A NOTE ON THE STRUCTURE OF A STOCHASTIC MODEL CONSIDERED BY V. M. DANDEKAR*

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Mr. Dandekar starts with the stochastic process $x_0, x_1, x_2, \ldots$, ad inf. which may be characterised by the following defining postulates :

Π_1 : Each x_i can take only the two values 0 (= failure) and 1 (= success) and the probability that $x_0 = 1$ is p.

Π_2 : For any m (or less) consecutive x_i's at most one can be 1.

Π_3 : If any $m-1$ (or more) consecutive x_i's are known to be zeros then the next x_i is 1 with probability p.

Π_4 : If $x_0 = 0$ then the conditional stochastic process $x_1, x_2, \ldots$ is the same as the original process $x_0, x_1, x_2, \ldots$.

Let $P_n = P(x_n = 1)$, $n = 0, 1, 2, \ldots$ ($P_0 = p$, $q = 1-p$) and let $\varphi(t)$ be the generating function $\sum\limits_0^\infty P_n t^n$.

The recurrence relation

$$P_n = pP_{n-m} + qP_{n-1},\quad n = 1, 2, \ldots$$

where $P_r = 0$ for negative r is easily verified.

Hence

$$\begin{aligned}\varphi(t) &= p + \sum_1^\infty P_n t^n \\ &= p + \Sigma\,(pP_{n-m} + qP_{n-1})t^n \\ &= p + pt^m\varphi(t) + qt\varphi(t),\end{aligned}$$

or

$$\varphi(t) = p/(1-qt-pt^m).$$

Therefore

$$\begin{aligned}\varphi(1/z) &= pz^m/(z^m - qz^{m-1} - p) \\ &= z\{pz^{m-1}/(z-\alpha_1)\ldots(z-\alpha_m)\}.\end{aligned}$$

It is easily checked that the zeros $\alpha_1, \alpha_2, \ldots, \alpha_m$ of the polynomial $z^m - qz^{m-1} - p$ are all distinct. Hence we have

$$\varphi(1/z) = z\sum_1^m c_i(z-\alpha_i)^{-1} \quad \text{where} \quad c_i = p\alpha_i\{m\alpha_i - (m-1)q\}^{-1},$$

or

$$\varphi(t) = \sum_1^m c_i(1-\alpha_i t)^{-1}.$$

Equating co-efficients of t^n we have

$$P_n = \sum_1^m c_i\alpha_i^n.$$

* Dandekar, V. M. (1955): Certain modified forms of binomial and Poisson distributions, *Sankhyā*, 15, Part 3.

Now, it is easily seen that one of the α_i's is unity and that all the others lie within the unit circle (provided $q > 0$).

Therefore $$\lim_{n\to\infty} P_n = \frac{p}{m-(m-1)q} = \frac{p}{1+(m-1)p} \text{ (if } q > 0).$$

Mr. Dandekar considers the following problem:

If we make an abrupt start on the stochastic process $x_0, x_1, x_2, \ldots$, i.e. if we take an x_n without knowing what n is and without knowing what happened to the previous x_i's then what is the probability that $x_n = 1$?

The question stated as above has no answer. If n were known then the answer is P_n. If we have a priori knowledge about n being a random variable then the answer is $\Sigma P_i q_i$ where $q_i = P(n = i)$. Mr. Dandekar arrives at the conclusion

$$P(x_n = 1) = p/\{1+(m-1)p\}$$

by an ingenious argument.

As we have noted before this is the limit of P_n as $n\to\infty$. When Mr. Dandekar makes an abrupt start on the stochastic process $(x_0, x_1, x_2, \ldots)$ he implicitly assumes that the process is in operation for an indefinitely long time. He then gets a new stochastic process $(y_0, y_1, y_2, \ldots)$ with the following characteristics:

Π_1' : The marginal distribution of each y_i is the same—each taking the two values 0 and 1 with probabilities $1-\pi$ and π respectively.

Π_2' : Same as Π_2 with x_i replaced by y_i.

Π_3' : Same as Π_3 with x_i replaced by y_i.

It is easy to verify that the above three properties may be taken as the defining postulates of the stochastic process $(y_0, y_1, y_2, \ldots)$. That $\pi = p/\{1+(m-1)p\}$ may then be proved as follows:

By Π_1', $P(y_{m-1} = 1) = \pi$. By Π_2', the event $y_{m-1} = 1$ can happen only if $y_0 = y_1 = \ldots = y_{m-2} = 0$ and by Π_3', $P(y_{m-1} = 1 | y_0 = y_1 = \ldots = y_{m-2} = 0) = p$. By Π_1' and Π_2', the probability that at least one of the first $(m-1)$ y_i's is 1 is $(m-1)\pi$.

Hence $$\pi = \{1-(m-1)\pi\}p$$

or $$\pi = p/\{1+(m-1)p\}.$$

If it is known that r $(r < m-1)$ consecutive y_i's are zeros then the probability that the next y_i is 1 is

$$1 - \frac{1-(r+1)\pi}{1-r\pi} = \pi/(1-r\pi).$$

(This follows from Π_1' and Π_2').

I wish to thank Professor C. R. Rao for drawing my attention to this problem.

Paper received : January, 1955.