n = \{t \in T : p(t,n) \neq \infty$ and $p(t,n+1) = \infty\}$, $n \geq 1$.

Notice that $\bigcup_{n \geq 0} R_n = T$, the R_n's are pairwise disjoint, and $R_n \in \underline{\underline{M}}$, $n \geq 0$.

For each $u, v \in$ Seq, fix a homeomorphism on $\Sigma(u)$ onto $\Sigma(v)$ say $h(u,v)$.

Define $f_o : R_o \times \Sigma \longrightarrow \Sigma$ by

$$f_o(t,\sigma) = h(\langle k \rangle, p(t,k))(\sigma) \text{ if } \sigma \in \Sigma(\langle k \rangle).$$

As the $p(t,n)$'s are $\underline{\underline{M}}$-measurable, f_o has the following properties :

(1) $f_o(t,.)$ is a homeomorphism of Σ onto P^t for $t \in R_o$,

(2) f_o is $\underline{\underline{M}}|R_o \otimes \underline{\underline{B}}_\Sigma$-measurable.

As any finite disjoint union of basic clopen sets $\Sigma(s)$ in Σ is a homeomorph of Σ, for each $n \geq 1$, we can similarly construct maps $f_n : R_n \times \Sigma \longrightarrow \Sigma$ with the above properties. Piecing together these maps we obtain a map q satisfying the required conditions.

4.3 Remark : If in Lemma 4.2 we assume that each P^t is clopen, then the condition $\{t : U \subseteq P^t\} \in \underline{\underline{M}}$ for U open in Σ always holds. To see this, notice that it suffices to have $\{t : U \cap (\Sigma - P^t) \neq \emptyset\} \in \underline{\underline{M}}$. Let $\{r_n\}_{n \geq 1}$ be a dense subset of Σ. Then this second condition holds iff $(\exists n)(r_n \in U \,\&\, r_n \notin P^t)$, as both U and $\Sigma - P^t$ are open. The result follows. The conclusion of the lemma consequently holds in this situation. This will be used in the sequel.

4.4 Lemma : Suppose $P \subseteq T \times \Sigma$ is in $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma}$, and has non-empty, open sections. Then P has an $\underline{\underline{M}}$-measurable selector.

Proof : Fix a countable dense set $\{r_n\}_{n \geq 1}$ in Σ. Put $T_n = \{t : (t, r_n) \in P\}$. Then $T_n \in \underline{\underline{M}}$, $n \geq 1$, and $\bigcup_{n \geq 1} T_n = T$ as each P^t is open and non-empty. As $\underline{\underline{M}}$ is a σ-field there are sets R_n such that $R_n \subseteq T_n$, $\bigcup_{n \geq 1} R_n = \bigcup_{n \geq 1} T_n$, and the R_n's are disjoint. Set $f = r_n$ on R_n, $n \geq 1$. Then f is an $\underline{\underline{M}}$-measurable selector.

4.5 Lemma : Let $S_o \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$ with S_o^t open for every t. Suppose $s_o(t) \in S_o^t$, $t \in T$. Then there is a sequence of $\underline{\underline{M}}$-measurable functions $s_n : T \longrightarrow X$, $n \geq 1$, such that

(i) $s_n(t) \in C^t \cap S_o^t$, for $t \in T$ and $n \geq 1$, and

(ii) $d(s_{n+1}(t), s_o(t)) < 1/3 . d(s_n(t), s_o(t))$, $n \geq 1$.

Proof : The proof is by induction on n. We will first prove the inductive step. Suppose s_n has been defined. Then consider

$$P = \{(t,x) : d(x,s_o(t)) < 1/3 . d(s_n(t),s_o(t)) \text{ and } (t,x) \in S_o\} .$$

As s_o and s_n are $\underline{\underline{M}}$-measurable, it follows from Lemma 4.1 that $P \in \underline{\underline{M}} \ (\underline{\overline{x}}) \ \underline{\underline{B}}_X$. Furthermore, P has non-empty and open vertical sections. Now put

$$Q = \{(t,\sigma) \in T \times \Sigma : h(t,\sigma) \in P^t\} .$$

Then $Q \in \underline{\underline{M}} \ (\underline{\overline{x}}) \ \underline{\underline{B}}_\Sigma$. Also Q has open sections as $h(t,.)$ is continuous for each $t \in T$, and further as $s_o(t)$ is a limit point of C^t for each t, Q^t is non-empty for each $t \in T$. By Lemma 4.4 there is a measurable selector $q : T \longrightarrow X$ for Q. Put $s_{n+1}(t) = h(t,q(t))$. The argument for the base step is even simpler. One has only to look at $P = \{(t,x) : (t,x) \in S_o\}$ and carry out the above argument.

4.6 Lemma : Let T be Polish (respectively an abstract set) and $\underline{\underline{M}}$ a countably generated sub σ-field of $\underline{\underline{B}}_T$ (respectively a σ-field on T closed under operation $(\underline{\underline{A}})$). Suppose $P \subseteq T \times \Sigma$, $P \in \underline{\underline{M}} \ (\underline{\overline{x}}) \ \underline{\underline{B}}_\Sigma$, and P^t is non-empty and open for each $t \in T$. Then there is $Q \subseteq P$ such that $Q \in \underline{\underline{M}} \ (\underline{\overline{x}}) \ \underline{\underline{B}}_\Sigma$, and Q^t is non-empty and clopen for each $t \in T$.

Proof : We will prove the lemma in the case when T is Polish. The other case is even simpler.

Put $T_s = \{t \in T : \Sigma(s) \subseteq P^t\}$ for $s \in Seq$.

Then T_s is a coanalytic set and $\bigcup_{s \in Seq} T_s = T$. Further, T_s is a union of atoms of $\underline{\underline{M}}$. By an invariant reduction principle (See [36]), there exist sets $R_s \in \underline{\underline{M}}$, $s \in Seq$, such that $R_s \cap R_{s'} = \emptyset$ for $s \neq s'$, $R_s \subseteq T_s$, and $\bigcup_{s \in Seq} R_s = \bigcup_{s \in Seq} T_s$. Now define

$$Q = \bigcup_{s \in Seq} R_s \times \Sigma(s).$$

This does the job.

In the other case, $T_s \in \underline{\underline{M}}$ as $\underline{\underline{M}}$ is closed under operation $(\underline{A})$ and the sets T_s can be directly disjointified.

4.7 Lemma : Let the assumptions of Lemma 4.6 be in force. Then there is a countable family $\{U_n\}_{n \geq 1}$ of subsets of $T \times \Sigma$ in $\underline{\underline{M}} \,(\underline{x})\, \underline{\underline{B}}_{\Sigma}$ satisfying :

(i) U_n^t is non-empty and clopen for each $t \in T$,

(ii) the family $\{U_n^t\}_{n \geq 1}$ is discrete in Σ, and consequently $\bigcup_{n \geq 1} U_n^t$ is closed in Σ for each $t \in T$,

(iii) $\Sigma - \bigcup_{n \geq 1} U_n^t$ is non-empty for each $t \in T$, and

(iv) $d(s_o(t), h(t,.)(U_n^t)) \downarrow 0$ as $n \longrightarrow \infty$.

Proof : We begin by fixing a set $S_o \subseteq T \times X$ such that $S_o \in \underline{\underline{M}} \,(\underline{x})\, \underline{\underline{B}}_X$, S_o^t is open, $s_o(t) \in S_o^t$, and $C^t - S_o^t \neq \emptyset$ for each $t \in T$. To see the existence of such a set one has only to make use of the fact that C is induced by the Caratheodory map h. For, fix $\sigma_o \in \Sigma$ and look at $f(\cdot,\sigma_o)$. Now put

$$S_o = \{(t,x) : d(x,s_o(t)) < d(x,f(t,\sigma_o))\}.$$

Then S_o has the required properties.

Now find $s_n : T \longrightarrow X$ as in Lemma 4.5. Now put, for $n \geq 1$,

$$V_n = \{(t,x) \in T \times X : d(x,s_n(t)) < 1/3.d(s_n(t),s_o(t)) \text{ and } (t,x) \in S_o\}.$$

Then $V_n \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$ and for each $t \in T$, V_n^t is open and non-empty, $C^t - V_n^t \neq \emptyset$, $d(s_o(t), V_n^t) \longrightarrow 0$ as $n \longrightarrow \infty$; and as for $m \geq n$, $d(V_n^t, V_m^t) \geq d(V_n^t, V_{n+1}^t) \geq 1/3\, d(s_n(t), s_o(t))$, we have, $\{V_n^t \cap C^t : n \geq 1\}$ is a family discrete in C^t. Put now

$$W_n = \{(t,\sigma) \in T \times \Sigma : h(t,\sigma) \in V_n^t\}.$$

Then, for $n \geq 1$, $W_n \in \underline{\underline{M}} \otimes \underline{\underline{B}}_\Sigma$ and W_n^t is open and non-empty for each $t \in T$. Apply Lemma 4.6 to obtain $U_n \in \underline{\underline{M}} \otimes \underline{\underline{B}}_\Sigma$ such that U_n^t is non-empty and clopen for each $t \in T$, and $U_n \subseteq W_n$. As the inverse image under a continuous map of a family discrete in the range of the map is discrete in the domain, one easily verifies that the family $\{U_n\}_{n \geq 1}$ obtained above satisfies conditions (i) - (iv). This completes the proof of the lemma.

The next lemma is contained in Sierpinski. For completeness, we outline a proof.

4.8 <u>Lemma</u> : There exist subsets C_n, $n \geq 0$ of Σ such that $C_n \cap C_m = \emptyset$ for $n \neq m$, each C_n is dense in Σ and each C_n is a one-one continuous image of Σ, and $\bigcup_{n \geq 0} C_n = \Sigma$.

<u>Proof</u> : Put $C_n = \{\sigma \in \Sigma : \sigma(2m) = n \text{ for all large } m\}$, $n \geq 1$, and $C_o = \Sigma - \bigcup_{n \geq 1} C_n$. It is easily seen that for $n \geq 1$, C_n is dense, condensed, and F_σ. Thus C_o is a G_δ. Also $C_o = \{\sigma \in \Sigma : (\forall n \geq 1)(\sigma(2m) \neq n \text{ infinitely often})\}$. Observe that C_o is both dense and boundary in Σ, so that by the theorem of Mazurkiewicz [15], there is a homeomorphism $f_o : \Sigma \xrightarrow{\text{onto}} C_o$. It is easily seen that

$$C_k = \bigcup_{n \geq 0} \{\sigma \in \Sigma : (\forall m \geq n)(\sigma(2m) = k) \ \& \ \sigma(2(n-1)) \neq k\} \cup \{\sigma \in \Sigma : (\forall m)(\sigma(2m) = k)\}, \quad k \geq 1,$$

Each of the sets within brackets is easily seen to be a homeomorph of Σ, and as the above union is a countable disjoint union, we can construct a map $f_k : \Sigma \xrightarrow{\text{onto}} C_k$, with f_k one-one, and continuous.

We are now in a position to prove the first theorem of this section.

4.9 <u>Theorem</u> : Let T be Polish and $\underline{\underline{A}}$ a countably generated sub σ-field of the Borel σ-field on T. Let $F : T \longrightarrow \mathbb{R}$ be an $\underline{\underline{A}}$-measurable multifunction taking dense-in-itself G_δ-values and further satisfying $Gr(F) \in \underline{\underline{A}} \otimes \underline{\underline{B}}_{\mathbb{R}}$. Then there is a map $f : T \times \Sigma \longrightarrow \mathbb{R}$ such that

(i) $f(t,\cdot)$ is a one-one, continuous map on Σ onto $F(t)$ for $t \in T$, and

(ii) $f(\cdot,\sigma)$ is an $\underline{\underline{A}}$-measurable selector for F.

Proof : As F satisfies all the hypotheses of Corollary 2.11 there is a map $h : T \times (\Sigma \dot{\cup} N) \longrightarrow \mathbb{R}$ such that

(a) $h(t,\cdot)$ is a one-one, continuous map on $\Sigma \dot{\cup} N$ onto $F(t)$ for each $t \in T$, and

(b) $h(\cdot,\sigma)$ is an $\underline{\underline{A}}$-measurable selector for F for each $\sigma \in \Sigma \dot{\cup} N$.

Get $\{C_n : n \geq 0\}$ as in Lemma 4.8. Thus, one may assume, that for each $n \geq 0$, there is a one-one Caratheodory map $h_n : T \times \Sigma(\langle n \rangle) \longrightarrow \mathbb{R}$ inducing $Gr(F) \bigcap \hat{h}(T \times C_n) = Gr(F_n)$, say, where $\hat{h}$ is the canonical map induced by h. Let $g_n : T \longrightarrow \mathbb{R}$ be defined by $g_n(t) = h(t,n+1)$, for $n \geq 0$. Then, as each $F(t)$ is dense-in-itself, it is apparent that $g_n(t)$ is a limit point of $F(t)$ and consequently of the dense subset $F_n(t)$ of $F(t)$, for each $t \in T$ (C_n being dense in Σ, $h(t,C_n) = F_n(t)$ is dense in $F(t)$).

Observe that to prove the theorem it suffices to prove that there is a one-one Caratheodory map $f_n : T \times \Sigma(\langle n \rangle) \longrightarrow \mathbb{R}$ inducing $Gr(F_n) \bigcup Gr(g_n)$, for then we could define

$$f : T \times \Sigma \longrightarrow \mathbb{R} \text{ by : } f(t,\sigma) = f_n(t,\sigma) \text{ if } \sigma \in \Sigma(\langle n \rangle).$$

We will now construct the map f_n. Notice that without loss of generality we may assume that $h_n : T \times \Sigma \longrightarrow \mathbb{R}$. We will then define $f_n : T \times \Sigma \longrightarrow \mathbb{R}$. Take h_n to be the map h fixed in the beginning of this section, $\underline{\underline{A}}$ to be the σ-field $\underline{\underline{M}}$ on T, and g_n to be the map $g_o : T \longrightarrow \mathbb{R}$.

Lemma 4.7 now applies to yield a family $\{U_m\}_{m \geq 1}$ satisfying the conditions therein.

Define $U_o \subseteq T \times \Sigma$ by $U_o^t = \Sigma - \bigcup_{m \geq 1} U_m^t$.

Then $U_o \in \underline{\underline{A}} \otimes \underline{\underline{B}}_\Sigma$ and U_o^t is clopen and non-empty for each $t \in T$. Also recall U_m^t is clopen for each t and $m \geq 1$.

By Lemma 4.2 and Remark 4.3, for each $m \geq 0$, there is an $\underline{\underline{A}} \otimes \underline{\underline{B}}_\Sigma$-measurable map $p_m : T \times \Sigma \longrightarrow \Sigma$ such that $p_m(t,.)$ is a homeomorphism of Σ onto U_m^t for each t.

As Σ is a homeomorph of the space A of irrationals in $(-1,1)$ together with 0, it is enough to define $f_n : T \times A \longrightarrow \mathbb{R}$. Fix $\{r_m\}_{m \geq 1}$ and $\{t_m\}_{m \geq 1}$, both sequences of distinct rationals such that $r_m \uparrow 0$ and $t_m \downarrow 0$. Take $r_1 = -1$, and $t_1 = 1$.

Put $A_{2m} = (r_m, r_{m+1}) \cap A, \ m \geq 1,$

$A_{2m+1} = (t_{m+1}, t_m) \cap A, \ m \geq 1.$

Each A_m is a homeomorph of Σ and we may therefore look upon the maps p_ℓ obtained above as defined on $T \times A_{\ell+2} \longrightarrow \Sigma$.

Define $f_n : T \times A \longrightarrow \mathbb{R}$ by cases as follows :

$f_n(t,\sigma) = h_n(t, p_{m-2}(t,\sigma))$, if $\sigma \in A_m$, $m \geq 2$.

$f_n(t,0) = g_n(t)$.

The map f_n has the required properties. The theorem follows immediately in view of the remarks made earlier.

4.10 Remark : As any zero-dimensional Polish space can be embedded as a G_δ in $\mathbb{R}$, Theorem 4.8 implies that the analogous result holds in any zero-dimensional space. However, our method does not go through for a general Polish space. The difficulty lies in our use of Theorem 2.10 and not in the methods developed in this section. We add that R.D. Mauldin and H. Sarbadhikari have independently obtained Theorem 4.9 [24].

We will now improve Theorem 3.11 to obtain a uniform version of the fact that a set is condensed Borel if and only if it is a one-one, continuous image of Σ. This result reads as follows :

4.11 Theorem : Let $(T, \underline{\underline{M}})$ be a measurable space where $\underline{\underline{M}}$ is a σ-field closed under operation $(\underline{\underline{A}})$. Let X be Polish. Suppose $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$ has condensed sections. Then there is a map $f : T \times \Sigma \longrightarrow X$ satisfying :

(i) f is $\underline{\underline{M}} \otimes \underline{\underline{B}}_\Sigma$-measurable,

(ii) $f(t,\cdot)$ is a one-one, continuous map on Σ onto B^t for each $t \in T$, and

(iii) if $\hat{f} : T \times \Sigma \longrightarrow T \times X$ is the canonical map induced by f, then

(a) $\hat{f}$ is $(\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma}, \underline{\underline{M}} \otimes \underline{\underline{B}}_X)$ - measurable,

(b) $\hat{f}^{-1}$ is $(\underline{\underline{M}} \otimes \underline{\underline{B}}_X|_B, \underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma})$ - measurable.

By arguing as in Theorem 3.12, any f satisfying the above can be shown to induce only $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_X$ with condensed sections.

<u>Proof</u> : As Lemmas 4.1 - 4.8 hold in this set-up also, arguments identical to the ones used in the proof of Theorem 4.9 show the existence of a map $f : T \times \Sigma \longrightarrow X$ satisfying conditions (i) and (ii). It suffices to verify (iii)(b). By the construction of the map in Theorem 4.8 it follows that it is enough to see that $f_n|_{T \times A_n}$ satisfies this condition for each n. By the definition of the map f_n it follows that this will be established the moment each p_ℓ constructed in the proof satisfies this condition (Note that now we have by Theorem 3.11 a map $h : T \times (\Sigma \dot{\cup} N) \longrightarrow X$ inducing B and satisfying condition (iii)(b)). But the validity of this property for p_n is immediate from Lemma 3.5. This completes the proof of Theorem 4.1 .

5. <u>Approximating C-sets in the product by sets in the product σ-field</u> : We have so far considered sets in the product of a measurable space $(T, \underline{\underline{M}})$ and $(Y, \underline{\underline{B}}_Y)$, with Y Polish, and have noticed that sets in $\underline{\underline{M}} \otimes \underline{\underline{B}}_Y$ have a particularly simple structure. Indeed, we have observed some of their pleasant properties. Recall now that if X is Polish then the class of C-sets in X, denoted by $\underline{\underline{S}}(X)$, is the smallest class containing the Borel sets and closed under complementation and Souslin's operation $(\underline{\underline{A}})$. Now let X and Y be Polish spaces, and consider the σ-field $\underline{\underline{S}}(X \times Y)$. These are in general complicated sets that cannot be related to any reasonable product structure. For instance, as observed by B.V. Rao [28], it is not true that every C-set in $\mathbb{R} \times \mathbb{R}$ is in $\underline{\underline{L}}(\mathbb{R}) \otimes \underline{\underline{L}}(\mathbb{R})$, $\underline{\underline{L}}(\mathbb{R})$ being the family of Lebesgue measurable subsets of $\mathbb{R}$.

We will now see, however, that any set in $\underline{\underline{S}}(X \times Y)$ can be " approximated " section-wise by sets in $\underline{\underline{S}}(X) \otimes \underline{\underline{B}}_Y$ in the sense of category and measure. That is, we will show that if $A \in \underline{\underline{S}}(X \times Y)$, then there are B and C in $\underline{\underline{S}}(X) \otimes \underline{\underline{B}}_Y$ such that $B \subseteq A \subseteq C$, and for each $x \in X$, $C^x - B^x$ is meager. A similar statement can be formulated and proved for measure. Many propositions about sets in $\underline{\underline{S}}(X \times Y)$ then reduce to ones about the simpler sets in $\underline{\underline{S}}(X) \otimes \underline{\underline{B}}_Y$.

In particular, various selection theorems for sets in $\underline{\underline{S}}(X \times Y)$ are an immediate consequence of such approximations.

One can, for example, obtain the following theorem proved by Burgess in [6]: Let $A \in \underline{\underline{S}}(X \times Y)$ be the graph of a G_δ-valued, $\underline{\underline{S}}(X)$-measurable multifunction into Y. Then A has an $\underline{\underline{S}}(X)$-measurable selection. While Burgess uses high power tools from game theory and the theory of inductive definability, our methods are essentially elementary (modulo some fairly deep results about coanalytic sets), and "classical" in spirit. Indeed, one way of looking at the content of this section would be to view it as a method for proving selection theorems for sets in $\underline{\underline{S}}(X \times Y)$ by induction, wherein a sufficiently strong induction hypothesis is stated (namely, the "approximation" theorem), resulting in unification and simplification. We might add that so complicated are Burgess' arguments that he writes down the proof only for the coanalytic case.

Vaught in [42] has shown that if $A \in \underline{\underline{S}}(X \times Y)$, then $\{x \in X : A^x \text{ is non-meager}\}$ etc. are sets in $\underline{\underline{S}}(X)$. Another consequence of our approximation theorem is that these computations follow from the corresponding ones for Borel sets. Similar computations also hold for measure and these follow again from the computations for Borel sets through our approximation theorem in the measure case.

We will actually prove most of our results level-wise through a hierarchy of C-sets.

We begin with the following simple observation that is implicit in what we have done so far.

5.1 Lemma : Let $(T, \underline{\underline{M}})$ be a measurable space, $\underline{\underline{M}}$ being a σ-field closed under operation $(\underline{\underline{A}})$, and let Y be Polish. Let $B \in \underline{\underline{M}} \,(\overline{x})\, \underline{\underline{B}}_Y$ have non-empty vertical sections. Then B has an $\underline{\underline{M}}$-measurable selection.

Proof : Arguments similar to the ones we have been using show that the problem reduces to that of finding an $\underline{\underline{S}}_{[0,1]}$-measurable selection for a Borel set in $[0,1] \times Y$. But this is ensured by the Von-Neumann selection theorem.

As we will prove our results level by level through a hierarchy of C-sets, we will now specify this hierarchy (due to Nikodym).

Let X be Polish. For each $\alpha < \omega_1$, the first uncountable ordinal define by transfinite recursion classes $\underline{\underline{A}}_\alpha(X)$ and $\underline{\underline{S}}_\alpha(X)$ as follows :

Put $\underline{\underline{A}}_0(X) = \underline{\underline{S}}_0(X)$ = Borel sets in X.

Suppose these classes have been defined for all $\alpha < \beta$.

$$\text{Put } \underline{\underline{A}}_\beta(X) = \Big\{ A \subseteq X : A \text{ is the result of operation } (\underline{\underline{A}}) \text{ on a system of sets } \{A_{n_1 n_2 \ldots n_k}\} \text{ with } A_{n_1 \ldots n_k} \in \bigcup_{\alpha < \beta} \underline{\underline{S}}_\alpha(X) \Big\}$$

$$\underline{\underline{S}}_\beta(X) = \sigma\text{-field generated by } \underline{\underline{A}}_\beta(X).$$

Observe that $\underset{=}{A}_1(X)$ is just the class of analytic sets in X, and $\underset{=}{S}_1(X)$ is the analytic σ-field on X. Furthermore,

$$\bigcup_{\alpha < \omega_1} \underset{=}{S}_\alpha(X) = \underset{=}{S}(X) .$$

We will need the next result to give us one half of the base step when we induct on sets in $\underset{=}{S}(X \times Y)$. The result is is in Kechris [14].

5.2 <u>Lemma</u> : Let $A \subseteq \omega^\omega$ be a Σ^1_1 (lightface), meager set. Then A is contained in the union of all closed, nowhere dense sets given by Δ^1_1 trees on ω. The relativized version also holds. (All the terms are the standard ones used in Moschovakis [25]).

We will need the following simple fact to reduce the other half of the base step to the zero-dimensional case.

5.3 <u>Lemma</u> : Let Y be Polish. Then there is a meager set N such that $Y - N$ is a zero-dimensional Polish space.

<u>Proof</u> : Let $\{V_n\}_{n \geq 1}$ be a base for Y. Put $N = \bigcup_{n \geq 1} (cl(V_n) - V_n)$. Then N is clearly a meager F_σ in Y. Furthermore, for each $n \geq 1$, $cl(V_n) \cap (Y - N) = V_n \cap (Y - N)$. Thus $Y - N$ is a zero-dimensional G_δ in Y, and therefore a zero-dimensional Polish space.

Towards proving our approximation theorem for an analytic set in the product, we will now describe a procedure for obtaining a dense G_δ set inside a comeager analytic set. The method is essentially in Sion's proof [34] that " analytic " sets in general topological spaces are capacitable in the sense of Choquet. We are interested here not so much in the next result, as in the method.

5.4 Lemma : Let $A \subseteq \Sigma$ be a comeager analytic set. Then A contains a dense G_δ set B.

Proof : As A is analytic, there is a continuous map f on Σ onto A. As Σ is a G_δ in $\mathbb{R}$ and $\mathbb{R}$ is σ-compact, we can look upon f as defined on a $K_{\sigma\delta}$ subset of $\mathbb{R}$ ($K_\sigma \equiv \sigma$-compact), i.e., we have continuous $f : D \xrightarrow{\text{onto}} A$, where $D = \bigcap_{m \geq 1} \bigcup_{n \geq 1} D(m,n)$, with $D(m,n)$ a compact subset of $\mathbb{R}$ for every $m,n \geq 1$. Without loss of generality we may assume that for each m, $D(m,n) \uparrow$ with n.

We will now define two systems $\{n_s : s \in Seq\}$ and $\{p_s : s \in Seq\}$ of positive integers satisfying :

(i) $n_s \in Seq$ and $lh(n_s) \geq lh(s)$ for $s \in Seq$.

(ii) $s \prec s' \Rightarrow n_s \prec n_{s'}$.

(iii) $lh(s) = lh(s')$ and $s \neq s' \Rightarrow \Sigma(n_s) \cap \Sigma(n_{s'}) = \emptyset$.

(iv) $\bigcup_{s \in S_k} \Sigma(n_s)$ is dense in Σ, for each $k \geq 1$.

(v) $lh(s) = k \Rightarrow f(D \cap D(1, p_{s\restriction 1}) \cap \ldots\ldots \cap D(k, p_s))$ is comeager in $\Sigma(n_s)$.

Suppose such systems have been defined. Put then

$U_k - \bigcup_{s \in S_k} \Sigma(n_s)$ and $B = \bigcap_{k \geq 1} U_k$. By (iv), each U_k is an open dense set, and B is therefore a dense G_δ. We will now check that $\bigcap_{k \geq 1} U_k \subseteq A$.

So let $y \in \bigcap_{k \geq 1} U_k$. By (iii), for each k, $\{\Sigma(n_s) : s \in S_k\}$ is a disjoint family. Consequently, there is a unique sequence $k_1, k_2, \ldots\ldots$ such that $y \in \Sigma(n_{\langle k_1 k_2 \ldots k_\ell \rangle})$ for all $\ell \geq 1$. Furthermore, by (i), $\delta(\Sigma(n_{\langle k_1 \ldots k_\ell \rangle})) \downarrow 0$, and by (v), for each $\ell \geq 1$, $f(D \cap D(1, p_{\langle k_1 \rangle}) \cap D(2, p_{\langle k_1 k_2 \rangle}) \cap \ldots\ldots \cap D(\ell, p_{\langle k_1 \ldots k_\ell \rangle}))$ is comeager in $\Sigma(n_{\langle k_1 \ldots k_\ell \rangle})$........($\underline{*}$). Observe that $D' = \bigcap_{\ell = 1}^{\infty} D(\ell, p_{\langle k_1 \ldots k_\ell \rangle}) \subseteq D$. It suffices now to see that $y \in f(D')$. Now, as f is continuous, $f(D')$ is compact. Thus, if $y \notin f(D')$, there is open $V \supseteq f(D')$ and i such that $V \cap \Sigma(n_{\langle k_1, \ldots, k_i \rangle}) = \emptyset$. Then $D' = \bigcap_{\ell = 1}^{\infty} D(\ell, p_{\langle k_1 \ldots k_\ell \rangle}) \subseteq f^{-1}(V)$.

As the $D(m,n)$'s are compact, there is $j \geq i$ such that $f(D \cap (\bigcap_{\ell = 1}^{j} D(\ell, p_{\langle k_1 \ldots k_\ell \rangle})))$ is contained in V. But this contradicts ($\underline{*}$).

It remains now to obtain the systems $\{n_s\}$ and $\{p_s\}$. We will obtain these by induction on $lh(s)$. Define $n_e = e$, and $p_e = 1$. Suppose n_s, p_s have been defined for all s with length $\leq \ell$. Fix $s \in S_\ell$. We have to define n_{sj}, p_{sj}, for each $j \geq 0$.

Define $R^s \subseteq$ Seq by :

$$R^s = \{u \in \text{Seq} : n_s \prec u \ \& \ lh(u) \geq \ell + 1 \ \& \ f(D \cap D(1, p_s \restriction 1) \cap \ldots \cap D(\ell, p_s) \cap D(\ell+1, m)) \text{ is comeager in } \Sigma(u) \text{ for some } m \geq 1, \text{ and } f(D \cap D(1, p_s \restriction 1) \cap \ldots \cap D(\ell, p_s) \cap D(\ell+1, m)) \text{ is not comeager in } \Sigma(v) \text{ for any } m \geq 1 \text{ and any } v \text{ such that } n_s \prec v \prec u \text{ and } v \neq u \}$$

[For example, R^e is, roughly speaking, the collection of those sequence numbers that code the "largest" basic clopen sets in which some $f(D \cap D(1,m))$ is comeager]. It follows that any two distinct sequence numbers appearing in this set must code disjoint neighbourhoods. It is also easily seen that the union of these neighbourhoods is dense in $\Sigma(n_s)$.

In general, we now define (fixing an enumeration $u_1, u_2, \ldots$ of Seq), for $j \geq 0$,

$$n_{sj} = u_\ell \text{ if } u_\ell \text{ is the } (j+1)^{st} \text{ sequence number appearing in } R^s, \text{ if there is one such.}$$

Since $\bigcup_{j \geq 0} \Sigma(n_{sj})$ is dense in $\Sigma(n_s)$, n_{sj} is defined for all $j \geq 0$. Now put

$$p_{sj} = \text{least } m \text{ such that } f(D \cap D(1, p_s \restriction 1) \cap \ldots \cap D(\ell, p_s) \cap D(\ell+1, m)) \text{ is comeager in } \Sigma(n_{sj}).$$

It is easily verified that $\{n_s\}$, $\{p_s\}$ so defined satisfy (i)-(v).

The next result is due to Kechris [14] and Vaught [42]. However the argument in Lemma 5.4 also yields this result.

5.5 <u>Lemma</u> : Let X and Y be Polish spaces and suppose $A \subseteq X \times Y$. Then for any open set V in Y, $A^*_V = \{x \in X : A^x$ is comeager in $V\}$ is analytic (coanalytic) if A is analytic (coanalytic).

<u>Proof</u> : Clearly it is enough to prove the result with $V = Y$, and as easy arguments show, for A analytic. Assume first that $Y = \Sigma$. Fix a continuous map f on D onto A, where $D \subseteq \mathbb{R}$ and $D = \bigcap_{m \geq 1} \bigcup_{n \geq 1} D(m,n)$, with each $D(m,n)$ compact in $\mathbb{R}$.

Observe that the argument in Lemma 5.4 shows that

A^x is comeager $\longleftrightarrow$ $\Big[\exists \{n_s : s \in \mathrm{Seq}\} \,\&\, \{p_s : s \in \mathrm{Seq}\}$ satisfying (i) - (iv) in the proof of Lemma 5.4 and satisfying, for each s,

(v) $(f(D \cap D(1, p_{s|1}) \cap \ldots \cap D(lh(s), p_s)))^x$ is comeager in $\Sigma(n_s)\Big]$

$\longleftrightarrow$ $\Big[\exists \{n_s\} \,\&\, \{p_s\}$ satisfying (i) - (iv) and for each s,

(v)' $(f(D \cap D(1, p_{s|1}) \cap \ldots \cap D(lh(s), p_s)))^x \cap \Sigma(n_s) \neq \emptyset\Big]$.

The first equivalence is clear. To see that the last condition implies that A^x is comeager notice that (v)' was all that was needed to show that the set B in the proof of Lemma 5.4 is contained in A. Finally, it is easy to see that the last condition is an analytic condition.

For general Y, apply Lemma 5.3, and the fact that any zero-dimensional, uncountable Polish space Y can be written as $\Sigma \bigcup Z$, with Z countable. A simple argument by cases when Z is meager, non-meager, and comeager in Y now yields the result.

We will now carry out the argument in Lemma 5.4 uniformly over the sections of an analytic set in the product of two Polish spaces.

5.6 <u>Lemma</u> : Let X and Y be Polish and let $A \subseteq X \times Y$ be an analytic set such that A^x is comeager for each $x \in E$, with E analytic. Then there is a set $B \subseteq A$ such that $B \in \underset{=}{S}_1(X) \underline{\otimes} \underset{=}{B}_Y$ such that B^x is a dense G_δ for each $x \in E$.

<u>Proof</u> : By virtue of Lemma 5.3 we may assume without loss of generality that Y is zero-dimensional. To begin with assume further that $Y = \Sigma$. Once again get a continuous map $f : D \xrightarrow{\text{onto}} A$ where $D \subseteq \Pi$ and $D = \bigcap_{m \geq 1} \bigcup_{n \geq 1} D(m,n)$ with $D(m,n)$ compact for each m, $n \geq 1$, and such that $D(m,n) \uparrow$ with n for each fixed m.

Let $u_1, u_2, \ldots$ be an enumeration of Seq. We will define two systems of $\underset{=}{S}_1(X)$-measurable functions defined on E into ω, $\{g_{n_1 n_2, \ldots, n_k}\}$ and $\{h_{n_1 n_2, \ldots, n_k}\}$ such that

(i) $g_s(x) \in \mathrm{Seq}$ and $\mathrm{lh}(g_s(x)) \geq \mathrm{lh}(s)$ for $s \in \mathrm{Seq}$, $x \in E$.

(ii) $s \prec s' \implies g_s(x) \prec g_{s'}(x)$, $x \in E$.

(iii) $lh(s) = lh(s')$ and $s \neq s' \implies \Sigma(g_s(x)) \cap \Sigma(g_{s'}(x)) = \emptyset$, $x \in E$.

(iv) $\bigcup_{s \in S_k} \Sigma(g_s(x))$ is dense in Σ, for each $k \geq 1$, $x \in E$.

(v) $lh(s) = k \implies (f(D \cap D(1, h_{s|1}(x)) \cap \ldots \cap D(k, h_s(x)))^x$

is comeager in $\Sigma(g_s(x))$, $x \in E$.

As before we will define such systems by induction on $lh(s)$. Put $g_e(x) = e$, $x \in E$ [we take $\Sigma(e) = \Sigma$].

$h_e(x) = 0$.

Suppose g_s, h_s have been defined for all $s \in \bigcup_{\ell \leq k} S_\ell$ satisfying the above properties. Fix $s \in S_k$. We have to define g_{sn}, h_{sn}, for each $n \geq 0$.

Let $E^\ell_{m_1,\ldots,m_k} = \{x \in E : h_{s|1}(x) = m_1, \ldots, h_s(x) = m_k,\ g_s(x) = u_\ell\}$.

Then by the induction hypothesis the $E^\ell_{m_1,\ldots,m_k}$'s are disjoint sets in $\underset{=}{S}_1(X)$ such that $\bigcup E^\ell_{m_1,\ldots,m_k} = E$, the union running over $\langle \ell, m_1, \ldots, m_k \rangle \in S_{k+1}$.

It suffices now to define $\{g_{sn}, h_{sn},\ n \geq 0\}$ on each $E^\ell_{m_1,\ldots,m_k}$ separately. So fix now $\langle \ell, m_1, \ldots, m_k \rangle \in S_{k+1}$.

Say that (n, u_j) satisfies $(*)$ at x if

$(f(D \cap D(1,m_1) \cap \ldots \cap D(k,m_k) \cap D(k+1,n)))^x$ is comeager in $\Sigma(u_j)$ and for all $u \in Seq$ with $lh(u) \geq k+1$ such that

$u_\ell \prec u \prec u_j \,\&\, u \neq u_j$ and $m \geq 1$, we have

$(f(D \cap D(1,m_1) \cap \dots \cap D(k,m_k) \cap D(k+1,m)))^x$ is not comeager in $\Sigma(u)$.

Define $R_j \subseteq E^{\ell}_{m_1,\dots,m_k}$ by :

$$R_j = \{x \in E^{\ell}_{m_1,\dots,m_k} : u_\ell \prec u_j \;\&\; lh(u_j) \geq k+1 \;\&\; \text{there is } n \geq 1 \text{ such that } (n, u_j) \text{ satisfies } (\bar{*}) \text{ at } x\}.$$

Now as f is continuous, $f(D \cap D(1,m_1) \cap \dots \cap D(k,m_k) \cap D(k+1,n))$ is an analytic subset of $X \times \Sigma$ for each n. By Lemma 5.5 each R_j is in $\underline{\underline{S}}_1(X)$.

Now define $g_{sn}(x) = u_j$ if j is the $(n+1)^{st}$ integer such that $x \in R_j$

and $h_{sn}(x) =$ least m such that $(m, g_{sn}(x))$ satisfies $(\bar{*})$ at x.

The proof of Lemma 5.4 shows that the above functions are well-defined and that $\{g_s\}$, $\{h_s\}$ so defined satisfy (i) - (v).

Now define $B_k \subseteq X \times \Sigma$ by :

$$B_k = \{(x,\sigma) : x \in E \text{ and } (\exists s \in S_k)\,(\exists n)\,(g_s(x) = u_n \;\&\; \sigma \in \Sigma(u_n)\} .$$

Then $B_k \in \underline{\underline{S}}_1(X) \,\bar{\otimes}\, \underline{\underline{B}}_\Sigma$ and B_k has dense, open sections on E.

Put $B = \bigcap_{k \geq 1} B_k$. Then $B \in \underline{\underline{S}}_1(X) \,\bar{\otimes}\, \underline{\underline{B}}_\Sigma$ and the argument in Lemma 5.4 shows that $B \subseteq A$. This proves the lemma when $Y = \Sigma$. For general Polish Y, the argument is as in Lemma 5.5.

5.7 <u>Remark</u> : Notice that the argument in Lemma 5.6 proves the following theorem of H. Sarbadhikari [30] : If B is a Borel set in $X \times Y$, with X, Y Polish, such that B^x is comeager for each x, then $B \supseteq \bigcap_{n \geq 1} B_n$, where each B_n is a Borel set with dense open sections. To see this, one has only to go over the above proof and make the necessary changes. To begin with, observe that the map f fixed in the beginning can now be taken to be one-one. As the image of a Borel set under a one-one Borel measurable map is Borel, and for Borel $F \subseteq X \times Y$ & open V in Y, $\{x : F^x$ is comeager in $V\}$ is Borel, the various sets appearing in the proof will now turn out to be Borel. The sets B_n in the proof will then do the job. It might be recalled that this result immediately yields a Borel uniformization for Borel sets with comeager, and therefore non-meager, sections by arguments that are by now standard. This result is also implicit in the result of Kechris (Lemma 5.2).

The next step is to observe that the counterpart of Lemma 5.6 for analytic sets with meager sections is a consequence of Lemma 5.2.

5.8 <u>Lemma</u> : Let X and Y be Polish, and $A \subseteq X \times Y$ an analytic set such that A^x is meager for all $x \in E$, with E coanalytic. Then there is $B \in \underset{\sim}{S}_1(X) \otimes \underset{\sim}{B}_Y$ such that $A \subseteq B$ and B^x is meager for all $x \in E$.

<u>Proof</u> : We shall give a simple effective-theoretic argument. Notice once again a crucial use of the local methods of the effective theory (all effective-theoretic notation and terminology is from Moschovakis [25]).

As before, it suffices to prove the result for $A \subseteq \omega^\omega \times \omega^\omega$. For simplicity assume that A is Σ^1_1. The relativized version can be argued similarly. Then, for each $x \in E$, A^x is a $\Sigma^1_1(x)$ meager set. Lemma 5.2 therefore applies to show that, for $x \in E$, A^x is contained in the (countable) union of all closed nowhere dense sets given by $\Delta^1_1(x)$ trees on ω

Let $d : \omega^\omega \times \omega \longrightarrow \omega^\omega$ be a Π^1_1 - recursive partial function that codes points in $\Delta^1_1(x)$, x running through ω^ω [25; 4D.2]. Then we may write

$$A \cap (E \times \omega^\omega) \subseteq \{(x,y) : (\exists n)\, (d(x,n)\downarrow \,\&\, d(x,n) \text{ codes a tree } T \text{ such that } [T] \text{ is nowhere dense } \&\, y \in [T])\}$$

$$= B', \text{ say}.$$

Let $C_n = \{x : d(x,n)\downarrow \,\&\, d(x,n) \text{ codes a tree whose body is nowhere dense}\}$, $n \geq 0$.

Then C_n is coanalytic and thus in $\underline{\underline{S}}_1(\omega^\omega)$.

Define $f_n : C_n \longrightarrow \omega^\omega$ by $f_n(x) = d(x,n)$. Then f_n is $\underline{\underline{S}}_1(\omega^\omega) \cap C_n$-measurable. Plainly,

$$B' = \bigcup_{n \geq 0} \{(x,y) : x \in C_n \,\&\, (f_n(x), y) \in F\}, \text{ where}$$

$F = \{(x,y) \in \omega^\omega \times \omega^\omega : (\forall m)(x(\bar{y}(n)) = 0)\}$. As F is closed, we have $B' \in \underline{\underline{S}}_1(\omega^\omega) \otimes \underline{\underline{B}}_{\omega^\omega}$. Thus, as E is coanalytic, $B = B' \cup ((\omega^\omega - E) \times \omega^\omega)$ does the job. The result for general X, Y Polish follows as before.

The next result yields the approximation theorem at the first level of the hierarchy of C - sets.

5.9 <u>Lemma</u> : Let X and Y be Polish. Let A be an analytic subset of $X \times Y$. Then there are B and C in $\underline{\underline{S}}_1(X) \textcircled{x} \underline{\underline{B}}_Y$ such that $B \subseteq A \subseteq C$ and $C^x - B^x$ is meager for each $x \in X$.

<u>Proof</u> : Fix a base $\{V_n\}_{n \geq 1}$ for Y.

Let $T_n = \{x \in X : A^x$ is comeager in $V_n\}$. As A^x satisfies the Baire property for each x,

$$\bigcup_{n \geq 1} T_n = \{x \in X : A^x \text{ is non-meager}\} \quad .....(\underline{*}).$$

For each $n \geq 1$, apply Lemma 5.6 with T_n playing the role of E and V_n playing the role of Y. Get B_n as in the Lemma. Put $B = \bigcup_{n \geq 1} (B_n \cap (T_n \times Y))$. Then $B \in \underline{\underline{S}}_1(X) \text{ (x) } \underline{\underline{B}}_Y$ and $(\underline{*})$ ensures that for each $x \in X$, $A^x - B^x$ is meager.

To get the set C one has only to carry out the above argument for the coanalytic set $(X \times Y) - A$. That this can be done is ensured by Lemma 5.8.

We will now state our main theorem for category.

5.10 (<u>Category</u>) <u>Approximation Theorem</u> : Let X and Y be Polish spaces. Let $A \in \underline{\underline{S}}_\alpha(X \times Y)$, $\alpha < \omega_1$. Then there are B and C in $\underline{\underline{S}}_\alpha(X) \textcircled{x} \underline{\underline{B}}_Y$ such that $B \subseteq A \subseteq C$ and $C^x - B^x$ is meager for each $x \in X$. In particular if $A \in \underline{\underline{S}}\ (X \times Y)$, then one can find

B and C in $\underline{\underline{S}}(X) \textcircled{x} \underline{\underline{B}}_Y$ will the above properties.

Proof : The argument is by induction on α. For $\alpha = 0$, $\underline{\underline{S}}_0(X \times Y)$ is the Borel σ-field on $X \times Y$. In this case there is nothing to prove.

Suppose then that the result is true for all $\beta < \alpha$. Let $\underline{\underline{F}}$ stand for the class of all sets in $\underline{\underline{S}}_\alpha(X \times Y)$ for which the result holds. It is easily seen that $\underline{\underline{F}}$ is closed under complements and countable unions. Thus, one need only check that $\underline{\underline{A}}_\alpha(X \times Y) \subseteq \underline{\underline{F}}$.

So fix $A \in \underline{\underline{A}}_\alpha(X \times Y)$. Then A is the result of operation $(\underline{A})$ performed on a system $\{A_{n_1 n_2 \dots n_k}\}$ with each $A_{n_1 \dots n_k} \in \bigcup_{\beta < \alpha} \underline{\underline{S}}_\beta(X \times Y)$. By the induction hypothesis, for each $(n_1, \dots, n_k)$ we have $B_{n_1 n_2 \dots n_k}$ and $C_{n_1 n_2 \dots n_k}$, both in $\bigcup_{\beta < \alpha} (\underline{\underline{S}}_\beta(X) \textcircled{x} \underline{\underline{B}}_Y)$, such that $B_{n_1 \dots n_k} \subseteq A_{n_1 \dots n_k} \subseteq C_{n_1 \dots n_k}$, and $C^x_{n_1 \dots n_k} - B^x_{n_1 \dots n_k}$ is meager for each $x \in X$. Let $B^* = \underline{A}(\{B_{n_1 \dots n_k}\})$ and $C^* = \underline{A}(\{C_{n_1 \dots n_k}\})$. Then $B^* \subseteq A \subseteq C^*$ and since any $y \in (C^*)^x - (B^*)^x$ is in some $C^x_{n_1 \dots n_k} - B^x_{n_1 \dots n_k}$, each of which is meager, we have $(C^*)^x - (B^*)^x$ is meager for each x. To complete the proof it suffices to get $B \subseteq B^*$, $C \supseteq C^*$ such that $B, C \in \underline{\underline{S}}_\alpha(X) \textcircled{x} \underline{\underline{B}}_Y$, $(B^*)^x - B^x$ is meager and $C^x - (C^*)^x$ is meager for each x. We will obtain B, the argument for C being similar.

Let $\underline{\underline{S}}^{\alpha}(X)$ be the σ-field generated by $\bigcup_{\beta < \alpha} \underline{\underline{S}}_{\beta}(X)$. Now $B^{*} = \underline{\underline{A}}(\{B_{n_1 \dots n_k}\})$ with each $B_{n_1 \dots n_k} \in \underline{\underline{S}}^{\alpha}(X) \otimes \underline{\underline{B}}_Y$. Thus, there is a countably generated sub σ-field $\underline{\underline{R}}^{\alpha}(X)$ of $\underline{\underline{S}}^{\alpha}(X)$ such that each $B_{n_1 \dots n_k}$ is in $\underline{\underline{R}}^{\alpha}(X) \otimes \underline{\underline{B}}_Y$. Let $m : (X, \underline{\underline{R}}^{\alpha}(X)) \longrightarrow [0,1]$ be the characteristic function of a countable generator for $\underline{\underline{R}}^{\alpha}$. Put $M = m(X)$, and for each $n_1, \dots, n_k$, let $B'_{n_1 \dots n_k} = \{(m(x),y) : (x,y) \in B_{n_1 \dots n_k}\}$. Then $B'_{n_1 \dots n_k} \in \underline{\underline{B}}_M \otimes \underline{\underline{B}}_Y$ for each $(n_1, \dots, n_k)$. Let $B^{**} = \{(m(x),y) \cdot (x,y) \in B^{*}\}$. Then

$$B^{**} = \underline{\underline{A}}(\{B'_{n_1 \dots n_k}\}).$$

It follows that there is an analytic set $A^{*} \subseteq [0,1] \times Y$ such that $A^{*} \cap (M \times Y) = B^{**}$. Apply Lemma 5.9 to get $A^{**} \subseteq A^{*}$, $A^{**} \in \underline{\underline{S}}_1([0,1]) \otimes \underline{\underline{B}}_Y$ such that for each $t \in [0,1]$, $(A^{*})^{t} - (A^{**})^{t}$ is meager.

Let $(m, id) : X \times Y \longrightarrow [0,1] \times Y$ be the map

$$(m,id)(x,y)) = (m(x),y)$$

Put $B = (m,id)^{-1}(A^{**})$. Observe that as m is a bimeasurable map of $(X, \underline{\underline{R}}^{\alpha})$ and $(M, \underline{\underline{B}}_M)$,

$$m^{-1}(\underline{\underline{A}}_1([0,1])) \subseteq \{A \subseteq X : A \text{ is the result of operation } (\underline{\underline{A}}) \text{ on sets in } \mathbb{R}^{\alpha}\}$$

$$\subseteq \underline{\underline{A}}_{\alpha}(X).$$

Consequently $m^{-1}(\underline{\underline{S}}_1([0,1])) \subseteq \underline{\underline{S}}_{\alpha}(X)$.

Thus $B \in \underline{\underline{S}}_\alpha(X) \; (\underline{\underline{x}}) \; \underline{\underline{B}}_Y$ and B clearly satisfies the other properties required of it.

We will now set down some of the consequences of Theorem 5.10.

5.11 Corollary : Let X and Y be Polish. Suppose $A \subseteq X \times Y$ and A^x is non-meager for each x. Then

(i) $A \in \underline{\underline{S}}(X \times Y) \Rightarrow A$ has an $\underline{\underline{S}}(X)$-measurable selection.

(ii) $A \in \underline{\underline{S}}_\alpha(X \times Y), \alpha < \omega_1 \Rightarrow A$ has an $\underline{\underline{S}}_\alpha(X)$-measurable selection.

Proof : To see (i), get $B \in \underline{\underline{S}}(X) \; (\underline{\underline{x}}) \; \underline{\underline{B}}_Y$ such that $A^x - B^x$ is meager, $B \subseteq A$, as in Theorem 5.10. As B^x is then nonempty for each x, Lemma 5.1 yields the result.

As for (ii), use Theorem 5.10 to get now $B \in \underline{\underline{S}}_\alpha(X) \; (\underline{\underline{x}}) \; \underline{\underline{B}}_Y$ satisfying the same properties. The result then follows from the next proposition that is worth putting down for the record.

5.12 Proposition : Let $(T, \underline{\underline{M}})$ be a measurable space and Y Polish. Suppose $B \in \underline{\underline{M}} \; (\underline{\underline{x}}) \; \underline{\underline{B}}_Y$ has non-meager sections. Then B has an $\underline{\underline{M}}$-measurable selection.

Proof . Let $m : T \longrightarrow [0,1]$ be the characteristic function of a generator for a countably generated sub σ-field $\underline{\underline{M}}_0$ of $\underline{\underline{M}}$ such that $B \in \underline{\underline{M}}_0 \; (\underline{\underline{x}}) \; \underline{\underline{B}}_Y$. Let $M = m(T)$ and B' be Borel in $[0,1] \times Y$ such that for $t \in T$, $(t,y) \in B$ iff $(m(t),y) \in B'$. Let $L = \{x \in [0,1] : (B')^x$ is non-meager$\}$. Then $M \subseteq L$, and L

is Borel. Put $B^* = B' \cap (L \times Y)$. Then as B^* is Borel and has non-meager sections on the Borel set L, by the theorem of H. Sarbadhikari [30] (See also Remark 5.7), B^* has a Borel selection g. Then $f = g \circ m$ is an $\underline{\underline{M}}$-measurable selection for B.

5.13 <u>Remark</u> : Observe also that an argument along the above lines also shows the following. Let $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_Y$. Then $\{t \in T : B^t$ is non-meager ([redacted] comeager)$\} \in \underline{\underline{M}}$. The next selection theorem has been obtained by Burgess [6].

5.14 <u>Corollary</u> : Let X and Y be Polish. Let $F : X \to Y$ be a multifunction such that $F(t)$ is non-meager in $cl(F(t))$ (in particular, we may take F to be G_δ-valued).

If F is $\underline{\underline{S}}(X)$-measurable and $Gr(F) \in \underline{\underline{S}}(X \times Y)$, then F has an $\underline{\underline{S}}(X)$-measurable selection.

<u>Proof</u> : The argument is via a useful technique due to R. Barua (See [2]). [redacted]

Define $G : X \to Y$ by $G(x) = cl(F(x))$.

Then G is a closed valued, $\underline{\underline{S}}(X)$-measurable multifunction. By Corollary 6.12 (next section), there is a map $g : X \times \Sigma$ into Y such that g is $\underline{\underline{S}}(X) \otimes \underline{\underline{B}}_\Sigma$-measurable and $g(x, .)$ is continuous, open, and onto $G(x)$, for each x.

Define $G' \subseteq X \times \Sigma$ by :

$$G' = \{(x, \sigma) : g(x, \sigma) \in F(x)\}.$$

As $Gr(F) \in \underline{\underline{S}}(X \times Y)$ and g is $\underline{\underline{S}}(X) \otimes \underline{\underline{B}}_\Sigma$-measurable, G' is in $\underline{\underline{S}}(X\times\Sigma)$. This holds because $\underline{\underline{S}}(X\times\Sigma)$ is closed under operation $(\underline{\underline{A}})$ and $\underline{\underline{S}}(X\times Y)$ is the smallest class closed under this operation and containing the Borel sets. Also as the inverse image of a non-meager set under a continuous, open map is non-meager, G' has non-meager sections. By Corollary 5.11, G' has an $\underline{\underline{S}}(X)$ - measurable selection g'. Then $f(x) = g(x, g'(x))$ is an $\underline{\underline{S}}(X)$ - measurable selection for F. ~~Here again one needs the~~

It should be pointed out that an argument along the above lines to prove a level-wise version of Corollary 5.14 would break down at the point where essential use has been made of the closure of $\underline{\underline{S}}(Z)$ under operation $(\underline{\underline{A}})$, for Polish Z.

5.15 Corollary : Let X and Y be Polish, with Y dense-in-itself. Suppose $A \in \underline{\underline{S}}(X\times Y)$ has non-meager sections. Then there is an $\underline{\underline{S}}(X\times Y)$ - measurable map f on $X\times Y$ onto A such that $f(x,.)$ maps Y onto A^x in a one-one fashion, and f^{-1} is $\underline{\underline{S}}(A)$ - measurable. In particular, A has $2^{\aleph_0}$ disjoint $\underline{\underline{S}}(X)$ - measurable selections.

Proof : By our approximation theorem, there is $B \subseteq A$, $B \in \underline{\underline{S}}(X) \otimes \underline{\underline{B}}_Y$ with non-meager, and therefore uncountable vertical sections. Theorem 3.11 applies (as $\underline{\underline{S}}(X)$ is closed under operation $(\underline{\underline{A}})$) to yield an $\underline{\underline{S}}(X) \otimes \underline{\underline{B}}_Y$ - measurable parametrization on $X\times Y$ onto B with the properties listed therein. A Cantor-Bernstein type argument exactly as in Cenzer and

Mauldin [9] now yields the result.

Next is the result of R.L. Vaught [42].

5.16 Corollary : Let X and Y be Polish and $A \subseteq X \times Y$. Then

(i) $A \in \underline{\underline{S}}(X \times Y) \Rightarrow \{x \in X : A^x$ is non-meager (comeager)$\} \in \underline{\underline{C}}(X)$.

(ii) $A \in \underline{\underline{S}}_\alpha(X \times Y) \Rightarrow \{x \in X : A^x$ is non-meager (comeager)$\} \in \underline{\underline{S}}_\alpha(X)$.

Proof : In either case get $B \subseteq A$, in the product σ-field as in Theorem 5.10. Clearly it is enough to perform the above computations for B. But these are valid in view of Remark 5.13.

5.17 Remark : We conclude this discussion by observing that our methods yield the following. Suppose $A \in \underline{\underline{S}}_\alpha(X \times Y)$, with X, Y Polish, and A^x is comeager for each x. Then there is $\{B_n : n \geq 1\} \subseteq \underline{\underline{S}}_\alpha(X) \otimes \underline{\underline{B}}_Y$ such that B_n^x is open and dense for each x and n, and $\bigcap_{n \geq 1} B_n \subseteq A$.

To see this get $B \subseteq A$ as in Theorem 5.10. An argument as in Proposition 5.12 shows that there is $\{B_n : n \geq 1\}$ with the above properties such that $\bigcap_{n \geq 1} B_n \subseteq B$, because of the validity of this result for Borel sets (See Remark 5.7).

5.18 We will now establish the measure-theoretic counterparts of Theorem 5.10 and its corollaries. But first we need a definition. Let X and Y be Polish. Call $\mu : X \times \underline{\underline{B}}_Y \longrightarrow \mathbb{R}$ a transition function if

(i) $\mu(x,.)$ is a probability measure on $\underline{\underline{B}}_Y$ for each $x \in X$,

(ii) $\mu(.,B)$ is $\underline{\underline{B}}_X$-measurable for each Borel B in Y.

An equivalent formulation is the following : Let $\underline{\underline{M}}(Y)$ be the class of all probability measures on $\underline{\underline{B}}_Y$ equipped with the weak topology. $\underline{\underline{M}}(Y)$ is then a Polish space. Let $\mu : X \times \underline{\underline{B}}_Y \longrightarrow \mathbb{R}$ satisfy condition (i) above. Define

$$\gamma : X \longrightarrow \underline{\underline{M}}(Y) \quad \text{by} \quad \gamma(x) = \mu(x,.).$$

Then μ is a transition function iff $\gamma : X \longrightarrow \underline{\underline{M}}(Y)$ is $(\underline{\underline{B}}_X, \underline{\underline{B}}_{\underline{\underline{M}}(Y)})$-measurable.

We will now formulate and write down less detailed proofs of the measure-analogues of the results we have proved for category.

Notice first that Sion's capacitability argument adapted to measure yields the following computation, implicit in Kechris [14].

5.19 <u>Lemma</u> : Let $A \subseteq X \times Y$ be analytic and $\mu : X \times \underline{\underline{B}}_Y$ into $\mathbb{R}$ a transition function. Then for any real r, $\{x : \mu(x, A^x) > r\}$ is analytic.

<u>Proof</u> : As before fix a continuous map f on D onto A, where $D = \bigcap_{m \geq 1} \bigcup_{n \geq 1} D(m,n)$, $D(m,n)$ compact in $\mathbb{R}$ for $m,n \geq 1$ and so that for each m, $D(m,n) \uparrow$ with n. An easy compactness argument as in Lemma 5.4 shows that, for fixed $\alpha \in \Sigma$,

$$f\Big(\bigcap_{k=1}^{\infty} D(k,\alpha(k))\Big) = \bigcap_{k=1}^{\infty} f\Big(D \cap \bigcap_{i=1}^{k} D(i,\alpha(i))\Big)$$

$$= \bigcap_{k=1}^{\infty} cl\Big(f\Big(D \cap \bigcap_{i=1}^{k} D(i,\alpha(i))\Big)\Big),$$

and thus,

$$\Big(f\Big(\bigcap_{k=1}^{\infty} D(k,\alpha(k))\Big)\Big)^{x} = \bigcap_{k=1}^{\infty} \Big(cl\Big(f\Big(D \cap \bigcap_{i=1}^{k} D(i,\alpha(i))\Big)\Big)\Big)^{x}, \quad \ldots(1)$$

for each x.

Also, as for $x \in X$, $\mu(x,.)$ is continuous from below, and for each k, $\Big(f\Big(D \cap \bigcap_{i=1}^{k} D(i,\alpha(i))\Big)\Big)^{x}$

$$= \bigcup_{n \geq 1} \Big(f\Big(D \cap \Big(\bigcap_{i=1}^{k} D(i,\alpha(i))\Big) \cap D(k+1,n)\Big)\Big)^{x},$$

we have,

$$\mu\Big(x,\Big(f\Big(D \cap \bigcap_{i=1}^{k} D(i,\alpha(i))\Big)\Big)^{x}\Big) > r \longleftrightarrow$$

$$\mu\Big(x,\Big(f\Big(D \cap \Big(\bigcap_{i=1}^{k} D(i,\alpha(i))\Big) \cap D(k+1,n)\Big)\Big)^{x}\Big) > r,$$

for some $n \geq 1$ ...(2)

Thus we may write,

$$\mu(x,A^{x}) > r \overset{(by(2))}{\longleftrightarrow} (\exists \alpha \in \Sigma)\Big[\mu\Big(x,\Big(f\Big(\bigcap_{k=1}^{\infty} D(k,\alpha(k))\Big)\Big)^{x}\Big) > r\Big]$$

$$\overset{(by(1))}{\longleftrightarrow} (\exists n \geq 1)(\exists \alpha \in \Sigma)\Big[(\forall k \geq 1)\Big(\mu\Big(x,\Big(cl\Big(f\Big(D \cap \bigcap_{i=1}^{k} D(i,\alpha(i))\Big)\Big)\Big)^{x}\Big) > r + \frac{1}{n}\Big].$$

But for each finite sequence $\langle \alpha(1),\ldots,\alpha(k)\rangle$ and k, $cl(f(D \cap \bigcap_{i=1}^{k} D(i,\alpha(i))))$ is closed, and the validity of the lemma for Borel sets shows that the condition inside brackets is actually a Borel condition. It follows that $\{x : \mu(x,A^x) > r\}$ is analytic.

Next is a preliminary version of Lemma 5.6.

5.20 <u>Lemma</u> : Let $A \subseteq X \times Y$ be analytic, with X,Y Polish. Suppose $\mu : X \times \underline{\underline{B}}_Y \longrightarrow \mathbb{R}$ is a transition function such that $\mu(x,A^x) > a$ for each $x \in E$, for some fixed $E \in \underline{\underline{S}}_1(X)$, and fixed $0 \leq a < 1$. Then there is a set $C \subseteq A$, $C \in \underline{\underline{S}}_1(X) \otimes \underline{\underline{B}}_Y$, such that C^x is compact and $\mu(x,C^x) \geq a$ for each $x \in E$.

<u>Proof</u> : We define sets $T(s)$, $s \in Seq$, to satisfy

(a) $T(e) = E$

(b) $T(s) \in \underline{\underline{S}}_1(X)$

(c) $T(s) \cap T(t) = \emptyset$ if $s \neq t$ & $lh(s) = lh(t)$

(d) $T(s) = \bigcup_{m \geq 1} T(sm)$

(e) $T(s) \subseteq \{x$ [redacted] $: \mu(x,(f(D \cap \bigcap_{i=1}^{k} D(i,s_i)))^x) > a\}$, where $lh(s)=k$.

Here f is the map on D onto A fixed in the proof of Lemma 5.19.

Suppose $T(s)$ has been defined to satisfy the above conditions for all $s \in Seq$ with $lh(s) \leq k$. Fix $s \in S_k$.

Put

$$T'(m) = \{x \in T(s) : \mu(x, (f(D \cap \bigcap_{i=1}^{k} D(i,s_i) \cap D(k+1,m)))^x) > a\} .$$

By Lemma 5.19, each $T'(m) \in \underline{\underline{S}}_1(X)$, and as in the proof of 5.19, $\bigcup_{m=1}^{\infty} T'(m) = T(s)$. Disjointify these sets to get $T''(m)$. Put

$$T(sm) = T''(m),\ m \geq 1 .$$

These sets satisfy the above conditions. Finally, take

$$C = \bigcap_{k=1}^{\infty} \bigcup_{s \in S_k} P_s , \text{ where}$$

$$P_s = (T(s) \times Y) \cap cl(f(D \cap \bigcap_{i=1}^{k} D(i,s_i))), s \in S_k .$$

It is clear that $C \in \underline{\underline{S}}_1(X)\ (\underline{\underline{X}})\ \underline{\underline{B}}_Y$. Fix $x \in E$. Then there is a unique $\alpha \in \Sigma$ such that $x \in T(\alpha|k)$ for every $k \geq 1$. Consequently,

$$C^x = \bigcap_{k=1}^{\infty} (cl(f(D \cap \bigcap_{i=1}^{k} D(i,\alpha(i)))))^x$$

$$= (f(\bigcap_{k=1}^{\infty} D(k,\alpha(k))))^x \text{ (by (1) in the proof of 5.19)}.$$

So C^x is compact, $C^x \subseteq A^x$, and

$$\mu(x,C^x) = \lim \mu(x,(f(D \cap \bigcap_{i=1}^{k} D(i,\alpha(i))))^x) \geq a .$$

We can now prove the anologue of Lemma 5.6.

5.21 <u>Lemma</u> : Let X, Y, and A be as above. Then there is a sequence $\{B_n\}_{n \geq 1}$ such that $B_n \in \underline{\underline{S}}_1(X)\ (\underline{\underline{X}})\ \underline{\underline{B}}_Y$, B_n^x is compact for each x, $B_n \subseteq A$, for $n \geq 1$ and $\mu(x, A^x - (\bigcup_{n \geq 1} B_n)^x) = 0$ for each x.

Proof : It is enough to show that for each $n \geq 1$, there is B_n such that $\mu(x, A^x - B_n^x) < \frac{1}{n}$, and satisfying the other properties. So fix $n \geq 1$.

Let $E_i = \left\{ x \in X : \frac{i}{n} < \mu(x, A^x) \leq \frac{i+1}{n} \right\}$, $i = 0, 1, \ldots, n-1$.

Clearly, the E_i's are disjoint, and by Lemma 5.19, they are in $\underline{\underline{S}}_1(X)$. Apply Lemma 5.20 to A with E replaced by E_i and $a = \frac{i}{n}$ to get C_i such that $C_i \subseteq A \cap (E_i \times Y)$, C_i^x is compact, and $\mu(x, C_i^x) \geq \frac{i}{n}$ for each $x \in E_i$. Then $B_n = \bigcup_{i=0}^{n-1} C_i$ does the job.

5.22 Remark : Note that (as in Remark 5.7) if in Lemmas 5.20 and 5.21 one assumes that A is Borel, then C and B_n, $n \geq 1$, will turn out to be Borel. This yields a proof of the theorem of Blackwell and Ryll-Nardzewski [4].

Next is a version of Lemma 5.3.

5.23 Lemma : Let X, Y, and μ be as above and let $C \subseteq X \times Y$ be coanalytic. Then there is a sequence $\{C_n\}_{n \geq 1}$ such that $C_n \in \underline{\underline{S}}_1(X) \otimes \underline{\underline{B}}_Y$, $C_n \subset C$, $n \geq 1$, and $\mu(x, C^x - (\bigcup_n C_n)^x) = 0$ for each x.

Proof : Just as we used an effective result of Kechris to prove Lemma 5.8, here we will use its measure analogue, which is also due to Kechris [14], namely :

Let μ be a probability measure on Σ such that the relation

$$R(k,s) \longleftrightarrow Seq(k) \,\&\, Seq(s) \,\&\, lh(s) = 2 \,\&\, \mu(\Sigma(k)) > \frac{s_0}{s_1}$$

is Δ^1_1. Let $\varepsilon > 0$. Then every Π^1_1 subset P of Σ contains a Δ^1_1, compact set Q_ε such that $\mu(P - Q_\varepsilon) < \varepsilon$. The relativized version also holds.

As in Lemma 5.21, it is enough to prove a version of Lemma 5.20. So let $E \in \underset{\sim}{S}_1(X)$ and suppose $\mu(x, C^x) > a$ for each $x \in E$. Furthermore, by going through Borel isomorphisms we may assume without loss of generality that $X = Y = \omega^\omega$.

Consider the relation

$$R(x,k,s) \longleftrightarrow Seq(k) \,\&\, Seq(s) \,\&\, lh(s) = 2 \,\&\, \mu(x, \Sigma(k)) > \frac{s_0}{s_1}$$

As $\mu(\cdot, \Sigma(k))$ is surely Borel measurable for each k, it is easily seen that R is Borel in $\omega^\omega \times \omega \times \omega$. Thus there is $z \in \omega^\omega$ such that R is $\Delta^1_1(z)$. Consequently, for each x, R^x is $\Delta^1_1(z,x)$. Also C, being coanalytic, is $\Pi^1_1(z')$ for some $z' \in \omega^\omega$. The relativized version of the result of Kechris quoted above now shows that for each $x \in E$, C^x contains a $\Delta^1_1(z,z',x)$ compact set $B(x)$ such that $\mu(x, B(x)) \geq a$, and therefore for each $x \in E$, there is a $\Delta^1_1(z,z',x)$ tree $T(x)$ on ω such that $[T(x)] \subseteq C^x$, $[T(x)]$ is compact and has $\mu(x,\cdot)$ measure $\geq a$. As in the proof of Lemma 5.8, now define

$$B_n = \{x : d(x,n)\downarrow \text{ and } d(x,n) \text{ codes a finitely splitting tree } T \text{ such that } [T] \subseteq C^x, \text{ and } \mu(x,[T]) \geq a\}, \quad n \geq 0.$$

It is easily verified (as C is coanalytic and $\mu(x,\cdot)$ is a transition function) that each B_n is coanalytic. Let $f_n : B_n \longrightarrow \omega^\omega$ be the $\underline{\underline{S}}_1(\omega^\omega)\cap B_n$ - measurable function defined by $f_n(x) = d(x,n)$.

Put $B = \bigcup_{n\geq 0}\{(x,\alpha) : f_n(x)(\bar{\alpha}(m)) = 0, \text{ for every } m\}$

It follows that $B \in \underline{\underline{S}}_1(\omega^\omega) \otimes \underline{\underline{B}}_{\omega^\omega}$, $B \subseteq C$, and $\mu(x,B^x) \geq a$ for each x. By remarks made earlier, this completes the proof.

The following is now obvious.

5.24 Lemma : Let X, Y, and μ be as above, and let $A \subseteq X\times Y$ be analytic. Then there are B and C such that $B, C \in \underline{\underline{S}}_1(X) \otimes \underline{\underline{B}}_Y$, $B \subseteq A \subseteq C$, and $\mu(x, C^x - B^x) = 0$ for each $x \in X$.

To see this one has only to argue as in Lemma 5.9.

5.25 Remark : Before proceeding any further we will have to overcome a technical difficulty that does not arise in the category case. Let $E \subseteq X$. Suppose μ is a transition function defined only on $E \times \underline{\underline{B}}_Y$. One can then look at the equivalent $\underline{\underline{B}}_E$ - measurable υ on $E \longrightarrow \underline{\underline{M}}(Y)$. By a well-known theorem on the extension of Borel measurable functions [15], υ has a $\underline{\underline{B}}_X$-measurable extension $v' : X \longrightarrow \underline{\underline{M}}(Y)$, which yields an equivalent μ' on $X \times \underline{\underline{B}}_Y$ into $\mathbb{R}$.

We put down now the fact that the theorem of Blackwell and Ryll-Nardzewski holds in an abstract setting.

5.26 Proposition : Let $(T, \underline{\underline{M}})$ be a measurable space Y Polish. Suppose $\mu : T \times \underline{\underline{B}}_Y \longrightarrow \mathbb{R}$ is such that for each $t \in T$, $\mu(t, .)$ is a probability measure on $\underline{\underline{B}}_Y$, and for each $B \in \underline{\underline{B}}_Y$, $\mu(., B)$ is $\underline{\underline{M}}$-measurable. Let $B \in \underline{\underline{M}} \otimes \underline{\underline{B}}_Y$ such that $\mu(t, B^t) > 0$ for each t. Then B has an $\underline{\underline{M}}$-measurable selection.

Proof : One need only imitate the argument in 5.12. There is however one subtle difference. As in 5.12, it suffices to prove the following : Let X and Y be Polish, $E \subseteq X$, and D Borel in $E \times Y$. Let λ be a transition function on $E \times \underline{\underline{B}}_Y$. Then if $\lambda(x, D^x) > 0$ for each $x \in E$, D has a $\underline{\underline{B}}_E$-measurable selection defined on E.

To see this, observe that by Remark 5.25, λ is the restriction to E of a transition function defined on $X \times \underline{\underline{B}}_Y$. Now proceed as in 5.12.

Virtually all the steps in the proof of Theorem 5.10 go through ditto to yield its measure analogue. Lemma 5.24 gives the base step, and a small additional argument using Remark 5.25 is needed to carry out the inductive step, just as in the proof of Proposition 5.26. Thus we have

5.27 <u>(Measure) Approximation Theorem</u> : Let X and Y be Polish and $\mu : X \times \underline{\underline{B}}_Y \longrightarrow \mathbb{R}$ a transition function. Let $A \in \underline{\underline{S}}_\alpha(X \times Y)$, $\alpha < \omega_1$. Then there are B and C in $\underline{\underline{S}}_\alpha(X) \; (\underline{x}) \; \underline{\underline{B}}_Y$ such that $B \subseteq A \subseteq C$, and $\mu(x, C^x - B^x) = 0$ for each $x \in X$. Furthermore, one can write $B = \bigcup_{n \geq 1} B_n$ with $B_n \in \underline{\underline{S}}_\alpha(X) \; (\underline{x}) \; \underline{\underline{B}}_Y$ such that B_n^x is compact for each x.

<u>Proof</u> : One need now check only the second statement. Get $B \in \underline{\underline{S}}_\alpha(X) \; (\underline{x}) \; \underline{\underline{B}}_Y$ as in the first part of the theorem. Observe now that the required result holds in complete generality. For, in the setting of Proposition 5.26, if $C \in \underline{\underline{M}} \; (\underline{x}) \; \underline{\underline{B}}_Y$, then arguing as in the proof of 5.26, we can, in view of Remark 5.22, by the usual Marczewski function argument get $C_n \in \underline{\underline{M}} \; (\underline{x}) \; \underline{\underline{B}}_Y$ such that $\bigcup_{n \geq 1} C_n = C$, $\mu(x, C^x - \bigcup_n C_n^x) = 0$, and C_n^x is compact for each x and n.

One can now, as before, conclude selection theorems such as

5.28 <u>Corollary</u> : Let X, Y, μ be as in Theorem 5.27. Suppose $A \subseteq X \times Y$ and $\mu(x, A^x) > 0$ for each x. Then

(i) $A \in \underline{\underline{S}}(X \times Y) \Rightarrow A$ has an $\underline{\underline{S}}(X)$-measurable selection.

(ii) $A \in \underline{\underline{S}}_\alpha(X \times Y), \alpha < \omega_1 \Rightarrow A$ has an $\underline{\underline{S}}_\alpha(X)$-measurable selection.

One has only to argue as in Corollary 5.11. Once again (ii) follows from (now) the fact that the theorem of Blackwell and Ryll-Nardzewski holds for an abstract measurable space.

One can also obtain analogues of Remark 5.13, and if $\mu(x,.)$ is continuous for each x, of Corollary 5.15. The measure version of the computation in Corollary 5.16 also holds.

5.29 <u>Corollary</u> : Let X, Y, and μ be as in Theorem 5.27. Then

(i) $A \in \underline{\underline{S}}(X \times Y) \Rightarrow \{x \in X : \mu(x, A^x) > r\} \in \underline{\underline{S}}(X)$ for each r

(ii) $A \in \underline{\underline{S}}_\alpha(X \times Y) \Rightarrow \{x \in X : \mu(x, A^x) > r\} \in \underline{\underline{S}}_\alpha(X)$, for each r.

An immediate consequence of this is a result of Shreve [32] proved in connection with Dynamic Programming. The result reads as follows :

5.30 <u>Corollary</u> : Let $f : [0,1] \longrightarrow [0,1]$ be $\underline{\underline{S}}_\alpha([0,1])$-measurable. Then $g : \underline{\underline{M}}([0,1]) \longrightarrow [0,1]$ defined by $g(\mu) = \int f d\mu$ is $\underline{\underline{S}}_\alpha(\underline{\underline{M}}([0,1]))$-measurable.

<u>Proof</u> : We have only to show that the function $\mu \longrightarrow \mu(C)$ is $\underline{\underline{S}}_\alpha(\underline{\underline{M}}([0,1]))$-measurable for each fixed C in $\underline{\underline{S}}_\alpha([0,1])$. Observe that if λ is defined on $\underline{\underline{M}}([0,1]) \times \underline{\underline{B}}_{[0,1]}$ by

$$\lambda(\mu, B) = \mu(B), \text{ then}$$

λ is a transition function (because of the validity of the result for Borel sets). Let $A = \underline{\underline{M}}([0,1]) \times C$. Then $A \in \underline{\underline{S}}_\alpha(\underline{\underline{M}}([0,1]) \times [0,1])$. Also

$$\{\mu : \mu(C) > r\} = \{\mu : \lambda(\mu, A^\mu) > r\}.$$

By Corollary 5.29, the last set is in the desired σ-field.

6. Continuous-open parametrizations : Let X be Polish and T an abstract set. In this section we will consider the existence of Caratheodory representations f defined on $T \times \Sigma$ (for sets in $T \times X$), such that $f(t,.)$ is continuous and open for each $t \in T$. We have come across such representations in the last section, where we have used such a representation for closed valued multifunctions to prove selection theorems (cf. Corollary 5.14). As the image of Σ under a continuous and open map is a G_δ, one can only hope to obtain such parametrizations for G_δ-valued multifunctions. Indeed, Sarbadhikari and Srivastava [31] have proved the following.

6.0 Theorem : Let T and X be Polish spaces and $\underline{\underline{A}}$ a countably generated sub σ-field of $\underline{\underline{B}}_T$. Suppose $F : T \longrightarrow X$ is a $\underline{\underline{B}}_T$-measurable, G_δ-valued multifunction such that $Gr(F) \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$. Then there is an $\underline{\underline{A}} \otimes \underline{\underline{B}}_X$-measurable map $f : T \times \Sigma \longrightarrow X$ such that for each $t \in T$, $f(t,.)$ is continuous, open, and onto $F(t)$.

Recall that Srivastava had earlier shown that in the above situation F has an $\underline{\underline{A}}$-measurable selection.

Having proved this, the above authors then asked whether Caratheodory maps of the above kind characterise multifunctions of the type specified in the hypothesis. We show in this section that this is not true, a fact that contrasts sharply with the situation for continuous, closed parametrizations [cf. Srivastava [37])

The content of this section is also motivated by the question whether Theorem 6.0 can be extended to the framework of the Kuratowski-Ryll-Nordzewski selection theorem. Debs [11] has considered this problem for Srivastava's selection theorem. By assuming that the graph is of a certain form, Debs has established a selection theorem for G_δ-valued multifunctions in the set-up of Kuratowski and Ryll-Nardzewski. We will show that Theorem 6.0 also holds in this situation. When specialized to closed valued multifunctions this yields the parametrization theorem we have used in proving Corollary 5.14. By the same argument the existence of such parametrizations shows that even if, in Theorem 6.0, $\underline{\underline{A}}$ is taken to be an arbitrary σ-field on T, F still has an $\underline{\underline{A}}$-measurable selection. Thus the Borel selection theorems of Blackwell and Ryll-Nardzewski, Sarbadhikari, Srivastava all hold for abstract σ-fields (in view of Propositions 5.12 and 5.26).

To return to the possible converse to Theorem 6.0, we show, specifically, that there is a Borel-measurable, G_δ-valued multifunction with analytic, non-Borel graph that admits such a representation. Do all such analytic sets then have Borel selections? We shall observe that there are analytic sets with sections that are " large " in every conceivable sense that do not contain coanalytic graphs. Burgess [7] (see also Maitra [20]) has since proved that the selection theorem implicit in 6.0 holds if Gr(F) is taken to be only coanalytic.

We will first prove a slight generalization of the selection theorem in Debs [11], primarily because the rewriting of Debs' proof involved in this, is exactly what we need to prove the representation theorem. The next definition is from Maitra and Rao [21].

6.1 <u>Definition</u> : We say $\underline{\underline{E}} \subseteq \underline{\underline{P}}(T)$ (T being an abstract set) satisfies the <u>weak reduction principle</u> (and we write $WRP(\underline{\underline{E}})$) if for any sequence of sets $E_1, E_2, \ldots$ from $\underline{\underline{E}}$ such that $\bigcup_{n=1}^{\infty} E_n = T$, we can find pairwise disjoint $E_1', E_2', \ldots$ from $\underline{\underline{E}}$ such that $E_i' \subseteq E_i$ for each i and $\bigcup_{i=1}^{\infty} E_i' = T$.

We remark here that it is well known that if $\underline{\underline{E}}$ is a field, then $WRP(\underline{\underline{E}}_\sigma)$.

An important family satisfying the weak reduction principle is, of course, the family of coanalytic subsets of a Polish space. The statement of our selection theorem is motivated partly by the desire to include this family.

6.2 We now fix some notation. In what follows, X will denote a Polish space with a metric d such that $\delta(X) < 1$. The topology on X will be denoted by $\underline{\underline{G}}$. We fix a base $\{V(n),\ n \in N\}$ for X such that $V(o) = X$ and $V(n) \neq \emptyset$ for each n. Also, in what follows, T will be a non-empty set and $\underline{\underline{E}}$ a family of subsets of T containing $\emptyset$ and T, closed under finite intersections and countable unions and such that, moreover, $WRP(\underline{\underline{E}})$.

Suppose that $F : T \longrightarrow X$ is a fixed $\underline{\underline{E}}$-measurable multifunction such that $Gr(F) \in (\underline{\underline{E}} \times \underline{\underline{U}})_{\sigma\delta}$. Set $G = Gr(F)$ and and write $G = \bigcap_{n=1}^{\infty} G_n$, where $G_n \supseteq G_{n+1}$ and

$G_n = \bigcup_{m=1}^{\infty} (E_{nm} \times U_{nm})$ with $E_{nm} \in \underline{\underline{E}}$ and $U_{nm} \in \underline{\underline{U}}$, $n,m \geq 1$.

The following lemma is implicit in [11].

6.5 Lemma : Let X be compact metric. If $H \subseteq T \times X$ and $H \in (\underline{\underline{E}} \times \underline{\underline{U}})_{\sigma}$, then for any closed set $C \subseteq X$, the set $\{t \in T : C \subseteq H^t\} \in \underline{\underline{E}}$.

Proof : Let $H = \bigcup_{n=1}^{\infty} E_n \times U_n$, with $E_n \in \underline{\underline{E}}$ and $U_n \in \underline{\underline{U}}$. Then, as C is compact, $\{t \in T : C \subseteq H^t\} = \bigcup (E_{n_1} \cap \ldots \cap E_{n_k})$ where the union runs through all finite sequences $(n_1, \ldots, n_k)$ such that $C \subseteq U_{n_1} \cup U_{n_2} \ldots \cup U_{n_k}$. Now use the closure properties of $\underline{\underline{E}}$.

6.4 Lemma : Let $\{T(u)\}$ and $\{U(u)\}$ be regular systems of sets belonging to $\underline{\underline{E}}$ and $\underline{\underline{U}}$ respectively such that :

(i) $T(e) = T$.

(ii) $T(u) = \bigcup_{n=1}^{\infty} T(un)$ for each $u \in Seq$.

(iii) $T(u) \cap T(v) = \emptyset$ if $u, v \in Seq$, $|u| = |v|$ and $u \neq v$.

(iv) $\delta(U(u)) < 2^{-|u|}$, for each u.

(v) $T(u) \neq \emptyset \Rightarrow U(u) \neq \emptyset$.

Put $M_k = \bigcup_{u \in S_k} (T(u) \times cl(U(u)))$ and $M = \bigcap_{k=1}^{\infty} M_k$.

Then M is the graph of an $\underline{\underline{E}}$-measurable function f.

Further, if each $T(u) \in (\underline{\underline{E}} \cap \underline{\underline{E}}^c)$, then f is $(\underline{\underline{E}} \cap \underline{\underline{E}}^c)_\sigma$-measurable.

Proof : Clearly M is the graph of a function $f : T \longrightarrow X$.

Now, if W is an open set in X, then

$$f^{-1}(W) = \bigcup \left\{ T(v) : cl(U(v)) \subseteq W \right\}$$

because

$$f(t) \in W \Longleftrightarrow (\exists v)(t \in T(v) \ \& \ cl(U(v)) \subseteq W).$$

Thus f is $\underline{\underline{E}}$-measurable (respectively $(\underline{\underline{E}} \cap \underline{\underline{E}}^c)_\sigma$-measurable).

The following is essentially the theorem in Debs [11], and subsumes the theorems of Kuratowski and Ryll-Nardzewski [17], Srivastava [36], Maitra and Rao [21], and, of course, Debs [11].

6.5 Theorem : Let X be a Polish space, T a non-empty set and $\underline{\underline{E}}$ a finitely multiplicative, countably additive family of subsets of T, containing $\emptyset$ and T and satisfying $WRP(\underline{\underline{E}})$. Let $F : T \longrightarrow X$ be an $\underline{\underline{E}}$-measurable multifunction with $Gr(F) \in (\underline{\underline{E}} \times \underline{\underline{U}})_{\sigma\delta}$, $\underline{\underline{U}}$ being the topology of X. Then F has an $(\underline{\underline{E}} \cap \underline{\underline{E}}^c)_\sigma$-measurable selection.

Proof : By taking a metric compactification of X, in which X is automatically a G_δ, we see that we can without loss of generality take X to be compact.

We shall obtain systems $\{T(s)\}$, $\{U(s)\}$, satisfying conditions (i) - (v) of Lemma 6.4 and further satisfying

(vi) $cl(U(s)) \subseteq G^t_k$, for $s \in S_k$ and $t \in T(s)$,

(vii) $U(s) \cap G^t \neq \emptyset$ for each $t \in T(s)$, and

(viii) $T(s) \in \underline{\underline{E}} \cap \underline{\underline{E}}^c$ for each $s \in S$.

Once such a system exists, Lemma 6.4 gives an $(\underline{\underline{E}} \cap \underline{\underline{E}}^c)_\sigma$-measurable function $f : T \longrightarrow X$ with graph M. By condition (vi), $M \subseteq C$. The map f is then the required selection for F.

It remains now to construct $\{T(s)\}$ and $\{U(s)\}$. The construction is by induction on $k = |s|$. For $k = 0$, put $T(e) = T$ and $U(e) = X$.

We will now define $T(sn)$, $U(sn)$, for all $n \geq 0$, assuming that $T(s)$ and $U(s)$ have been defined for fixed $s \in S_k$.

Put $A^m(s) = T(s) \cap \{t : V(m) \cap F(t) \neq \emptyset\} \cap \{t : cl(V(m)) \subseteq G^t_{k+1}\}$,

if $V(m) \subseteq U(s)$ and $\delta(V(m)) < \frac{1}{2^{k+1}}$,

$= \emptyset$, otherwise.

As F is $\underline{\underline{E}}$-measurable, $\{t : V(m) \cap F(t) \neq \emptyset\} \in \underline{\underline{E}}$.

By Lemma 6.5, X being compact, we have $\{t \in T : cl(V(m)) \subseteq G^t_{k+1}\} \in \underline{\underline{E}}$. By the closure properties of $\underline{\underline{E}}$, therefore, $A^m(s) \in \underline{\underline{E}}$ for each $m \geq 0$. Further, the induction hypothesis and (vii) yield $\bigcup_{m \geq 0} A^m(s) = T(s)$.

Now, as WRP($\underline{\underline{E}}$), we can obtain a pairwise disjoint family $\{B^m(s) : m \geq 0\} \subseteq \underline{\underline{E}}$ "reducing" $\{A^m(s)\}_{m \geq 0}$ i.e., the $B^n(s)$ satisfy $B^m(s) \subseteq A^m(s)$ and $\bigcup B^m(s) = \bigcup A^m(s)$. Now observe that, $\underline{\underline{E}}$ being closed under countable unions, we actually have $B^m(s) \in \underline{\underline{E}} \cap \underline{\underline{E}}^c$ for each m.

Put $T(sn) = B^n(s)$, $n \geq 0$ and

$$U(sn) = V(n) \text{ , if } T(sn) \neq \emptyset$$
$$= \emptyset \text{ , otherwise .}$$

It is easily seen that $\{T(s)\}$ and $\{U(s)\}$ so defined satisfy conditions (i) - (viii). This completes the proof of the theorem.

6.6 Remark : If $\underline{\underline{M}}$ is a field on T and we take $\underline{\underline{E}} = \underline{\underline{M}}_\sigma$, then, we have WRP($\underline{\underline{E}}$). The theorem in [11] follows.

That the multifunction F in Theorem 6.0 has an $\underline{\underline{A}}$-measurable selection is also immediate. The argument that $Gr(F)$ is of the required form is implicit in Lemma 3.8 of [36], for by this we may write $Gr(F) = \bigcap_{n \geq 1} B_n$, where each $B_n \in \underline{\underline{A}} \otimes \underline{\underline{B}}_X$ with B_n^t open for each t. As usual, we may assume without loss of generality that X is compact. For each $n \geq 1$, then,

$$B_n = \bigcup_{m \geq 1} (\{t \in T : V(m) \subseteq B_n^t\} \times V(m)).$$

But $\{t : V(m) \subseteq B_n^t\} = T - \pi_T(V(m) \cap ((T \times X) - B_n))$, which X, and therefore, $((T \times X) - B_n)^t$ being compact, is a Borel set by the result of Kunugui and Novikov [16]. Furthermore, this set is a union of $\underline{\underline{A}}$-atoms, and thus, by 2.3, belongs to $\underline{\underline{A}}$. This shows that $Gr(F)$ has the required representation .

To see that the theorem of Maitra and Rao follows, suppose F is closed valued. Let $r_1, r_2, \ldots$ be dense in X. Then

$$Gr(F) = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} (F^{-1}(S(r_i, \tfrac{1}{n})) \times S(r_i, \tfrac{1}{n})),$$

where $S(r_i, \frac{1}{n})$ is the open sphere with centre r_i and radius $\frac{1}{n}$. Again $Gr(F)$ has the required form.

6.7 Remark : If T is Polish, for $\underline{\underline{E}}$ we can take the family of sets of the additive class α, for any ordinal $\alpha > 0$, or the family of coanalytic sets to obtain selections of the respective classes, and in the last case, a Borel selection.

The question naturally arises : Is every coanalytic set with G_δ sections a countable intersection of coanalytic sets with open sections, for then any such set would have the form specified in Theorem 6.5. However, J.R. Steel [41] has given an example of a coanalytic set with dense G_δ sections which does not admit such a representation. (See Section 7 of this thesis in this connection).

6.8 We will now establish the parametrization theorem promised in the introduction. The notation we have fixed so far continues. However, we will make two additional assumptions. We will require, firstly, that $\underline{\underline{E}} = \underline{\underline{M}}_\sigma$, where $\underline{\underline{M}}$ is a field on T. We will also add, for technical reasons, the innocuous assumption that each

$V(n)$ appears in $\{V(n)\}_{n\geq 0}$ infinitely often.

While our proof follows the general lines of the proof of the Theorem in [31], there are some technical difficulties to be overcome. The crux of the argument lies in

6.9 Lemma : Let X be compact. For each $s \in \mathrm{Seq}$, there is a map $p(\cdot,s) : T \longrightarrow N$ such that :

a) $p(\cdot,s)$ is $\underline{\underline{E}}$-measurable.

b) $\delta(V(p(t,s))) < \frac{1}{2^k}$ for $s \in S_k$ and $t \in T$.

c) $cl(V(p(t,sn))) \subseteq G^t_{k+1} \cap V(p(t,s))$, $n \geq 0$.

d) $F(t) \cap V(p(t,s)) \neq \emptyset$.

e) $F(t) \cap V(p(t,s)) \subseteq \bigcup_{n=0}^{\infty} V(p(t,sn))$.

f) $F(t) \subseteq V(p(t,e))$.

Proof : The proof will be by induction on $|s|$. Define $p(t,e) \equiv 0$. Suppose $p(t,s)$ has been defined for $s \in S_k$. We shall, as always, define $p(t,sn)$, for $n \geq 0$.

Put $R_m = \emptyset$, if $\delta(V(m) \geq \frac{1}{2^{k+1}}$.

$$= \{t \in T : F(t) \cap V(m) \neq \emptyset,\ cl(V(m)) \subseteq V(p(t,s)) \text{ and } cl(V(m)) \subseteq G^t_{k+1}\}$$

if $\delta(V(m)) < \frac{1}{2^{k+1}}$.

By Lemma 6.4, $\{t : cl(V(m)) \subseteq G^t_{k+1}\} \in \underline{\underline{E}}$.

Now, $\{t : cl(V(m)) \subseteq V(p(t,s))\} = \bigcup \{t : p(t,s) = \ell\}$,

the union running over all ℓ such that $cl(V(m)) \subseteq V(\ell)$.
As $p(\cdot,s)$ is $\underline{\underline{E}}$-measurable, we have,
$\{t : cl(V(m)) \subseteq V(p(t,s))\} \in \underline{\underline{E}}$. Consequently, by the closure properties of $\underline{\underline{E}}$, F being $\underline{\underline{E}}$-measurable, each $R_m \in \underline{\underline{E}}$.

Let $R_m = \bigcup_{\ell \geq 0} Q_{ml}$, with $Q_{ml} \in \underline{\underline{M}}$.

Let $i \longrightarrow (m_i, \ell_i)$ be a one-one mapping from ω onto $\omega \times \omega$.
Put $P_i = Q_{m_i \ell_i} \in \underline{\underline{M}}$.

Observe now that $\bigcup_{m=0}^{\infty} R_m = T$. Furthermore, as the base $\{V(n)\}$ has been chosen so that each $V(n)$ appears infinitely often, it follows that, for each fixed $t \in T$, $\{m : t \in R_m\}$ is infinite, and consequently, that $\{i : t \in P_i\}$ is infinite.

Define $p(t,sn) = m_i$, if i is the $(n+1)^{st}$ integer j such that $t \in P_j$.

As $\{i : t \in P_i\}$ is infinite for every t, $p(\cdot,sn)$ is defined on the whole of T for each $n \geq 0$. Now,

$$p(t,so) = m \longleftrightarrow (\exists i)\,[m = m_i \text{ and } t \in P_i \text{ and } (\forall j < i)\,(t \notin P_j)].$$

As each $P_i \in \underline{\underline{M}}$ and $\underline{\underline{M}}$ is a field, it follows that
$\{t : p(t,so) = m\} \in \underline{\underline{M}}_\sigma = \underline{\underline{E}}$.

Further, for $n \geq 1$,

$$p(t,sn) = m \Longleftrightarrow (\exists i)\Big[m_i = m \text{ and } t \in P_i \text{ and } (\exists j_1 < j_2 < \dots < j_n < i)(t \in P_{j_1} \cap \dots \cap P_{j_n} \text{ and } (\forall j)(j < i \text{ and } j \notin \{j_1, \dots, j_n\} \longrightarrow t \notin P_j))\Big].$$

Here again the expression within brackets is a Boolean combination of the P_i's. Thus, as $\underline{\underline{M}}$ is a field, we have,

$$\{t : p(t,sn) = m\} \in \underline{\underline{M}}_\sigma = \underline{\underline{E}}.$$

Consequently, $p(.,sn)$ is $\underline{\underline{E}}$-measurable for each $n \geq 0$. One easily checks that the system of functions $p(.,s)$ as defined above satisfies conditions (a) - (f). This proves the lemma.

We now prove the representation theorem :

6.10 <u>Theorem</u> : Let X be a Polish space, T a non-empty set and $\underline{\underline{M}}$ a field on T. Put $\underline{\underline{E}} = \underline{\underline{M}}_\sigma$. Let $F : T \longrightarrow X$ be an $\underline{\underline{E}}$-measurable multifunction with $Gr(F) \in (\underline{\underline{E}} \times \underline{\underline{U}})_{\sigma\delta}$, $\underline{\underline{U}}$ being the topology of X. Then there exists a map $f : T \times \Sigma \longrightarrow X$ satisfying :

(i) for each $t \in T$, $f(t,.) : \Sigma \longrightarrow X$ is continuous, open, and onto $F(t)$,

(ii) for each $\sigma \in \Sigma$, the map $f(.,\sigma) : T \longrightarrow X$ is $\underline{\underline{E}}$-measurable.

<u>Proof</u> : As before, without loss of generality we take X to be compact. By Lemma 6.9, we have a system $p(.,s)$ of functions

satisfying conditions (a) - (f).

Define $f : T \times \Sigma \longrightarrow X$ by :

$$f(t,\sigma) = \bigcap_{k=1}^{\infty} cl(V(p(t,\sigma|k))) .$$

Plainly, for each t and σ, the intersection on the right reduces to a singleton. The map f is therefore well-defined. Fix $\sigma \in \Sigma$. By (c) $f(\cdot,\sigma)$ is a selection for F. Fix open W in X. Then

$$f(t,\sigma) \in W \longleftrightarrow (\exists k)(cl(V(p(t,\sigma|k))) \subseteq W)$$

$$\longleftrightarrow (\exists s)(\exists k)(p(t,\sigma|k) = s \text{ and } cl(V(s)) \subseteq W)$$

As the maps $p(t,\sigma|k)$ are $\underline{\underline{E}}$-measurable, and $\underline{\underline{E}}$ is closed under countable unions, $f(\cdot,\sigma)$ is $\underline{\underline{E}}$-measurable. Conditions (b), (c), and (e) ensure that $f(t,\cdot)$ is a continuous-open map on Σ onto $F(t)$. This proves the theorem.

6.11 Remark : The above proof does not go through under the weaker assumptions made on $\underline{\underline{E}}$ in Theorem 6.5. Lemma 6.9 makes essential use of the fact that $\underline{\underline{E}}$ is of the type $\underline{\underline{M}}_\sigma$, with $\underline{\underline{M}}$ a field on T. Maitra and Rao [22] is of interest in this context.

Furthermore, Theorem 6.0 is a consequence by virtue of the observations made in Remark 6.7. One can also, as in Remark 6.7, take for $\underline{\underline{E}}$ the family of sets of additive class α on a metric space T, for $\alpha > 0$, to get selections of the corresponding class.

Now let T, X, $\underline{\underline{M}}$, and $\underline{\underline{E}}$ be as above, and suppose $F : T \longrightarrow X$ is an $\underline{\underline{E}}$-measurable, closed valued multifunction. Then as argued in Remark 6.6, $Gr(F)$ is of the required form. Thus we have the following improvement of the main result in Srivastava [35] (In [35], the maps $f(t,.)$ need not be open).

6.12 Corollary : Let T, X, $\underline{\underline{E}}$ be as above. Let $F : T \longrightarrow X$ be an $\underline{\underline{E}}$-measurable, closed valued multifunction. Then there is $f : T \times \Sigma \longrightarrow X$ satisfying conditions (i) and (ii) of Theorem 6.10.

6.13 An interesting consequence of this is that arguing as in Corollary 5.14, via the trick due to R. Barua one sees that Srivastava's selection theorem holds for arbitrary measurable spaces. Burgess has recently shown (See also A. Maitra [20]) in [7] that the following is true : Let X be an analytic space and Y Polish. Suppose $F : X \longrightarrow Y$ is a $\underline{\underline{B}}_X$-measurable multifunction such that (i) $F(x)$ is non-meager in $cl(F(x))$ for each $x \in X$, and (ii) $Gr(F)$ is relatively coanalytic in $X \times Y$. Then F has a $\underline{\underline{B}}_X$-measurable selection. This generalises and unifies the theorems of Sarbadhikari and Srivastava. We can now prove

6.14 Corollary : Let $(T, \underline{\underline{M}})$ be a measurable space, and X Polish. Suppose $F : T \longrightarrow X$ is an $\underline{\underline{M}}$-measurable multifunction such that (i) $F(t)$ is non-meager in $cl(F(t))$ for each t, and (ii) $Gr(F) \in \underline{\underline{M}} \; (\underline{\underline{x}}) \; \underline{\underline{B}}_X$. Then F has an $\underline{\underline{M}}$-measurable selection.

Proof : Put $G = Gr(F)$ and define a multifunction $H : T \longrightarrow X$ by $H(t) = cl(F(t))$. The H is closed valued and $\underline{\underline{M}}$-measurable. As $\underline{\underline{M}}$ is a σ-field, 6.12 applies to yield $h : T \times \Sigma \longrightarrow X$ satisfying

(a) $h(t,.)$ is continuous, open and onto $H(t)$, $t \in T$,

(b) $h(.,\sigma)$ is an $\underline{\underline{M}}$-measurable selection for H, $\sigma \in \Sigma$.

It follows that h is $\underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma}$-measurable.

Let $G' = \{(t,\sigma) \in T \times \Sigma : h(t,\sigma) \in G^t\} \in \underline{\underline{M}} \otimes \underline{\underline{B}}_{\Sigma}$.

Then, as the inverse image under a continuous, open map of a non-meager set is non-meager, and G^t is non-meager in $H(t)$, it follows that $(G')^t$ is non-meager for each $t \in T$. Thus Proposition 5.12 applies to G' to yield an $\underline{\underline{M}}$-measurable selection for G', say g. Then $h(t,g(t))$ is an $\underline{\underline{M}}$-measurable selection for F.

6.15 Counterexamples : We will now take up the question of the converse to Theorem 6.0. So suppose T and X are Polish spaces and $F : T \longrightarrow X$ is a multifunction induced by a $\underline{\underline{B}}_T \otimes \underline{\underline{B}}_{\Sigma}$-measurable, continuous-open Caratheodory map $f : T \times \Sigma \longrightarrow X$ as in Theorem 6.0. Then, as observed in [31], F is $\underline{\underline{B}}_T$-measurable. Moreover, by a theorem of Hausdorff, continuous, open images of absolute G_δ sets are absolute G_δ's. Consequently, F is G_δ valued. The question has been posed in [31] whether in this situation $Gr(F)$ is necessarily in $\underline{\underline{B}}_T \otimes \underline{\underline{B}}_X$. An answer in the affirmative would yield a characterization of multifunctions of the type specified in Theorem 6.0 in terms of such representations. We remark here that in [37], it has been shown that such multifunctions are indeed characterised by Caratheodory maps $f : T \times \Sigma \longrightarrow X$ where each $f(t,.)$ is a continuous, closed map onto $F(t)$, for each t.

We shall see below that there is an analytic, non-Borel set that is induced by a continuous-open Caratheodory map. On the other hand, if, in Theorem 6.0, it is assumed that $Gr(F)$ is analytic while the other conditions remain in force, it need not even contain a coanalytic graph.

6.16 <u>Example 1</u> : Let T, X be uncountable Polish spaces. Let A be an analytic, non-Borel subset of T. Fix $x_o \in X$ such that x_o is not an isolated point.

Let $G \subseteq T \times X$ be defined by :

$$G = (A \times \{x_o\}) \cup (T \times (X - \{x_o\})).$$

Then G is an analytic, non-Borel subset of $T \times X$. Consider the multifunction $F : T \longrightarrow X$ defined by :

$$F(t) = G^t .$$

Then (i) As each $F(t)$ is dense in X, F is $\underline{\underline{B}}_T$-measurable.

[The assumption that x_o is not isolated is needed to ensure that $F(t)$ is dense]

(ii) Each $F(t)$, being open, is a G_δ in X.

(iii) $Gr(F) = G$ is an analytic, non-Borel set.

We will now show that F is induced by a continuous, open Caratheodory map $f : T \times \Sigma \longrightarrow X$ providing us with our counter-example. This will be done by obtaining a subset H of $T \times X \times \Sigma$ satisfying :

(a) for each $t \in T$, $(X - \{x_0\}) \times \Sigma \subseteq H^t$,

(b) H is a G_δ in $T \times X \times \Sigma$,

(c) $\pi_{T \times X}(H) = G$.

Suppose, first, that such an H has been obtained. Consider the multifunction $K : T \longrightarrow X \times \Sigma$ given by :

$$K(t) = H^t.$$

Then, by (a), K is $\underline{\underline{B}}_T$-measurable. By (b), K is G_δ-valued and $Gr(K) = H \in \underline{\underline{B}}_T \otimes \underline{\underline{B}}_{X \times \Sigma}$. Thus K satisfies the hypotheses of Theorem 1.2. There is therefore a Caratheodory map $h : T \times \Sigma \longrightarrow X \times \Sigma$ inducing K such that for each t, $h(t,.)$ is a continuous, open map onto $K(t)$. Look at the map $f : T \times \Sigma \longrightarrow X$ defined by :

$$f(t,\sigma) = \pi_X(h(t,\sigma)).$$

Then, clearly, for each $\sigma \in \Sigma$, h being $\underline{\underline{B}}_T$-measurable, so is $f(.,\sigma) : T \longrightarrow X$. As π_X is continuous and (c) holds, we have

$$f(t,.) : \Sigma \longrightarrow X \text{ is continuous and onto } F(t).$$

Let $U = U_1 \times U_2 \subseteq X \times \Sigma$, with U_1 open in X and U_2 open in Σ. Fix $t \in T$. Then $\pi_X(U \cap H^t)$ is either U_1 or $U_1 - \{x_0\}$. In either case $\pi_X(U \cap H^t)$ is open. It follows that π_X is an open map on the range of $h(t,.)$, and hence so is its composition with the open map $h(t,.) : \Sigma \longrightarrow X \times \Sigma$. Thus, the map f gives the required representation.

Finally, it remains to find H. As A is analytic there is a closed subset C of $T \times \Sigma$ such that $\pi_T(C) = A$.

Let $$H = (T \times (X - \{x_0\}) \times \Sigma) \cup (\{(t,x,\sigma) \in T \times X \times \Sigma : (t,\sigma) \in C \;\&\; x = x_0\}).$$

This H satisfies (a), (b), and (c).

6.17 <u>Example 2</u> : We will now show there is an analytic set in $\Sigma \times \Sigma$ with dense, open sections with no Borel selection. Thus the theorem of Burgess referred to earlier is the best possible.

Let $C \subseteq \Sigma \times \Sigma \times \Sigma$ be a coanalytic set in $\Sigma \times \Sigma \times \Sigma$ which is universal for all coanalytic sets in $\Sigma \times \Sigma$. To fix ideas, we assume that the sections of C obtained by fixing the first coordinate run through the coanalytic subsets of $\Sigma \times \Sigma$. Consider $D = \{(x,z) \in \Sigma \times \Sigma : (x,x,z) \in C\}$. Then D is coanalytic in $\Sigma \times \Sigma$. By Kondo's uniformization theorem [16] there is a coanalytic uniformization for D, i.e., there is coanalytic $B \subseteq D$ such that B^x is a singleton whenever $D^x \neq \emptyset$.

Let $A = (\Sigma \times \Sigma) - B$, and define a multifunction $F : \Sigma \longrightarrow \Sigma$ by $F(x) = A^x$. Then,

(a) As each A^x is dense in Σ, F is $\underline{\underline{B}}_\Sigma$-measurable. In fact, A^x is either Σ or Σ minus a point.

(b) $F(x)$ is open in X, for each x.

(c) $Gr(F) = A$ is analytic in $\Sigma \times \Sigma$.

However, F admits no Borel selection. Indeed, A has no coanalytic uniformization, a fortiori, no Borel uniformization. For if not, let $E \subseteq A$ be a coanalytic set uniformizing A. As C is universal, there is $x^* \in \Sigma$ such that $E = C^{x^*}$. Now, these being a unique $y^* \in \Sigma$ such that $(x^*, y^*) \in E$, we have $D^{x^*} = y^*$. Consequently $(x^*, y^*) \in B$. But $E \subseteq A$. So $(x^*, y^*) \in A = (\Sigma \times \Sigma) - B$, a patent absurdity !

As the sections of A, in the above example, are of measure one under any continuous probability on Σ, are comeager, and are dense G_δ's, it follows that the uniformization results of Blackwell and Ryll-Nardzewski [4], Sarbadhikari [30], and Srivastava [36] do not extend to analytic sets.

7. Analytic sets with σ-compact sections : We will now see that a refinement of the method of 6.17 yields a solution to a problem of J.R. Steel, a problem that has its genesis in the following question posed by C. Dellacherie : Suppose $A \subseteq \omega^\omega \times \omega^\omega$ is analytic (Σ^1_1) and has σ-compact vertical sections. Can A be written as $\bigcup_{n \geq 0} A_n$, where each A_n is analytic and has compact vertical sections. In [41], Steel showed that Dellacherie's question admits a negative answer. Steel then formulated the following question : Suppose $A \subseteq \omega^\omega \times \omega^\omega$ is analytic and has σ-compact vertical sections. Does there exist an analytic set B such that $B \subseteq A$, B has compact vertical sections and $\pi(B) = \pi(A)$, where π denotes projection to the first coordinate. We prove in this section that the answer to Steel's question is also negative.

For convenience, we follow the notation and terminology of Moschovakis [25].

7.1 Theorem : There exists a Σ^1_1 set $A \subseteq \omega^\omega \times \omega^\omega$ having countable (and hence σ-compact) vertical sections such that whenever B is Σ^1_1, $B \subseteq A$, and B has compact vertical sections then $\pi(B) \neq \pi(A)$.

Proof : Fix a Σ^1_1 set $R \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ which is universal for the Σ^1_1 subsets of $\omega^\omega \times \omega^\omega$. By [25 ; 4D.2], fix a Π^1_1-recursive partial function $d : \omega \times \omega^\omega \longrightarrow \omega^\omega$ which parametrizes points in $\Pi^1_1(\alpha) \cap \omega^\omega$.

Define $P \subseteq \omega^\omega \times \omega$ by :

$$P(\alpha,s) \longleftrightarrow Seq(s) \ \& \ (\forall\beta)(R(\alpha,\alpha,\beta) \longrightarrow (\exists i < lh(s)) \ (d((s)_i,\alpha)\downarrow \ \& \ d((s)_i,\alpha) = \beta)).$$

Clearly P is Π^1_1. By Easy Uniformization Theorem [25, 4B.4] there is a Π^1_1 set G which uniformizes P. Let $f : \omega^\omega \longrightarrow \omega$ be the (partial) function whose graph is G. Then f is a Π^1_1-recursive partial function. Next define

$$Q(\alpha,\beta) \longleftrightarrow f(\alpha)\downarrow \ \& \ (\exists i) \ (i < lh(f(\alpha)) \ \& \ d((f(\alpha))_i,\alpha)\downarrow \ \& \ d((f(\alpha))_i,\alpha) = \beta).$$

Now check that

(a) Q is Π^1_1,

(b) $Q_\alpha = \{\beta : Q(\alpha,\beta)\}$ is finite,

and (c) if $R_{\alpha,\alpha} = \{\beta : R(\alpha,\alpha,\beta)\}$ is finite, then $R_{\alpha,\alpha} \subseteq Q_\alpha$.

Fact (c) is an immediate consequence of the Effective Perfect Set theorem [25, 4F.1].

Finally, define

$$A(\alpha,\beta) \longleftrightarrow (\exists n) \ ((\forall i)(\beta(i) = n) \ \& \ \neg Q(\alpha,\beta)).$$

Then A is Σ^1_1 and has countable vertical sections. Moreover, since each Q_α is finite, it follows that $\pi(A) = \omega^\omega$.

Suppose now that $B \subseteq A$, B is Σ^1_1 and B has compact vertical sections. Find a code α_0 in ω^ω for B, that is, find α_0 such that $B = R_{\alpha_0}$ $(= \{(\alpha,\beta) : R(\alpha_0, \alpha, \beta)\})$. Since $B_\alpha \subseteq A_\alpha$, A_α is discrete, and B_α is compact, it follows that B_α is finite. In particular, $B_{\alpha_0} = R_{\alpha_0,\alpha_0}$ is finite. So, by (c), $B_{\alpha_0} \subseteq Q_{\alpha_0}$. But also $B_{\alpha_0} \subseteq A_{\alpha_0} \subseteq \omega^\omega - Q_{\alpha_0}$. It follows that $B_{\alpha_0} = \emptyset$.

and hence $\pi(B) \neq \omega^{\omega} = \pi(A)$. This completes the proof.

7.2 <u>Remark</u> : The above is actually a proof that the following weak 'reduction' property fails for analytic sets ;

If A_n, $n \geq 0$, are analytic subsets of ω^{ω} with $\bigcup_{n\geq 0} A_n = \omega^{\omega}$, then there exist analytic sets B_n, $n \geq 0$, such that $(\forall n)\,(B_n \subseteq A_n)$, $\limsup B_i = \emptyset$, and $\bigcup_{n\geq 0} B_n = \omega^{\omega}$.

7.3 <u>Remark</u> : An alternative counterexample to 6.17 may be obtained as follows. With the same notation as in the proof of the theorem, define $A^* = (\omega^{\omega} \times \omega^{\omega}) - Q$. Then A^* is Σ^1_1 and its vertical sections, being cofinite, are of measure one under any continuous probability on ω^{ω}, are comeager and are dense G_δ's. But, as is easy to see, A^* does not admit a Borel (Δ^1_1) uniformization. Again, therefore, we see that the uniformization results of Blackwell and Ryll-Nardzewski [4], Sarbadhikari [30], and Srivastava [36] do not extend to analytic sets.

Of course, if one only wants an analytic set with large sectionsnot containing a Borel graph, one could look at

$$A = \{(\alpha,\beta) \in \omega^{\omega} \times \omega^{\omega} : \beta \notin \Delta^1_1(\alpha)\}.$$

It is easily checked that A is Σ^1_1, has cocountable sections, and contains no Borel graph.

8. A Δ^1_1 "parametrization" criterion : Recursion theory translates the problem of uniformizing a Borel set by a Borel graph into a 'local' condition. This is embodied in the so called Δ-uniformization criterion which reads as follows : Suppose $P \subseteq \omega^\omega \times \omega^\omega$ is a Δ^1_1 set. Then P is uniformized by a Δ^1_1 graph if and only if P^x contains a $\Delta^1_1(x)$ point, whenever $P^x \neq \emptyset$, $x \in \omega^\omega$. Is there an analogue for Borel parametrizations ? This is the question taken up in this section.

For notational simplicity we will work in spaces of type ω^ω. In effective language the problem of parametrizing a Borel set by a Borel measurable function is the following : Let $P \subseteq \omega^\omega \times \omega^\omega$ be a Δ^1_1 set with non-empty vertical sections. When is there a one-one Δ^1_1-recursive map $f : \omega^\omega \times \omega^\omega \longrightarrow \omega^\omega \times \omega^\omega$ such that $f(x,\cdot)$ maps ω^ω onto $\{x\} \times P^x$ for each $x \in \omega^\omega$? We will show below that the existence of such an f is equivalent to

(a) for each x, there is a $\Delta^1_1(x)$-isomorphism of ω^ω and P^x,

and (a la Mauldin) to

(b) for each x, there is a $\Delta^1_1(x)$, compact, and perfect set $Q^{(x)} \subseteq P^x$.

Condition (b) provides a criterion and the equivalence of (a) is proved via (b). As the choice of the criterion suggests, the content of this section is, in a sense, just an effectivization of Mauldin's article [23].

As usual, for our notation and definitions from the effective theory, we follow Moschovakis [25].

Recall that $\alpha \in 2^\omega$ codes a tree T on ω if $\alpha(s) = 0 \longleftrightarrow s \in T$. Call a tree T on ω <u>good</u> if it has no finite branches, is finitely splitting, and if

(i) Every node u in T has at least two distinct strict extensions of equal length, and

(ii) for each $u \in T$, if v_1, v_2, v_3 extend u, and if $(\forall s)(s \not\subsetneq v_1 \longleftrightarrow s \not\subsetneq v_2 \longleftrightarrow s \not\subsetneq v_3)$, then $v_1 = v_2$, or $v_2 = v_3$, or $v_1 = v_3$.

In other words T is good iff its body, $[T]$, looks precisely like a Cantor set.

8.1 <u>Lemma</u> : Let $P \subseteq \omega^\omega \times \omega^\omega$ be Δ^1_1. Suppose P^x contains a $\Delta^1_1(x)$, compact, perfect set for each $x \in \omega^\omega$. Then there is a Δ^1_1-recursive function f on $\omega^\omega \longrightarrow 2^\omega$ such that for each x, $f(x)$ codes a good tree T with $[T] \subseteq P^x$.

<u>Proof</u> : As each P^x contains a $\Delta^1_1(x)$, compact, perfect set using [25;4F.10], it is easily checked that there is for each x, a $\Delta^1_1(x)$ point $\alpha_x \in 2^\omega$ such that α_x codes a good tree with body contained in P^x. Standard arguments using the Δ-uniformization criterion now yield the result.

8.2 <u>Proposition</u> : Let $P \subseteq \omega^\omega \times \omega^\omega$ be Δ^1_1. Suppose P^x contains a $\Delta^1_1(x)$, compact and perfect set for each x. Then there is a

Δ^1_1-recursive map g on $\omega^\omega \times \omega^\omega \longrightarrow \omega^\omega$ such that $g(x,.)$ is a $\Delta^1_1(x)$-recursive isomorphism of ω^ω into P^x, $x \in \omega^\omega$.

Proof : As ω^ω and 2^ω are Δ^1_1-recursively isomorphic it suffices to define g on $\omega^\omega \times 2^\omega$. Let f be as in Lemma 8.1. Fix $x \in \omega^\omega$, so that $f(x)$ codes a good tree T with $[T] \subseteq P^x$. Give 2^ω the lexicographic order and $[T]$ the lexicographic order derived from ω^ω. Then, by the definition of a good tree, there is an order preserving isomorphism between 2^ω and $[T]$. Define g so that $g(x,.)$ is this isomorphism for each x. As f is Δ^1_1-recursive, it is easily checked (by writing out an appropriate clumsy expression) that g is Δ^1_1-recursive.

Next is the criterion for Δ^1_1-parametrizations.

8.3 Proposition : Let P be as in 8.2. Then there is a Δ^1_1-recursive map h on $\omega^\omega \times \omega^\omega \longrightarrow \omega^\omega$ such that $h(x,.)$ is a Δ^1_1-recursive isomorphism of ω^ω and P^x, $x \in \omega^\omega$.

Proof : This strengthened version of 8.2 follows from 8.2 by carrying out an effective Cantor-Bernstein argument. This we will accomplish by considering an appropriate pos Δ^1_1 set relation operative on ω^ω and looking at its fixed point.

Let g be as in Proposition 8.2. This yields a Δ^1_1-isomorphism g_1 on $\omega^\omega \times \omega^\omega$ into P defined by : $g_1(x,\alpha) = (x, g(x,\alpha))$. We also have a Δ^1_1- isomorphism on P into $\omega^\omega \times \omega^\omega$, namely the identity map. Call this g_2.

We will now put down a set relation operative on $\omega^{\omega} \times \omega^{\omega}$ whose associated operator is the one that arises in the proof of the classical Cantor-Bernstein theorem (we follow Dedekind's (See [1])), i.e., we will formalise the operator

$$\psi(S) = (\omega^{\omega} \times \omega^{\omega}) - g_2(P - g_1(S)), \; S \subseteq \omega^{\omega} \times \omega^{\omega} .$$

Define, therefore, for $x, \alpha \in \omega^{\omega}$, and $A \subseteq \omega^{\omega} \times \omega^{\omega}$,

$$\emptyset(x,\alpha,A) \longleftrightarrow (x,\alpha) \notin g_2(P - g_1(A))$$

$$\longleftrightarrow (\forall \beta)(((x,\beta) \in P \;\&\; (x,\beta) \notin g_1(A)) \longrightarrow (x,\alpha) \neq g_2(x,\beta))$$

$$\longleftrightarrow (\forall \beta)\,(((x,\beta) \notin P) \vee ((\exists ! \gamma)((x,\gamma) \in A \;\&\; (x,\beta) = g_1(x,\gamma)))$$
$$\vee\; (\alpha \neq \beta)).$$

$$\longleftrightarrow (\forall \beta)\,[((x,\beta) \notin P) \vee (\alpha \neq \beta)]$$
$$\vee((x,\beta) \in g_1(\omega^{\omega} \times \omega^{\omega}) \;\&\; (\forall z)(((x,\beta) = g_1(x,z)) \longrightarrow (x,z) \in A)).$$

As P, $g_1(\omega^{\omega} \times \omega^{\omega})$ are Δ^1_1, and g_1 is Δ^1_1-recursive, the above is easily seen to be a pos Π^1_1 set relation. It is also seen to be pos Σ^1_1 through the equivalence

$$\emptyset(x,\alpha,A) \longleftrightarrow ((x,\alpha) \notin g_2(P)) \vee ((\exists z)\,(z \in A \;\&\; (x,\alpha) = g_2(g_1(z))).$$

Further, the proof of the classical Cantor-Bernstein theorem shows

$$\emptyset^{\infty}(x,\alpha) \longleftrightarrow \emptyset^{\omega_o}(x,\alpha) .$$

As $\omega_o < \omega_1^{CK}$, by a well-known theorem (See [8]), $\emptyset^{\infty}(x,\alpha)$ is a Δ^1_1 set. Define now h' by

$$h'(x,\alpha) = g_1(x,\alpha) \quad \text{if} \quad \emptyset^{\infty}(x,\alpha)$$
$$= (x,\alpha) \qquad \text{otherwise},$$

and take $h(x,\alpha) = \pi_2(h'(x,\alpha))$.

Then h is Δ^1_1-recursive and the proof of the classical Cantor-Bernstein theorem shows that $h(x,.)$ maps ω^ω onto P^x in a one-one fashion.

We can now prove

8.4 <u>Theorem</u> : Suppose $P \subseteq \omega^\omega \times \omega^\omega$ is Δ^1_1. Then the following are equivalent :

(a) there is a Δ^1_1-recursive map f on $\omega^\omega \times \omega^\omega \longrightarrow \omega^\omega$, such that $f(x,.)$ is one-one on ω^ω onto P^x, for each x,

(b) for each x, there is a $\Delta^1_1(x)$-isomorphism of ω^ω and P^x, and

(c) for each x, there is a $\Delta^1_1(x)$, compact, and perfect set $Q^{(x)} \subseteq P^x$.

<u>Proof</u> : In view of 8.3 it remains now to show only (b) $\Rightarrow$ (c). We will prove the absolute version, i.e., we will show that if g is a Δ^1_1-isomorphism of ω^ω and a Δ^1_1 set $R \subseteq \omega^\omega$, then R contains a Δ^1_1, compact, perfect set.

Fix then g and R. Following Mauldin [loc. cit.], define a measure μ on ω^ω by

$\mu(N_s) = \lambda(g^{-1}(N_s))$, where N_s is the basic clopen set in ω^ω consisting of all sequences which agree with the sequence number s, and λ is the canonical measure on 2^ω given by the product of the $(\frac{1}{2}, \frac{1}{2})$ measures on $\{0, 1\}$. As g is a Δ^1_1-isomorphism, μ is indeed a measure on ω^ω, and further $\mu(R) = 1$.

We will now invoke a result in Kechris [14;4.3.2] to complete the proof. To apply this one has to check that the relation

$$S(u,v) \longleftrightarrow \mathrm{Seq}(u) \ \& \ \mathrm{Seq}(v) \ \& \ \mathrm{lh}(v) = 2 \ \& \ \mu(N_u) > v_0/v_1$$

is Δ^1_1 [Recall v_i denotes the $(i+1)^{st}$ coordinate of v]. But this follows immediately from the definition of μ, the fact that g is Δ^1_1-recursive, and a computation in Kechris [14;2.2.3]. The result cited states in particular that any Δ^1_1 set of μ-measure one contains a Δ^1_1, compact, and perfect set. This completes the proof.

9. Effective continuous - open representations : We will now prove an effective analogue of Theorem 6.0. In [20] A. Maitra showed that there is a basis theorem for $\Delta^1_1, \underset{\sim}{\Pi}^0_2$ subsets of ω^ω which yields Srivastava's selection theorem for G_δ-valued multifunctions. To do this a local analogue of Borel measurability is essential and this he formulated as follows : Let Γ be a point class (i.e., Γ is a collection of subsets of finite products of ω^ω and ω). Let $A \subseteq \omega^\omega$. Then say that A is Γ-normal if the predicate $R_A \subseteq \omega$, defined by

$$R_A(s) \longleftrightarrow N_s \cap A \neq \emptyset \quad \text{is in } \Gamma, \text{ where}$$

$N_s = \{\alpha \in \omega^\omega : \bar{\alpha}(lh(s)) = s\}$, $s \in$ Seq.

Then in [20] it was shown that if a nonempty set A is Π^1_1, Δ^1_1-normal, and $\underset{\sim}{\Pi}^0_2$ (indeed, if it is non-meager in its closure) then it contains a Δ^1_1 point. We will now show that if A is nonempty, Δ^1_1, Δ^1_1-normal and $\underset{\sim}{\Pi}^0_2$, then there is a Δ^1_1-recursive map f on ω^ω onto A which is continuous and open (in an effective sense made precise in Theorem 9.3). We will actually prove a uniform version of this for subsets of $\omega^\omega \times \omega^\omega$.

When specialized to $\underset{\sim}{\Pi}^0_1$ sets this easily extends to the higher odd levels of the analytical hierarchy. Such a parametrization turns out to be the principal tool needed to extend the above basis theorem to these higher levels. All this is the content of an article of R. Barua and the author [2].

We will need the following results (from Louveau [18]). The second is a strengthened version of the Δ-uniformization criterion and follows easily from it.

9.1 Lemma : If $P \subseteq \omega^\omega$ is Δ^1_1 and $\underset{\sim}{\Pi}^0_2$, then P is $\Pi^0_2(\delta)$ for some $\delta \in \Delta^1_1 \cap \omega^\omega$.

This is the effective content of Saint-Raymond's boldface theorem.

9.2 Lemma : Let $P \subseteq \omega^\omega \times \omega^\omega$ be Π^1_1. Let $B = \{\alpha \in \omega^\omega : (\exists \beta \in \Delta^1_1(\alpha))\, P(\alpha,\beta)\}$. If A is a Σ^1_1 subset of B, then there is a Δ^1_1-recursive function $f : \omega^\omega \longrightarrow \omega^\omega$ such that $(\forall \alpha \in A)(P(\alpha, f(\alpha)))$.

Next is our main theorem.

9.3 Theorem : Let $P \subseteq \omega^\omega \times \omega^\omega$ be Δ^1_1 such that for each α, P_α is non-empty and $\underset{\sim}{\Pi}^0_2$. Further, assume that the set

$$A = \{(\alpha, s) : P_\alpha \cap N_s \neq \emptyset\} \text{ is } \Delta^1_1 .$$

Then there is a total function $f : \omega^\omega \times \omega^\omega \longrightarrow \omega^\omega$ such that

(a) for each α, f_α takes ω^ω onto P_α,

(b) f is Δ^1_1-recursive,

(c) there is a total function $\emptyset : \omega^\omega \longrightarrow \omega^\omega$ such that $\emptyset$ is Δ^1_1-recursive and for each α, f_α is recursive in $\emptyset(\alpha)$ (and hence continuous), and

(d) there is a total function $\psi : \omega^\omega \times \omega \longrightarrow \omega$ such that ψ is Δ^1_1 recursive and for each α and v, $f(\alpha, N_v) = P_\alpha \cap N_{\psi(\alpha,v)}$ (and hence f_α is open).

We shall first prove a lemma. For each space $\mathcal{X} = (\omega^\omega)^k \times \omega^\ell$, fix a good Π^0_2 universal set for $\underset{\sim}{\Pi}{}^0_2$ subsets of $\mathcal{X}$ and write G (unambiguously) for each of them. This can be done by [25; 3H.1].

<u>Lemma</u> : There is a total Δ^1_1-function $h : \omega^\omega \longrightarrow \omega^\omega$ such that $P(\alpha,\beta) \longleftrightarrow G(h(\alpha),\beta)$.

<u>Proof</u> : Let $d : \omega^\omega \times \omega \longrightarrow \omega^\omega$ be a Π^1_1-recursive partial function which parametrizes $\Delta^1_1(\alpha) \cap \omega^\omega$. Define

$$T(\alpha,k) \longleftrightarrow d(\alpha,k)\downarrow \ \& \ (\forall\beta)(P(\alpha,\beta) \longleftrightarrow G(d(\alpha,k),\beta)).$$

Clearly T is Π^1_1. Using the relativized version of Louveau's theorem and properties of good universal sets, one can check that $(\forall\alpha)(\exists k)\,T(\alpha,k)$. Consequently, by the Δ-selection principle, there is a total Δ^1_1-function $g : \omega^\omega \longrightarrow \omega$ such that $(\forall\alpha)T(\alpha,g(\alpha))$. Now set $h(\alpha) = d(\alpha,g(\alpha))$. Plainly, h is Δ^1_1 and does the job.

<u>Proof of theorem 9.3</u> : $G(\alpha,\beta)$ being Π^0_2, we may write

$$G(\alpha,\beta) \longleftrightarrow (\forall m)(\exists t)(\beta \in N_t \ \& \ Q(\alpha,t,m)),$$

where $Q \subseteq \omega^\omega \times \omega \times \omega$ is Σ^0_1.

Define $R(\alpha,u) \longleftrightarrow \mathrm{Seq}(u)\ \&$

$$(\forall i < lh(u))(\mathrm{Seq}(u_i) \ \& \ lh(u_i) \geq i \ \& \ A(\alpha,u_i)) \ \&$$
$$(\forall i < lh(u)-1)(N_{u_{(i+1)}} \subseteq N_{u_i}) \ \&$$
$$(\forall i < lh(u))(\exists t)(Q(h(\alpha),t,i) \ \& \ N_{u_i} \subseteq N_t).$$

Clearly R is Δ^1_1.

Next define, $Q^*(\alpha,u,v) \longleftrightarrow R(\alpha,u) \ \& \ \mathrm{Seq}(v) \ \& \ \mathrm{lh}(u) = \mathrm{lh}(v) \ \&$

$$(\exists s)\,[\,\mathrm{Seq}(s) \ \& \ \mathrm{lh}(s) = \mathrm{lh}(u) \ \&$$

$$(\forall i < \mathrm{lh}(s))\,\{\mathrm{Seq}(s_i) \ \& \ \mathrm{lh}(s_i) = v_i + 1 \ \& \ s_{i,\mathrm{lh}(s_i)-1} = u_i \ \&$$

$$(\forall j < \mathrm{lh}(s_i))\, R(\alpha, u\restriction i * \langle s_{i,j}\rangle) \ \&$$

$$(\forall j < \mathrm{lh}(s_i) - 1)(s_{i,j} < s_{i,j+1}) \ \&$$

$$(\forall p < u_i)(\forall j < \mathrm{lh}(s_i))(p \neq s_{i,j} \longrightarrow \neg R(\alpha, u\restriction i * \langle p\rangle))]$$

Plainly Q^* is Δ^1_1. It is easy to see that for each α,β and n, there is a unique u such that $Q^*(\alpha,u,\overline{\beta}(n+1))$. Consequently, we can define a total function $f : \omega^\omega \times \omega^\omega \longrightarrow \omega^\omega$ by

$$f(\alpha,\beta) = \delta \longleftrightarrow (\forall n)(\exists u)\,(Q^*(\alpha,u,\overline{\beta}(n+1)) \ \& \ \delta \in N_{u_n}).$$

It is a straightforward verification that for each α, $f(\alpha,\cdot)$ is onto P_α. Next the neighbourhood diagram G^f is given by

$$G^f(\alpha,\beta,s) \longleftrightarrow (\exists n)(\exists u)\,(Q^*(\alpha,u,\overline{\beta}(n+1)) \ \& \ N_{u_n} \subseteq N_s).$$

G^f is Δ^1_1 and hence f is Δ^1_1.

Denote the function $f(\alpha,\cdot)$ by f_α. Clearly $G^{f_\alpha} = G^f_\alpha$. So G^{f_α} is $\Delta^1_1(\alpha)$. But plainly G^{f_α} is Σ^0_1. Fix a set $H \subseteq \omega^\omega \times \omega^\omega \times \omega$ such that H is Σ^0_1 and a good universal set for $\underset{\sim}{\Sigma}{}^0_1 \restriction \omega^\omega \times \omega$. Following the proof of the above lemma we can get a Δ^1_1-recursive total function $\emptyset : \omega^\omega \longrightarrow \omega^\omega$ such that

$$G^{f_\alpha}(\beta,s) \longleftrightarrow H(\emptyset(\alpha),\beta,s).$$

Hence f_α is recursive in $\emptyset(\alpha)$, for each α. Finally an easy computation shows that for each α and sequence number v,

$$f(\alpha, N_v) = P_\alpha \cap N_{u_{lh(v)-1}},$$

where u is the unique natural number such that $Q^*(\alpha, u, v)$. We can therefore define a total function $\psi : \omega^\omega \times \omega \longrightarrow \omega^\omega$ by

$$\psi(\alpha, v) = s \longleftrightarrow [\mathrm{Seq}(v) \;\&\; (\exists u)(Q^*(\alpha, u, v) \;\&\; s = (u)_{lh(v)-1})]$$
$$\vee [\neg \mathrm{Seq}(v) \;\&\; s = 0].$$

It then follows immediately that ψ is Δ^1_1 and for each α and v, we have

$$f_\alpha(N_v) = P_\alpha \cap N_{\psi(\alpha, v)}.$$

So in particular, f_α is open.

It is easily seen that Π^1_1, non-Δ^1_1 sets cannot admit such a representation. On the other hand, an example along the lines of 6.16 can easily be constructed to show that such representations do not characterise sets as in the theorem.