Stochastic quantization of a dissipative dynamical system and its hydrodynamical interpretation

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In the framework of stochastic quantization the quantum theory of an open system is envisaged here. A hydrodynamical model has been suggested for the motion of a quantum particle in the presence of frictional dissipative force. Finally, a comparison is made between quantum conservative and nonconservative systems.

I. INTRODUCTION

Stochastic quantization procedure[1] is possibly the only scheme for quantization of dissipative system, as physically meaningful Lagrangian or Hamiltonian, is not available in this case[2] and so the canonical or path integral quantization scheme cannot help in this regard. Instead, Nelson's stochastic formalism can neatly be developed in absence of Lagrangian or Hamiltonian as it is an effectively stochastic version of Newton's equation of motion. Some works have already been done for the stochastic version of quantum dissipative system[3–5] and all such works lead to the nonlinear Schrödinger–Langevin equation heuristically derived by Kostin.[6] In most of the above works the nonconservation (dissipation on antidissipation) arises due to the interaction between the considered open system and the external system, the nature of interaction is usually assumed phenomenologically. Particularly, in case of dissipation, energy flows irreversibly from the observed system to the external world. The main feature of the earlier works is to derive a Schrödinger equation incorporating dissipative potential term and a random potential term from classical dynamical behavior of an open system described by a classical Langevin equation.

In the present paper we are also interested in deriving a Schrödinger–Langevin equation from the classical Langevin equation by Nelson's stochastic formalism, but after modifying Nelson's approach on the basis of our geometry of internal space-time as perceived by us in realizing the internal symmetry of relativistic quantum particles,[7] quantum gravity phenomena,[8,9] and geometric phase factor like Berry phase[10] in nonrelativistic quantum mechanics.

On the basis of our earlier work[11] on relativistic generalization of Nelson's stochastic quantization procedure introducing an anisotropy in the internal space-time we have seen in a recent paper[12] on stochastic quantization in deriving Schrödinger equation for conservative systems that the element responsible for the generation of the fermion number in the relativistic domain is manifested as "quantum potential" in the nonrelativistic domain. In our formalism, that very crucial thing is the anisotropic nature of the internal space-time of a relativistic quantum particle and it is better to say it is the main architect for the quantization in general. This very nature of internal geometry when incorporated in the nonrelativistic region Nelson's osmotic velocity becomes much more conceivable. Therefore, our interest automatically goes parallel in studying the quantum mechanics for nonconservative system, which is the main feature of the present paper.

In Sec. II, we shall quote the results of stochastic quantization in our formalism for conservative systems for the comparison with the nonconservative systems.

In Sec. III, the Schrödinger–Langevin equation is being derived from classical Langevin equation by our modified Nelson's formalism and a hydrodynamical model is prescribed for the motion of quantum particle in the presence of dissipative force.

In Sec. IV, the discussions of our derived results are being made and a comparison between quantum conservative and nonconservative systems has also been discussed.

II. STOCHASTIC QUANTIZATION OF CONSERVATIVE DYNAMICAL SYSTEM AND ITS HYDRODYNAMICAL ANALOG

Before deriving the Schrödinger–Langevin equation from Nelson's formalism we will quote, in this section, our results of quantization in case of conservative system[12] by our modified Nelson's formalism to have better realization of nonrelativistic quantum mechanics. In the above work it is assumed that the quantum particle has an extension and this extension is given by the internal variable $\xi(\xi_0)$ in addition with the external variable $\mathbf{X}(t)$ and the internal Brownian motion is considered along with the external. This has been done in analogy with the geometry of the space-time structure in the relativistic domain to explain the internal symmetry[7] of a relativistic quantum particle and to get a quantum field theory from a stochastic field theory.[11] In the relativistic domain each external space-time point X_μ is associated with an attached vector ξ_μ and this ξ_μ can be written in terms of two spinorial variables $\bar{\Theta}^A$, $\Theta^{A'}$ which correspond to two internal helicities corresponding to the particle and antiparticle configurations and make the internal space anisotropic in nature. The metric of the space-time in our geometry is designated by $g_{\mu\nu}(X,\Theta,\bar{\Theta})$ by which some problems of quantum gravity is shown to be well explained.[8,9]

To have better insight in the nonrelativistic case we start with the assumption of internal Brownian motion in the internal ξ space along with the external X space. We therefore assume $Q_i(t,\xi_0)$, $i=1,2,...,n$ to be the configuration variable and, along with Nelson's assumptions, we further assume that $Q(t,\xi_0)$ is separable; i.e.,

$$Q(t,\xi_0)=q(t)q(\xi_0).$$

The process $Q_i(t,\xi_0)$ is supposed to satisfy the stochastic differential equations

$$dQ_i(t,\xi_0)=b_i(Q(t,\xi_0),t,\xi_0)dt+d\omega_i(t) \tag{2.1}$$

and

$$dQ_i(t,\xi_0)=b_i'(Q(t,\xi_0),t,\xi_0)d\xi_0+d\omega_i(\xi_0), \tag{2.2}$$

which represents Brownian motion in external and internal space, respectively, b_i and b_i' are corresponding velocity fields and $d\omega_i$'s are independent Brownian motion such that $d\omega_i(t)(d\omega_i(\xi_0))$ do not depend on $Q(S,S')$ for $S \leqslant t(S' \leqslant \xi_0)$. Expectations of which have the values

$$\langle d\omega_i(t)\rangle=0$$

$$\langle d\omega_i(t)d\omega_j(t')\rangle=(\hbar/m)\delta_{ij}(t-t')dt\,dt',$$

$$\langle d\omega_i(\xi_0)\rangle=0, \tag{2.3}$$

$$\langle d\omega_i(\xi_0)d\omega_j(\xi_0')\rangle=(\hbar/\pi_0)\delta_{ij}(\xi_0-\xi_0')d\xi_0\,d\xi_0'.$$

Here, π_0 is a constant having the dimension of mass.

To make the description symmetrical in both external and internal time we have the backward equations:

$$dQ_i(t,\xi_0)=b_i^*(Q(t,\xi_0),t,\xi_0)dt+d\omega_i^*(t), \tag{2.4}$$

$$dQ_i(t,\xi_0)=b_i'^*(Q(t,\xi_0),t,\xi_0)d\xi_0+d\omega_i^*(\xi_0), \tag{2.5}$$

where ω^* has the same properties as ω except $d\omega_i^*(t)(d\omega_i^*(\xi_0))$ are independent of $Q(S,S')$ with $S \geqslant t(S' \geqslant \xi_0)$.

The mean forward and mean backward derivatives are given by

$$D_tQ_i(t,\xi_0)=\lim_{\Delta t\to 0^+}E_t\frac{Q_i(t+\Delta t,\xi_0)-Q_i(t,\xi_0)}{\Delta t}$$

$$=b_i(Q(t,\xi_0),t,\xi_0),$$

$$D_{\xi_0}Q_i(t,\xi_0)=\lim_{\Delta\xi_0\to 0^+}E_{\xi_0}\frac{Q_i(t,\xi_0+\Delta\xi_0)-Q_i(t,\xi_0)}{\Delta\xi_0}$$

$$=b_i'(Q(t,\xi_0),t,\xi_0),$$

$$D_t^*Q_i(t,\xi_0)=\lim_{\Delta t\to +0}E_t\frac{Q_i(t,\xi_0)-Q_i(t-\Delta t,\xi_0)}{\Delta t}$$

$$=b_i^*(Q(t,\xi_0),t,\xi_0),$$

$$D^*\xi_0Q_i(t,\xi_0)=\lim_{\Delta\xi_0\to +0}E_{\xi_0}\frac{Q_i(t,\xi_0)-Q_i(t,\xi_0-\Delta\xi_0)}{\Delta\xi_0}$$

$$=b_i'^*(Q(t,\xi_0),t,\xi_0), \tag{2.6}$$

where E_t, E_{ξ_0} are the conditional expectations with respect to σ algebra Σ, generated by the random variables $Q_i(t,\xi_0)$, $i=1,2,...,n$.

In general, for sufficiently regular function $F(Q(t,\xi_0),t,\xi_0)$ we have

$$D_tF(Q(t,\xi_0),t,\xi_0)=\left(\frac{\partial}{\partial t}+b\cdot\nabla+\nu\Delta\right)F(Q(t,\xi_0),t,\xi_0),$$

$$D_{\xi_0}F(Q(t,\xi_0),t,\xi_0)=\left(\frac{\partial}{\partial\xi_0}+b'\cdot\nabla_\xi+\nu'\Delta_\xi\right)F(Q(t,\xi_0),t,\xi_0),$$

$$D_t^*F(Q(t,\xi_0),t,\xi_0)=\left(\frac{\partial}{\partial t}+b^*\cdot\nabla-\nu\Delta\right)F(Q(t,\xi_0),t,\xi_0),$$

$$D_{\xi_0}^*F(Q(t,\xi_0),t,\xi_0)=\left(\frac{\partial}{\partial\xi_0}+b'^*\cdot\nabla_\xi-\nu'\Delta_\xi\right)$$

$$\times F(Q(t,\xi_0),t,\xi_0), \tag{2.7}$$

where ∇ is the gradient operator and

$$\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial q_i^2}$$

∇_ξ and Δ_ξ are the same with respect to $q(\xi_0)$.

Equation of continuity

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(v\rho)=0 \tag{2.8}$$

is straightforwardly obtained from forward and backward Fokker–Planck equation for measurable probability density $\rho(q(t),t)$ of the random variable $Q(t,\xi_0)$. Here,

$$v=\tfrac{1}{2}(D_t+D_t^*)Q(t,\xi_0)=\tfrac{1}{2}(b+b^*) \tag{2.9}$$

is the current velocity in the external space.

Since distribution $\rho\,d^3x$ is invariant on space-time Nelson[1] showed that

$$u=\nu\,\mathrm{grad}\ln\rho, \tag{2.10}$$

where

$$u=\tfrac{1}{2}(b-b^*). \tag{2.11}$$

Defining mean acceleration a of the process $Q(t,\xi_0)$ as

$$a=\tfrac{1}{2}(D_tD_t^*+D_t^*D_t)Q(t,\xi_0) \tag{2.12}$$

we obtain by (2.6) and (2.12)

$$\frac{\partial v}{\partial t}=a-(v\cdot\nabla)v+(u\cdot\nabla)u+\nu\Delta u, \tag{2.13}$$

where $\nu=\hbar/2m$.

In this theory, the dynamics is given by Newton's law, i.e.,

$$a=F/m=-1\nabla V/m, \tag{2.14}$$

where F is the conservative external force derivable from potential V.

Therefore, (2.13) takes the form

$$\frac{\partial v}{\partial t}=-(1/m)\nabla V-(v\cdot\nabla)v+(u\cdot\nabla)u+(\hbar/2m)\Delta u. \tag{2.15}$$

Defining wave function

$$\Psi=e^{R+iS/\hbar},\ \text{where}\ \rho=|\Psi|^2. \tag{2.16}$$

and

$$u = (\hbar/m)\operatorname{grad} R, \quad v = (1/m)\operatorname{grad} S \tag{2.17}$$

we arrived at the usual Schrödinger equation from Eq. (2.15).

Equation (2.15) is of particular interest in the sense that it gives hydrodynamical analog of nonrelativistic motion of a quantum particle.

By the help of relations (2.17), (2.15) can be rewritten in the form

$$m\frac{Dv}{Dt} = -\nabla\left(V - \frac{1}{2}mu^2 - \frac{\hbar}{2}\nabla\cdot u\right), \tag{2.18}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (v\cdot\nabla).$$

Equation (2.18) is Euler's equation of motion for perfect fluid except the pressure potential is replaced by $-\frac{1}{2}mu^2 - (\hbar/2)\nabla\cdot u$ which is known as the quantum potential in the literature.

In our modified formalism there are two other velocity fields b' and b'^* appeared in Eqs. (2.2) and (2.5). So we can define two other velocities

$$v' = \tfrac{1}{2}(b' + b'^*) = (\hbar/\pi_0)\nabla_\xi R \tag{2.19}$$

and

$$u' = \tfrac{1}{2}(b' - b'^*) = (1/\pi_0)\nabla_\xi S \tag{2.20}$$

called as internal current and osmotic velocities.

Introducing internal variable $q(\xi_0)$ in addition to the external variable $q(t)$ we are working here with six-dimensional space, but to incorporate the effect of internal space in the physically measurable three-dimensional space we use the simplest form of mapping between $q(t)$ and $q(\xi_0)$ as

$$q(\xi_0) = Cq(t), \tag{2.21}$$

where C is a suitable parameter.

By (2.21) we get

$$\nabla R = C\nabla_\xi R$$

$$\therefore\ u = \frac{\hbar}{m}\nabla R = \frac{\hbar C}{m}\nabla_\xi R = \frac{C\pi_0}{m}v' \quad [\text{by } (2.19)] = C'v'. \tag{2.22}$$

So, external osmotic velocity is effectively nothing but the internal current velocity. Similarly, it can be shown that internal osmotic velocity is effectively the external current velocity. It is very consistent in the sense that an extended particle in motion should have two independent velocities one external and another internal.

Due to the above, it appears to us that the "quantum potential," which is the hidden element for quantization, is the manifestation of internal motion of a quantum particle. Moreover, if we look into the last two terms in the right-hand side of (2.18) i.e.,

$$\nabla\left(\frac{1}{2}mu^2 + \frac{\hbar}{2}\nabla\cdot u\right) = \nabla\left(\frac{1}{2}mu^2\right) + \frac{\hbar}{2}\nabla^2 u \,[\because\ \nabla\times u = 0]$$

the first term gives an internal normal stress $\frac{1}{2}mu^2$ and second term is like an internal viscous force arises due to intermolecular friction within the internal structure of the body. The coefficient of viscosity is here proportional to diffusion coefficient.

On the basis of above discussions we may infer that motion of a nonrelativistic quantum particle can be thought of as an invicid motion of an extended fluid particle having nonstationary random behavior within. The force interacting among the constituents of a nonrelativistic quantum particle may be imagined by the proper realization of last two terms of the right-hand side of Eq. (2.18).

III. STOCHASTIC QUANTIZATION OF NONCONSERVATIVE DYNAMICAL SYSTEMS AND ITS HYDRODYNAMICAL ANALOG

Here, to study the stochastic quantization of nonconservative system we are particularly interested in dissipative system i.e., in this section we shall derive the Schrödinger–Langevin equation from the classical Langevin equation by adopting quantization procedure as advocated by Nelson.[1]

In classical dynamics the motion of a particle in the presence of a frictional dissipative force is given by the equation

$$m\frac{d^2x(t)}{dt^2} = -\nabla V(x(t),t) - \gamma\frac{dx(t)}{dt}, \tag{3.1}$$

where V stands for the potential corresponding to conservative external force and γ the friction coefficient arises due to the interaction with the surroundings into which energy dissipate irreversibly. In general there is a random force $F(t)$ in addition $-\gamma[dx(t)/dt]$ in the above equation to describe the interaction with the surrounding, we neglected it for simplification.

In our modified formalism, Eq. (3.1) can be rewritten in terms of configuration variable $Q_i(t,\xi_0)$, $i=1,2,\ldots,n$ as

$$m\ddot{Q}(t,\xi_0) = -\operatorname{grad} V(Q(t,\xi_0),t,\xi_0) - \gamma\dot{Q}(t,\xi_0), \tag{3.2}$$

where our first assumption is that the particle trajectory will be given by a diffusion process $Q(t,\xi_0)$ which satisfies the stochastic differential equations (2.1), (2.2), (2.4), and (2.5) along with the separability condition on $Q(t,\xi_0)$, i.e., $Q(t,\xi_0) = q(t)q(\xi_0)$. Here, also (2.3) and (2.6)–(2.11) hold automatically.

Our second assumption is that the Langevin equation (3.2) can be expressed in terms of mean velocity and the mean acceleration given by Eqs. (2.9) and (2.12). So (3.2) becomes

$$ma = -\operatorname{grad} V(Q(t,\xi_0),t,\xi_0) - \gamma v. \tag{3.3}$$

By the help of (2.12) and (2.6) we have

$$\frac{\partial v}{\partial t}=a-(v\cdot\nabla)v+(u\cdot\nabla)u+\frac{\hbar}{2m}\Delta u. \qquad (3.4)$$

So by (3.3), (3.4) becomes

$$m\frac{\partial v}{\partial t}=-\nabla V-\gamma v-m(v\cdot\nabla)v+m(u\cdot\nabla)u+\frac{\hbar}{2}\Delta u. \qquad (3.5)$$

Finally defining the wave function

$$\Psi=e^{R+iS/\hbar} \quad \text{where} \quad \rho=|\Psi|^2$$

and $u=(\hbar/m)\text{grad } R$, $v=(1/m)\text{grad } S$, from Eq. (3.5) we arrive at

$$i\hbar\frac{\partial\Psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V+\frac{\gamma S}{m}\right)\Psi$$

$$=\left(-\frac{\hbar^2}{2m}\nabla^2+V+\frac{i\gamma}{2m}\log\frac{\Psi^*}{\Psi}\right)\Psi. \qquad (3.6)$$

This nonlinear equation coincides with that of Kostin[6] and this equation is known as the Schrödinger–Langevin equation which is the nonrelativistic equation of motion of a quantum particle in presence of a dissipative frictional force.

To have a hydrodynamical analog of this type of quantum dissipative system as in the case of conservative system we go back to Eq. (3.5), which can be rewritten in the form

$$m\frac{Dv}{Dt}+\gamma v=-\nabla\left(V-\frac{1}{2}mu^2-\frac{\hbar}{2}\nabla\cdot u\right). \qquad (3.7)$$

This equation is nothing but a same hydrodynamical motion as in the conservative case except the frictional term γv in the left-hand side. This extra term retards the flow as in the case of any external resistance.

The crucial element, which is very apparent from Eq. (3.7) is that, the "quantum potential" is same as that of conservative case. As we have discussed earlier the quantum behavior of a particle is hidden within the quantum potential so we can easily infer that difference between quantum conservative and nonconservative dynamical system is not quantum in nature but as that of in the classical case. We therefore suggest again that the motion of a nonrelativistic quantum particle in presence of a frictional force can be thought of as an invicid motion of an extended fluid particle having nonstationary random behavior within and constrained to move in a resisting medium which ultimately helps it to approach equilibrium.

IV. DISCUSSION

A comparison between Eqs. (2.18) and (3.7) might give us a very interesting result and leads us to a better understanding of quantum conservative and nonconservative systems. In the discussion just after Eq. (3.7) we have noted that the motion of a quantum particle in presence of friction will be such that the particle likes to dissipate energy maintaining its internal behavior unchanged and will approach equilibrium. But to have a connection between conservative and nonconservative systems we now try to find out the mathematical interconnection of Eqs. (2.18) and (3.7).

Equation (3.7) provokes us to write a hydrodynamical equation of a perfect fluid particle having varying mass with that of a potential of the same form. Obviously, that equation will be written as

$$\frac{D}{Dt}[m(t)v]=-\nabla\left[V-\frac{1}{2}m(t)u^2-\frac{\hbar}{2}\nabla\cdot u\right] \qquad (3.8)$$

or

$$v\frac{\partial m(t)}{\partial t}+m(t)\frac{Dv}{Dt}=-\nabla\left[V-\frac{1}{2}m(t)u^2-\frac{\hbar}{2}\nabla\cdot u\right].$$

If we take $m(t)=m_0e^{\beta t}$ we get

$$m_0\frac{Dv}{Dt}+\gamma v=-\nabla\left[Ve^{-\beta t}-\frac{1}{2}m_0u^2-\frac{\hbar}{2}e^{-\beta t}\nabla\cdot u\right], \qquad (3.9)$$

where $\gamma=m_0\beta$.

A minute observation of Eqs. (3.9) and (3.7) i.e., of Eqs. (3.8) and (3.7) will possibly allow us to infer that the quantum particle prefers to maintain its internal character (which we believe is responsible for the quantum properties like its isospin, strangeness, fermion number, etc. in the relativistic domain) in presence of external resistance like friction by emitting energy from within to the surroundings, if somehow an arrangement is made such that it is compelled to move without the exchange of energy with the surroundings, the role of friction can be thought of as if the particle is accumulating mass continuously along with the increase in friction within [note that nonconservative system with mass m_0 is equivalent with conservative system with mass $m_0e^{\beta t}$ and nonconservative system with coefficient $(\hbar/2)e^{-\beta t}$ is equivalent to the conservative system with coefficient $(\hbar/2)$ assuming external conservative force is absent].

From the above paragraph we can possibly naively state that to study the nonrelativistic motion of quantum particle in a presence frictional force, a connection between mass and energy may be smelled although it is the thing which should be realized in the relativistic domain. Moreover, we can arrive at another conclusion that the internal structure of the quantum particle is nonrigid in nature. This is very consistent with our earlier work on quantum uncertainty.[13]

Finally, we will like to mention that from the study of quantum particle in the relativistic domain we have realized that the internal geometry is anisotropic in nature, i.e., equivalent to vorticity in hydrodynamics. The nonrelativistic quantum description is obtained in the sharp point limit and the geometrical and topological properties of a relativistic particle are then squeezed through the so-called quantum potential.

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