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INDEPENDENCE OF TIME AND CAUSE OF FAILURE IN THE MULTIPLE DEPENDENT COMPETING RISKS MODEL

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Abstract: Several components having lifelengths which are not necessarily independent are "competing" to fail first. We show that the time and identity of first failure are mutually independent if and only if the cause specific hazard rates of the components are proportional to each other.

Key words and phrases: Competing risks, dependent failure times, joint survival function, proportional hazard functions, cause-specific hazard rate function, bivariate symmetry, absolutely continuous bivariate exponential function, parallel systems.

1. Some General Results

The following characterization is well known in the theory of competing risks. (See Armitage(1959), Allen(1963) and Sethuraman (1965, Theorem 3).)

Theorem 1. *Let $X_1, \ldots, X_n$ be independent random variables with survival functions $\bar{F}_1, \ldots, \bar{F}_n$. Let*

$$X_0 = \min\{X_1, \ldots, X_n\} \quad \text{and} \quad I = i \quad \text{if } X_0 = X_i \ (i = 1, \ldots, n).$$

Then X_0 and I are independent if and only if there exist positive real numbers $\beta_2, \ldots, \beta_n$ such that

$$\bar{F}_i(t) = (\bar{F}_1(t))^{\beta_i} \quad \text{for} \quad i = 2, \ldots, n.$$

In other words, the time of first failure and the cause of first failure are mutually independent if and only if the hazard rates are proportional.

In this note we extend this characterization from the independent case to the dependent case. It is well known in competing risk analysis that the assumption of independence of the competing random variables is not always appropriate. For example, the heart condition of a patient may very well depend on the condition of his other organs. Thus dependence models are both realistic and suitable. For a review of competing risk theory, see Gail (1982).

Assume that $(X_1, \ldots, X_n)$ have joint survival function $\bar{F}(x_1, \ldots, x_n)$ and joint density $f(x_1, \ldots, x_n)$ w.r.t. Lebesgue measure for $0 \leq x_i < \infty$, $i = 1, 2, \ldots, n$. Let $\bar{F}_0(t) = \bar{F}(t, \ldots, t)$ denote the survival function of X_0 and $f_0(t)$ the corresponding density. Let

$$\begin{aligned} p_i(t) &= P[I = i | X_0 = t] \\ &= P[X_i \leq \min(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) | X_0 = t] \\ &= \frac{\int_t^\infty \cdots \int_t^\infty \int_t^\infty \cdots \int_t^\infty f(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_n) \prod_{j \neq i} du_j}{f_0(t)}. \end{aligned}$$

Since $f_0(t) = -\frac{\partial}{\partial t} \bar{F}(t, \ldots, t)$ it follows by the chain rule of differentiation that

$$\sum_{i=1}^{n} p_i(t) = 1 \qquad \text{for} \quad t \geq 0.$$

Let

$$r_i(x_i | X_1 > x_1, \ldots, X_{i-1} > x_{i-1}, X_{i+1} > x_{i+1}, \ldots, X_n > x_n)$$

denote the conditional failure rate of X_i given $X_j > x_j$ for $j = 1, \ldots, n; j \neq i$. Then by definition,

$$\begin{aligned} & r_i(x_i | X_1 > x_1, \ldots, X_{i-1} > x_{i-1}, X_{i+1} > x_{i+1}, \ldots, X_n > x_n) \\ &= \frac{\int_{x_1}^\infty \cdots \int_{x_{i-1}}^\infty \int_{x_{i+1}}^\infty \cdots \int_{x_n}^\infty f(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_n) \prod_{j \neq i} du_j}{\bar{F}(x_1, \ldots, x_n)}. \end{aligned}$$

It follows that

$$g_i(t) = r_i(t | X_j > t, j = 1, \ldots, i-1, i+1, \ldots, n) = p_i(t) r_{F_0}(t), i = 1, 2, \ldots, n, \tag{1}$$

and as a result

$$\sum_{i=1}^{n} g_i(t) = r_{F_0}(t),$$

where $r_{F_0}(t)$ denotes the failure rate of X_0. $g_i(t)$ is called the *cause specific hazard rate function corresponding to the* ith *risk.* It may, alternatively, be computed from

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} P[t \leq X_0 \leq t + \Delta t, I = i | X_0 = t] = g_i(t).$$

See, for example, Kalbfleisch and Prentice (1980).

Theorem 2. *X_0 and I are independent if and only if there exist positive constants $\beta_2, \ldots, \beta_n$ such that*

$$p_i(t) = \beta_i p_1(t) \quad \textit{for all } t \geq 0, i = 2, \ldots, n,$$

or equivalently,

$$g_i(t) = \beta_i g_1(t) \quad \textit{for all } t \geq 0, i = 2, \ldots, n.$$

Proof. X_0 and I are independent if and only if

$$p_i(t) = P[I = i] = \Theta_i, \text{ say, for all } t \geq 0 \text{ and for all } i = 1, \ldots, n.$$

Since

$$\sum_{i=1}^{n} p_i(t) = \sum_{i=1}^{n} \Theta_i = 1,$$

it follows that X_0 and I are independent if and only if

$$p_i(t) = \beta_i p_1(t) \quad \text{for all } t \geq 0, \text{ where } i = 2, \ldots, n \text{ and } \beta_i = \Theta_i/\Theta_1.$$

The conclusion of the theorem follows by using (1).

Remark. Let $X_0^* = \max\{X_1, \ldots, X_n\}$, $I^* = i$ if $X_0^* = X_i$, $i = 1, \ldots, n$, and $p_i^*(t) = P[I^* = i | X_0^* = t]$. Then we can prove a result similar to Theorem 2:

Theorem 2′. *X_0^* and I^* are independent if and only if there exist positive constants $\beta_2^*, \ldots, \beta_n^*$ such that*

$$p_i^*(t) = \beta_i^* p_1^*(t) \quad \textit{for all } t \geq 0, i = 2, \ldots, n,$$

or equivalently,

$$g_i^*(t) = \beta_i^* g_1^*(t) \quad \textit{for all } t \geq 0, i = 2, \ldots, n,$$

where

$$g_i^*(t) = \frac{\left(\frac{\partial}{\partial t_i} F(t_1, \ldots, t_n)\right)_{t_j = t, j = 1, \ldots, n}}{F(t, \ldots, t)}.$$

Although this result may not be directly usable in competing risk analysis, it is useful in the reliability analysis of parallel systems having n dependent components.

2. Applications

(a) Let (X_1, X_2) have a bivariate symmetric distribution. Then it is easy to see that

$$r_1(t|X_2 > t) = r_2(t|X_1 > t) \quad \text{for all } t \geq 0,$$

so that Theorem 2 holds with $n = 2$ and $\beta_2 = 1$. Bagai, Deshpandé and Kochar (1989a, b) proposed tests for the equality of two risks in the competing risks model with independent lifelengths. It follows from Theorem 2 that the null distributions of the test statistics remain unchanged when the null hypothesis is extended to include dependence with bivariate symmetry. This is also true for the tests proposed by Froda (1987).

(b) Consider the case of the absolutely continuous bivariate exponential distribution of Block and Basu (1974) with joint pdf:

$$f(x,y) = \begin{cases} \frac{\lambda_1\lambda(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2} e^{-\lambda_1 x-(\lambda_2+\lambda_{12})y} & \text{if } 0 < x < y \\ \frac{\lambda_2\lambda(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_{12})x-\lambda_2 y} & \text{if } x > y > 0, \end{cases}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. It is easy to show that for $t > 0$,

$$\begin{aligned} f_0(t) &= \lambda e^{-\lambda t}, \\ p_1(t) &= \lambda_1/(\lambda_1 + \lambda_2), \quad \text{and} \\ p_2(t) &= \lambda_2/(\lambda_1 + \lambda_2) = (\lambda_2/\lambda_1)p_1(t). \end{aligned}$$

Thus the conditions of Theorem 2 are satisfied. As a consequence X_0 and I are independent.

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