The Passage From Random Walk to Diffusion in Quantum Probability

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Abstract

The notion of a quantum random walk in discrete time is formulated and the passage to a continuous time diffusion limit is established. The limiting diffusion is described in terms of solutions of certain quantum stochastic differential equations.

QUANTUM RANDOM WALK; TOY COHERENT VECTOR; COHERENT VECTOR; WEYL OPERATOR; CREATION; GAUGE AND ANNIHILATION PROCESSES; QUANTUM STOCHASTIC DIFFERENTIAL EQUATION; QUANTUM DIFFUSION; QUANTUM MARKOV SEMIGROUP; BOSON FOCK SPACE

1. Introduction

Suppose $\xi_1, \xi_2, \cdots$ is a sequence of i.i.d. real-valued random variables with mean 0 and variance unity. For any fixed positive integer n and $x \in \mathbb{R}$ consider the stochastic process

$$S_n(t, x) = \begin{cases} x & \text{if } 0 \leqq nt < 1, \\ x + n^{-\frac{1}{2}}(\xi_1 + \xi_2 + \cdots + \xi_j) & \text{if } j \leqq nt < j+1, \\ j = 1, 2, \cdots. \end{cases}$$

It is a well-known result of classical probability theory that the probability measure of the stochastic process $\{S_n(t, x), t \geqq 0\}$ converges in a suitable topology to the probability measure of the process $\{x + w(t), t \geqq 0\}$ as $n \to \infty$ where $\{w(t), t \geqq 0\}$ is the standard Brownian motion process. In other words the standard Brownian motion process starting from x at time 0 can be obtained as the limit of a random walk starting from x after a suitable rescaling of length and time. Our aim is to indicate how such a passage from discrete-time random walks to continuous-time diffusions could take place in the context of quantum probability where evolution takes place in the algebra of operators in a Hilbert space according to automorphisms induced by unitary operators. To this end we first reformulate the above-mentioned classical example in the language of operators in tensor products of Hilbert spaces and then show how this leads to quantum diffusions by using the ideas of boson stochastic calculus developed in [3] and outlined at greater length in [7].

2. Classical random walk in the language of quantum probability

Let μ be a probability measure on $\mathbb{R}$. If $\xi_1, \xi_2, \cdots$ is a sequence of i.i.d. random variables with distribution μ then the path described by the random walk starting from x and governed by the law μ is described by the sequence

$$x, x+\xi_1, x+\xi_1+\xi_2, \cdots, x+\xi_1+\xi_2+\cdots+\xi_n, \cdots.$$

To describe such a dynamics in the language of unitary operators in a Hilbert space we introduce the special Hilbert space

$$\begin{aligned}\mathcal{H} &= L_2(\mathbb{R}) \otimes L_2(\mu) \otimes L_2(\mu) \otimes \cdots \\ &= L_2(\mathbb{R}) \otimes L_2(P)\end{aligned} \tag{2.1}$$

where $P = \mu \times \mu \cdots$. It may be noted that the countable tensor product of copies of $L_2(\mu)$ is taken with respect to the sequence of unit vectors each of which is the constant function 1 in $L_2(\mu)$. Define the unitary operators $W(m, n)$ in $\mathcal{H}$ for each $0 \leqq m < n < \infty$ by putting

$$[W(m, n)u](x, \omega) = u(x - \{\xi_{m+1} + \cdots + \xi_n\}, \omega) \tag{2.2}$$

where $\omega = (\xi_1, \xi_2, \cdots)$ denotes the sample point of the i.i.d. sequence $\{\xi_j\}$ of random variables. Introduce the shift θ by putting

$$\theta(\xi_1, \xi_2, \cdots) = (\xi_2, \xi_3, \cdots) \tag{2.3}$$

and define the induced isometry S on $\mathcal{H}$ by

$$(Su)(x, \omega) = u(x, \theta\omega). \tag{2.4}$$

Then we have

$$\begin{aligned} W(m, n)W(l, m) &= W(l, n) \quad \text{for} \quad l < m < n \\ SW(m, n) &= W(m+1, n+1)S. \end{aligned} \tag{2.5}$$

If $u_0, v_0 \in L_2(\mathbb{R})$ then

$$\langle u_0 \otimes 1, W(n, m)v_0 \otimes 1\rangle_{\mathcal{H}} = \langle u_0, T^{n-m}v_0\rangle$$

where 1 is the constant function in $L_2(P)$, $\langle \cdot, \cdot \rangle$ denotes inner product and T denotes the operator in $L_2(\mathbb{R})$ defined by

$$(Tu_0)(x) = \int u_0(x-y)\mu(dy).$$

Thus the stationary Markov contraction semigroup $\{T^n, n \in \mathbb{Z}_+\}$ which describes an irreversible evolution in $L_2(\mathbb{R})$ is derived from a reversible evolution of unitary operators $\{W(m, n), 0 \leqq m < n < \infty\}$ in the larger Hilbert space $L_2(\mathbb{R}) \otimes L_2(P)$ satisfying the covariance condition in (2.5) under the shift isometry S. The family $\{W(m, n)\}$ describes a discrete Heisenberg dynamics in which an observable X_m at time m evolves to the observable

$X_n = W(m, n)^{\dagger} X_m \; W(m, n)$ at time $n > m$, where † denotes adjoint. In particular, if X_0 is the multiplication operator defined by

$$(X_0 u)(x, \omega) = xu(x, \omega)$$

then $W(0, n)^{\dagger} X_0 W(0, n) = X_n$ is the multiplication operator defined by

$$(X_n u)(x, \omega) = (x + \xi_1 + \cdots + \xi_n) u(x, \omega)$$

and $W(m, n)^{\dagger} X_m W(m, n) = X_n$ for all $m < n$. In other words the unitary evolution $\{W(m, n)\}$ implements the change of position from $x + \xi_1 + \cdots + \xi_m$ at time m to $x + \xi_1 + \cdots + \xi_n$ at time n of the classical random walk and hence gives a description of the random walk in the language of quantum probability.

To make the passage from discrete to continuous time we assume that $\mathbb{E}\xi_j = 0$, $\mathbb{E}\xi_j^2 = 1$ and proceed as follows: for any $0 \leqq s < t < \infty$ define the unitary operators $W_n(s, t)$ by

$$
\begin{aligned}
[W_n(s, t)u](x, \omega) &= u(x - n^{-\frac{1}{2}}\{\xi_{j+1} + \cdots + \xi_k\}, \omega) \\
&\qquad \text{if} \quad j/n \leqq s < \frac{j+1}{n} \leqq k/n \leqq t < \frac{k+1}{n}, \\
&= u(x, \omega) \quad \text{otherwise.}
\end{aligned}
\tag{2.6}
$$

In view of the rescaling of time we denote the shift isometry S defined by (2.4) as $S_{1/n}$ and write $S_{j/n} = \{S_{1/n}\}^j$. Then

$$W_n(t, u) W_n(s, t) = W_n(s, u) \quad \text{if} \quad s < t < u$$

$$S_{j/n} W_n(s, t) = W_n(s + j/n, t + j/n) S_{j/n}, \qquad j = 0, 1, 2, \cdots. \tag{2.7}$$

We may compare (2.7) with (2.5) and say that $\{W_n(s, t)\}$ is a continuous-time evolution by unitary operators covariant under shifts by j/n units of time for $j = 0, 1, 2, \cdots$. It is to be noted that $W_n(s, t)$ is strongly right-continuous in t for each fixed s. We now wish to analyse the asymptotic behaviour of $\{W_n(s, t)\}$ as $n \to \infty$.

Borrowing the language of Meyer in [7] we introduce the interesting and fruitful notion of 'toy coherent vectors' in $L_2(P)$. For any f in the space $C_c(\mathbb{R}_+)$ of all complex-valued continuous functions on $\mathbb{R}_+ = [0, \infty)$ with compact support define the associated *toy coherent vectors* $\psi_n(f)$ by

$$\psi_n(f) = \bigotimes_{j=1}^{\infty} (1 + n^{-\frac{1}{2}} f(j/n) \xi_j), \qquad n = 1, 2, \cdots \tag{2.8}$$

in $L_2(P)$. Then $\{\psi_n(f), f \in C_c(\mathbb{R}_+)\}$ is a total family in $L_2(P)$ for each n and

$$\langle \psi_n(f), \psi_n(g) \rangle = \prod_{j=1}^{\infty} \{1 + n^{-1}(\bar{f}g)(j/n)\} \tag{2.9}$$

where $\langle \cdot, \cdot \rangle$ denotes inner product which is conjugate linear in the first variable. It is significant that

$$\lim_{n\to\infty} \langle \psi_n(f), \psi_n(g)\rangle = \exp\int_0^\infty \bar{f}g \tag{2.10}$$

where $\int_a^b f$ denotes the Riemann integral of f over $[a, b]$. Furthermore for $u_0, v_0 \in L_2(\mathbb{R})$ and toy coherent vectors $\psi_n(f)$, $\psi_n(g)$ we have

$$\begin{aligned}
&\langle u_0 \otimes \psi_n(f), W_n(s,t)v_0 \otimes \psi_n(g)\rangle \\
&= \int \mathbb{E}\bar{u}_0(x)v_0(x - n^{-\frac{1}{2}}(\xi_{j+1} + \cdots + \xi_k)) \\
&\quad \times \prod_{r=1}^{\infty} [1 + n^{-\frac{1}{2}}\overline{f(r/n)\xi_r}][1 + n^{-\frac{1}{2}}g(r/n)\xi_r]dx \\
&\text{if } \frac{j}{n} \leqq s < \frac{j+1}{n} \leqq \frac{k}{n} \leqq t < \frac{k+1}{n}.
\end{aligned} \tag{2.11}$$

Define the unitary Fourier transform $\mathscr{F}$ on $L_2(\mathbb{R})$ by

$$(\mathscr{F}u)(x) = (2\pi)^{-\frac{1}{2}}\int \exp(-ixy)u(y)dy = \hat{u}(x), \qquad u \in L_2(\mathbb{R})$$

and denote the characteristic function of the distribution μ by ϕ. By Plancherel's theorem (2.11) becomes

$$\begin{aligned}
&\langle u_0 \otimes \psi_n(f), W_n(s,t)v_0 \otimes \psi_n(g)\rangle \\
&= \int \overline{\hat{u}_0(y)}\hat{v}_0(y) \prod_{r=j+1}^{k} \{\phi(n^{-\frac{1}{2}}y) - in^{\frac{1}{2}}\phi'(n^{-\frac{1}{2}}y)(\bar{f}+g)(r/n) \\
&\quad - n^{-1}\phi''(n^{-\frac{1}{2}}y)\bar{f}g(r/n)\} \\
&\quad \times \prod_{r\notin\{j+1,\cdots,k\}} [1 + n^{-1}\bar{f}g(r/n)]dy
\end{aligned} \tag{2.12}$$

where j, k, s, t satisfy the conditions in (2.11). Using the mean value theorem for real and imaginary parts of ϕ and letting $n \to \infty$ in (2.12) we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \langle u_0 \otimes \psi_n(f), W_n(s,t)v_0 \otimes \psi_n(g)\rangle \\
&= \int \bar{\hat{u}}_0(y)\hat{v}_0(y) \exp\left\{-\tfrac{1}{2}(t-s)y^2 + iy\int_s^t (\bar{f}+g) + \int_0^\infty \bar{f}g\right\}.
\end{aligned} \tag{2.13}$$

Elementary properties of the standard Brownian motion process $\{w(t)\}$ enable us to express the right-hand side of (2.13) as

$$\int \overline{\hat{u}_0(y)}\hat{v}_0(y)\mathbb{E}\exp\left\{iy[w(t) - w(s)] + \int_0^\infty (\bar{f}+g)dw - \tfrac{1}{2}\int_0^\infty (\bar{f}^2 + g^2)\right\}dy.$$

Now an application of Plancherel's theorem to the above integral shows that

$$\lim_{n\to\infty} \langle u_0 \otimes \psi_n(f), W_n(s, t)v_0 \otimes \psi_n(g)\rangle \tag{2.14}$$
$$= \int \overline{u_0(x)}v_0(x-[w(t)-w(s)])\overline{\psi(f)}\psi(g)dxdP_0(w)$$

where P_0 is the probability measure of the standard Brownian motion and

$$\psi(f)(w) = \exp\left(\int_0^\infty f dw - \frac{1}{2}\int_0^\infty f^2\right) \tag{2.15}$$

is the *coherent vector* associated with f in $L_2(P_0)$. Now consider the Hilbert space

$$\mathcal{H}' = L_2(\mathbb{R}) \otimes L_2(P_0)$$

and introduce the unitary operators $W(s, t)$, $0 \leqq s \leqq t < \infty$ defined by

$$[W(s, t)u](x, w) = u(x - [w(t) - w(s)], w). \tag{2.16}$$

Then we can express (2.14) as

$$\lim_{n\to\infty} \langle u_0 \otimes \psi_n(f), W_n(s, t)v_0 \otimes \psi_n(g)\rangle \tag{2.17}$$
$$= \langle u_0 \otimes \psi(f), W(s, t)v_0 \otimes \psi(g)\rangle$$

for all $u_0, v_0 \in L_2(\mathbb{R})$, $f, g \in C_c(\mathbb{R}_+)$. Furthermore

$$W(t, u)W(s, t) = W(s, u) \quad \text{for} \quad 0 \leqq s < t < u < \infty. \tag{2.18}$$

If S_h denotes the shift isometry defined in $\mathcal{H}'$ by

$$(S_h u)(x, w) = u(x, \theta_h w),$$
$$(\theta_h w)(t) = w(t+h) - w(h) \quad \text{for all} \quad h \geqq 0$$

then

$$S_h W(s, t) = W(s+h, t+h)S_h. \tag{2.19}$$

Equations (2.18) and (2.19) imply that $\{W(s, t)\}$ is a unitary evolution in $\mathcal{H}'$ covariant under shift in time. In the sense of (2.17) the unitary evolution $\{W_n(s, t)\}$ defined by the rescaled random walk in (2.6) approaches the covariant unitary evolution $\{W(s, t)\}$ defined by the standard Brownian motion in (2.16).

The asymptotic result (2.17) suggests the following notion of convergence for a sequence of unitary evolutions in varying Hilbert spaces.

Definition. Let $\mathcal{H}_n$, $n = \infty, 1, 2, \cdots$ be a family of Hilbert spaces and let $\mathcal{I}$ denote an index set. Suppose that for each fixed n there is a total family $\{\psi_{n,\alpha}, \alpha \in \mathcal{I}\}$ of vectors in $\mathcal{H}_n$ and a family $\{W_n(s, t), 0 \leqq s \leqq t < \infty\}$ of unitary operators in $\mathcal{H}_n$ such that $W_n(s, t)$ is strongly right-continuous in t for fixed s, $W_n(s, s) = 1$ and $W_n(t, u)W_n(s, t) = W_n(s, u)$ for $0 \leqq s \leqq t \leqq u < \infty$. We say that $\{W_n(\cdot, \cdot)\psi_{n\cdot}\}$ *converges* to $\{W_\infty(\cdot, \cdot), \psi_{\infty,\cdot}\}$ as $n \to \infty$ if

$$\lim_{n\to\infty} \langle \psi_{n,\alpha}, W_n(s, t)\psi_{n,\beta}\rangle = \langle \psi_{\infty,\alpha}, W_\infty(s, t)\psi_{\infty,\beta}\rangle$$

for each $\alpha, \beta \in \mathcal{I}$ and $0 \leqq s \leqq t < \infty$.

Remark. In the example that we have discussed it is only apparent that the Hilbert space $\mathcal{H}$ defined by (2.1) is not varying. In view of the scaling of time involved in (2.6) the number of times the tensor product of $L_2(\mu)$ is taken in an interval of unit length is equal to n at the nth stage. In the limit we obtain therefore the continuous tensor product $L_2(P_0)$.

The notion of convergence of operators in varying Hilbert spaces introduced here has also been independently arrived at recently by A. Bach and L. Accardi in their seminars at the University of Rome II.

3. Passage to Weyl operators from a quantum random walk

Since all the discussions hereafter will be concerned with Hilbert spaces we fix our notations as follows. All the Hilbert spaces will be tacitly assumed to be complex and separable with inner product $\langle\cdot,\cdot\rangle$ which is conjugate linear in the first variable. For any Hilbert space $\mathfrak{h}$ we denote by $\mathcal{B}(\mathfrak{h})$ the C^*-algebra of all bounded operators on $\mathfrak{h}$. For any operator $A \in \mathcal{B}(\mathfrak{h})$ we write $A^\dagger$ for its adjoint. By $C_c(\mathbb{R}_+, \mathfrak{h})$ we mean the space of all continuous maps from $[0, \infty)$ into $\mathfrak{h}$ with compact support. Suppose $\mathfrak{h} = \mathfrak{h}_1 \otimes \mathfrak{h}_2 \otimes \cdots$ is a countable tensor product of Hilbert spaces $\mathfrak{h}_j$, $j = 1, 2, \cdots$ with respect to the sequence $\{\Omega_j\}$ of unit vectors $\Omega_j \in \mathfrak{h}_j$, $j = 1, 2, \cdots$. If A is an operator in $\mathfrak{h}_{j_1} \otimes \mathfrak{h}_{j_2} \otimes \cdots \otimes \mathfrak{h}_{j_k}$ the *ampliation* $\tilde{A}$ of A to $\mathfrak{h}$ is the unique operator satisfying the relations

$$\begin{aligned}\langle u_1 \otimes u_2 \otimes \cdots, \tilde{A}v_1 \otimes v_2 \otimes \cdots\rangle \\ = \langle u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_k}, Av_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_k}\rangle \prod_{i \notin \{j_1, \cdots, j_k\}} \langle u_i, v_i\rangle\end{aligned}$$

for all sequences $\{u_n\}$, $\{v_n\}$ such that $u_n, v_n \in \mathfrak{h}_n$ for each n and $u_n = v_n = \Omega_n$ for all but a finite number of n's.

We start with the simplest case and construct a 'quantum random walk' in

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \tag{3.1}$$

where the countable tensor product is taken with respect to the sequence $\{\Omega_j\}$, $\Omega_j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $\mathbb{C}^2$, $j = 1, 2, \cdots$. We may look upon $\mathbb{C}^2$ as the L_2 space of a coin-tossing experiment or as the Hilbert space for describing the states of a two-level observable like spin. If l, z are complex numbers such that $|l| \leqq 1$, $|z| = 1$ then the matrix

$$U = \begin{pmatrix} (1-|l|^2)^{\frac{1}{2}} & -\bar{l}z \\ l & (1-|l|^2)^{\frac{1}{2}}z \end{pmatrix} \tag{3.2}$$

is unitary and inner automorphism by U on the algebra of 2×2 matrices can be interpreted as a one-step transition in a quantum random walk.

Now suppose that l, z are two complex-valued continuous functions on $\mathbb{R}_+$ such that l is bounded and $|z(t)| \equiv 1$. Define the unitary matrix-valued functions $U_n(t)$ by

$$U_n(t) = \begin{pmatrix} (1-n^{-1}|l(t)|^2)^{\frac{1}{2}} & -n^{-\frac{1}{2}}\bar{l}(t)z(t) \\ n^{-\frac{1}{2}}l(t) & (1-n^{-1}|l(t)|^2)^{\frac{1}{2}}z(t) \end{pmatrix} \tag{3.3}$$

for all large n. Let V_{jn} denote the unitary operator which is the ampliation of the unitary operator $U_n(j/n)$ on the jth copy of $\mathbb{C}^2$ in (3.1). For any $0 \leqq s \leqq t < \infty$ let

$$\begin{aligned} W_n(s, t) &= V_{kn}V_{k-1n} \cdots V_{j-1n} \\ &\quad \text{if } \frac{j}{n} \leqq s < \frac{j+1}{n} \leqq \frac{k}{n} \leqq t < \frac{k+1}{n} \\ &= 1 \quad \text{otherwise.} \end{aligned} \tag{3.4}$$

Then $W_n(s, t)$ is a unitary operator in $\mathscr{H}$ for each $0 \leqq s \leqq t < \infty$ and

$$W_n(t, u)W_n(s, t) = W_n(s, u) \quad \text{for all} \quad 0 \leqq s \leqq t \leqq u < \infty.$$

For fixed s, $W_n(s, t)$ is strongly right continuous in t. Thus $\{W_n(s, t)\}$ describes a quantum dynamical evolution where transition takes place at times $1/n, 2/n, \cdots$.

For any complex-valued continuous function f with compact support in $\mathbb{R}_+$, i.e. $f \in C_c(\mathbb{R}_+)$ define the *toy coherent vector* $\psi_n(f)$ in $\mathscr{H}$ by

$$\psi_n(f) = \bigotimes_{j=1}^{\infty} [1 \oplus n^{-\frac{1}{2}}f(j/n)] \tag{3.5}$$

where $\mathbb{C}^2$ is expressed as $\mathbb{C} \oplus \mathbb{C}$. Then

$$\begin{aligned}&\langle \psi_n(f), W_n(s,t)\psi_n(g)\rangle \\ &= \prod_{r=j+1}^{k} \langle [1 \oplus n^{-\frac{1}{2}} f(r/n)], U_n(r/n)[1 \oplus n^{-\frac{1}{2}} g(r/n)]\rangle \\ &\quad \times \prod_{r \notin \{j+1,\cdots,k\}} [1 + n^{-1}(\bar{f}g)(r/n)] \quad \text{if} \quad j/n \leqq s < \frac{j+1}{n} \leqq k/n \leqq t < \frac{k+1}{n}.\end{aligned}$$

By elementary algebra we get

$$\begin{aligned}&\langle \psi_n(f), W_n(s,t)\psi_n(g)\rangle \\ &= \prod_{r \notin \{j+1,\cdots,k\}} [1 + n^{-1}\bar{f}g(r/n)] \\ &\quad \times \prod_{r=j+1}^{k} \{(1 - n^{-1}|l(r/n)|^2)^{\frac{1}{2}} + n^{-1}[\bar{f}l - gz\bar{l} + \bar{f}gz(1 - n^{-1}|l|^2)^{\frac{1}{2}}(r/n)\} \\ &\quad \text{if} \quad j/n \leqq s < \frac{j+1}{n} \leqq k/n \leqq t < \frac{k+1}{n}.\end{aligned}$$

Thus

$$\begin{aligned}&\lim_{n\to\infty} \langle \psi_n(f), W_n(s,t)\psi_n(g)\rangle \\ &= \exp\left\{\int_0^\infty \bar{f}g + \int_s^t (\bar{f}l + \bar{f}g(z-1) - \bar{l}zg - \tfrac{1}{2}|l|^2)\right\}. \end{aligned} \tag{3.6}$$

We shall now express the right-hand side of the above equation as $\langle \psi(f), W(s,t)\psi(g)\rangle$ in a suitable Hilbert space where $\{W(s,t)\}$ is a strongly continuous unitary evolution and $\psi(f)$, $\psi(g)$ are suitable vectors. To this end we introduce some definitions.

For any Hilbert space $\mathfrak{h}$ we denote by $\mathfrak{h}^{\circledS n}$ its n-fold symmetric tensor product and define the boson Fock space $\Gamma(\mathfrak{h})$ as

$$\Gamma(\mathfrak{h}) = \mathbb{C} \oplus \mathfrak{h} \oplus \cdots \oplus \mathfrak{h}^{\circledS n} \oplus \cdots. \tag{3.7}$$

For any $u \in \mathfrak{h}$ let

$$\psi(u) = 1 \oplus u \oplus (2!)^{-\frac{1}{2}} u^{\otimes^2} \oplus \cdots \oplus (n!)^{-\frac{1}{2}} u^{\otimes^n} \oplus \cdots \tag{3.8}$$

denote the *coherent vector* associated with u in $\Gamma(\mathfrak{h})$. For any $u \in \mathfrak{h}$ and unitary operator U on $\mathfrak{h}$ we define the unitary *Weyl operator* $W(u, U)$ in $\Gamma(\mathfrak{h})$ by putting

$$W(u, U)\psi(v) = \exp\{-\tfrac{1}{2}\|u\|^2 - \langle u, Uv\rangle\}\psi(Uv + u), \qquad v \in \mathfrak{h} \tag{3.9}$$

and extending to $\Gamma(\mathfrak{h})$ (see [3]). Then the Weyl operators obey the commutation relations

$$W(u, U)W(v, V) = \exp\{-iIm\langle u, Uv\rangle\}W(u + Uv, UV). \tag{3.10}$$

When $\mathfrak{h} = L_2(\mathbb{R}_+)$, $u = \chi_{[s,t]}l$ and U is multiplication by $1 - \chi_{[s,t]} + \chi_{[s,t]}z$ where $\chi_{[s,t]}$ is the indicator function of the interval $[s, t]$ in $\mathbb{R}_+$ and l, z are as in (3.3) the Weyl operator $W(s, t) = W(u, U)$ satisfies

$$\begin{aligned} &W(t, u)W(s, t) = W(s, u) \quad \text{for all} \quad 0 \leqq s \leqq t \leqq u < \infty, \\ &\langle \psi(f), W(s, t)\psi(g)\rangle \\ &\quad = \exp\left\{\int_0^\infty \bar{f}g + \int_s^t (\bar{f}l + \bar{f}g(z-1) - \bar{l}zg - \tfrac{1}{2}|l|^2)\right\} \end{aligned} \tag{3.11}$$

for all $f, g \in L_2(\mathbb{R}_+)$. Combining (3.6) and (3.11) we conclude that $\{W_n(s, t), \psi_n(f), 0 \leqq s \leqq t < \infty, f \in C_c(\mathbb{R}_+)\}$ converges as $n \to \infty$ to $\{W(s, t), \psi(f), 0 \leqq s \leqq t < \infty, f \in C_c(\mathbb{R}_+)\}$ in the boson Fock space $\Gamma(L_2(\mathbb{R}_+))$ in the sense of the definition at the end of Section 2.

Remark 1. The family of unitary operators $\{W(t), t \geqq 0\}$ where $W(t) = W(0, t)$ described above in $\Gamma(L_2(\mathbb{R}_+))$ obeys the quantum stochastic differential equation

$$dW = \{l dA^\dagger + (z - 1)d\Lambda - \bar{l}z dA - \tfrac{1}{2}|l|^2 dt\}W \tag{3.12}$$

in terms of the quantum processes $A^\dagger$, Λ and A called respectively the *creation*, *gauge* and *annihilation* processes in [3]. The process $A + A^\dagger$ and $-i(A - A^\dagger)$ are two non-commuting classical Brownian motions in the vacuum state $\psi(0)$ and the pair $\{A(t) + A^\dagger(t), -i[A(t) - A^\dagger(t)], t \geqq 0\}$ is the quantum Brownian motion first introduced in [1]. The process $N_\lambda(t) = \Lambda(t) + \lambda^{\frac{1}{2}}[A(t) + A^\dagger(t)] + \lambda t$, $t \geqq 0$ is a classical Poisson process of intensity λ realized as a commuting family of self-adjoint operators in $\Gamma(L_2(\mathbb{R}_+))$ in the vacuum state $\psi(0)$. For more details in this direction the reader may refer to [3], [7].

Remark 2. It is possible to generalize the asymptotic result concerning the random walk process $W_n(s, t)$ defined by (3.4) to the case when $\mathbb{C}^2$ is replaced by $\mathbb{C} \oplus \mathfrak{k}$ and the Hilbert space

$$\mathcal{H} = (\mathbb{C} \oplus \mathfrak{k}) \otimes (\mathbb{C} \oplus \mathfrak{k}) \otimes \cdots \tag{3.13}$$

is considered, the countable tensor product being taken with respect to the unit vector $1 \oplus 0$ in each copy of $\mathbb{C} \oplus \mathfrak{k}$. We proceed as follows. For any $l \in \mathfrak{k}$ with $\|l\| < 1$ and unitary operator Z in $\mathfrak{k}$ we introduce the unitary operator U in $\mathbb{C} \oplus \mathfrak{k}$ in analogy with (3.2) by putting

$$U = \begin{pmatrix} (1 - \|l\|^2)^{\frac{1}{2}} & -\langle l| Z \\ |l\rangle & (1 - |l\rangle\langle l|)^{\frac{1}{2}} Z \end{pmatrix} \tag{3.14}$$

where we have used the bra and ket notation of Dirac. Let $\mathcal{U}(\mathfrak{k})$ denote the group of unitary operators on $\mathfrak{k}$. Suppose now l is a $\mathfrak{k}$-valued bounded continuous function on $\mathbb{R}_+$ and Z is a strongly continuous $\mathcal{U}(\mathfrak{k})$-valued function on $\mathbb{R}_+$. Define $U_n(t)$ to be the unitary operator in (3.14) when l is replaced by $n^{-\frac{1}{2}}l(t)$ and Z by $Z(t)$. Let V_{jn} denote the unitary operator in $\mathcal{H}$ which is the ampliation of $U_n(j/n)$ on the jth copy $C \oplus \mathfrak{k}$ in (3.13). Now define $W_n(s, t)$ exactly as in (3.4). For any $\mathfrak{k}$-valued continuous function f on $\mathbb{R}_+$ with compact support define the toy coherent vector $\psi_n(f)$ by (3.5). Then it is straightforward to establish the asymptotic result

$$\lim_{n\to\infty} \langle \psi_n(f), W_n(s, t)\psi_n(g)\rangle$$

$$= \exp \int_0^\infty \langle f(\cdot), g(\cdot)\rangle + \int_s^t \{\langle f(\cdot), l(\cdot)\rangle + \langle f(\cdot), (Z(\cdot) - 1)g(\cdot)\rangle$$

$$- \langle l(\cdot), Z(\cdot)g(\cdot)\rangle - \tfrac{1}{2}\|l(\cdot)\|^2\}$$

$$= \langle \psi(f), W(\chi_{[s,t]}l, 1 - \chi_{[s,t]} + \chi_{[s,t]}Z)\psi(g)\rangle$$

where $W(l, z)$ is the Weyl operator on $\Gamma(L_2(\mathbb{R}_+) \otimes \mathfrak{k})$ when $L_2(\mathbb{R}_+) \otimes \mathfrak{k}$ is interpreted as the Hilbert space of $\mathfrak{k}$-valued square-integrable functions on $\mathbb{R}_+$. This shows that the random walk described by the evolution $\{W_n(s, t)\}$ converges in the sense of the definition at the end of Section 2 to the evolution $\{W(s, t)\}$ described by the Weyl operators

$$W(s, t) = W(\chi_{[s,t]}l, 1 - \chi_{[s,t]} + \chi_{[s,t]}Z).$$

It can once again be shown that $W(t) = W(0, t)$ obeys a quantum stochastic differential equation which is a multi- (possibly infinite-) dimensional version of (3.12).

It is significant to note that all classical stochastic processes with independent increments can be described in terms of the operators $W(s, t)$ described above (see [8], [9]).

4. Passage to a quantum diffusion from a quantum random walk

By analogy with the discussion of Section 2 we consider the Hilbert space

$$\mathcal{H} = \mathfrak{h}_0 \otimes \{(\mathbb{C} \oplus \mathfrak{k}) \otimes (\mathbb{C} \oplus \mathfrak{k}) \otimes \cdots\} \tag{4.1}$$

where $\mathfrak{h}_0$ and $\mathfrak{k}$ are fixed Hilbert spaces and the countable tensor product in $\{\ \}$ is taken with respect to the sequence of unit vectors $1 \otimes 0$ in each copy of $\mathbb{C} \oplus \mathfrak{k}$. For any operator $L : \mathfrak{h}_0 \to \mathfrak{h}_0 \otimes \mathfrak{k}$ such that $\|L\| < 1$ and unitary operators Z and V on $\mathfrak{h}_0 \otimes \mathfrak{k}$ and $\mathfrak{h}_0$ respectively we construct a unitary operator U on

$\mathfrak{h}_0 \otimes (\mathbb{C} \oplus \mathfrak{k})$ by identifying it with $\mathfrak{h}_0 \oplus (\mathfrak{h}_0 \otimes \mathfrak{k})$ and defining U in the matrix notation as

$$U = \begin{pmatrix} (1 - L^\dagger L)^{\frac{1}{2}} & -L^\dagger Z \\ L & (1 - LL^\dagger)^{\frac{1}{2}} Z \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \otimes 1 \end{pmatrix}. \tag{4.2}$$

It is instructive to compare this with (3.14) wherein $\mathfrak{h}_0 = \mathbb{C}$, $V = 1$ and $Lu = u\,|l\rangle$, $u \in \mathbb{C}$. Inner automorphism by U in the algebra of operators on $\mathfrak{h}_0 \otimes (\mathbb{C} \oplus \mathfrak{k})$ enables us to make the transition $X \to U^\dagger (X \otimes 1) U$ from an operator X on $\mathfrak{h}_0$ to an operator on $\mathfrak{h}_0 \otimes (\mathbb{C} \oplus \mathfrak{k})$. If V_j is the ampliation of U from $\mathfrak{h}_0 \otimes$ (jth copy of $\mathbb{C} \oplus \mathfrak{k}$) to $\mathscr{H}$ in (4.1) and

$$W(m, n) = V_n V_{n-1} \cdots V_{m+1} \quad \text{for} \quad 0 \leqq m < n \tag{4.3}$$

then $\{W(m, n)\}$ is a unitary evolution in discrete time. If S is the shift isometry defined by

$$Su \otimes \{\xi_1 \otimes \xi_2 \otimes \cdots\} = u \otimes \{1 \otimes \xi_1 \otimes \xi_2 \otimes \cdots\}$$

then

$$SW(m, n) = W(m+1, n+1) S. \tag{4.4}$$

Let

$$\Omega = (1 \oplus 0) \otimes (1 \oplus 0) \otimes \cdots$$

in $\otimes_{j=1}^\infty (\mathbb{C} \oplus \mathfrak{k})$. Then we have the relations

$$\langle u \otimes \Omega, W(m, n) v \otimes \Omega \rangle = \langle u, \{(1 - L^\dagger L)^{\frac{1}{2}} V\}^{n-m} v \rangle, \tag{4.5}$$

$$\begin{aligned} &\langle u \otimes \Omega, W(m, n)^\dagger X \otimes 1 W(m, n) v \otimes \Omega \rangle \\ &= \langle u, T^{n-m}(X) v \rangle \end{aligned} \tag{4.6}$$

for all $u, v \in \mathfrak{h}_0$ and any bounded operator X in $\mathfrak{h}_0$ where T is the completely positive map on the C^* algebra $\mathscr{B}(\mathfrak{h}_0)$ of all bounded operators in $\mathfrak{h}_0$ defined by

$$T(X) = V^\dagger \{(1 - L^\dagger L)^{\frac{1}{2}} X (1 - L^\dagger L)^{\frac{1}{2}} + L^\dagger X \otimes 1_{\mathfrak{k}} L\} V. \tag{4.7}$$

Equations (4.5) – (4.7) justify calling the unitary evolution $\{W(m, n)\}$ defined by (4.3) a *homogeneous quantum random walk*.

Now let $L(t)$, $Z(t)$, $H(t)$ be operator-valued functions on $\mathbb{R}_+$ such that $L(t)$ is an operator from $\mathfrak{h}_0$ into $\mathfrak{h}_0 \otimes \mathfrak{k}$, $Z(t)$ is a unitary operator in $\mathfrak{h}_0 \otimes \mathfrak{k}$ and $H(t)$ is a self-adjoint operator on $\mathfrak{h}_0$. We assume that

(i) $\sup_t (\|L(t)\| + \|H(t)\|) < \infty$

(ii) $L(\cdot)$, $Z(\cdot)$ and $H(\cdot)$ are continuous in the uniform topology.

On the basis of the operator-valued functions L, Z, H we construct a sequence of evolutions as follows. Since $n^{-\frac{1}{2}} \|L(t)\| < 1$ for all $t \geqq 0$ and sufficiently large n we define the unitary operator $U_N(t)$ in $\mathfrak{h}_0 \otimes (\mathbb{C} \oplus \mathfrak{k})$ through the expression in (4.2) by substituting $n^{-\frac{1}{2}}L(t)$, $Z(t)$, $\exp\{-in^{-1}H(t)\}$ for L, Z, V respectively. As in (3.4) define V_{jn} in $\mathcal{H}$ to be the ampliation of the unitary operator $U_n(j/n)$ from the tensor product of $\mathfrak{h}_0$ and the jth copy of $\mathbb{C} \oplus \mathfrak{k}$ to $\mathcal{H}$ in (4.1) and determine the unitary evolution $W_n(s, t)$ according to (3.4). For any $\mathfrak{k}$-valued continuous function f on $\mathbb{R}_+$ with compact support define the associated toy coherent vector $\psi_n(f) = \otimes_{j=1}^{\infty} [1 \oplus n^{-\frac{1}{2}} f(j/n)]$ in $(\mathbb{C} \oplus \mathfrak{k}) \otimes (\mathbb{C} \oplus \mathfrak{k}) \otimes \cdots$. Then we have the following theorem.

Theorem 4.1. There exists a strongly continuous unitary evolution $\{W(s, t)\}$ in the Hilbert space $\mathfrak{h}_0 \otimes \Gamma(L_2(\mathbb{R}_+) \otimes \mathfrak{k})$ such that

$$\lim_{n \to \infty} \langle u \otimes \psi_n(f), W_n(s, t) v \otimes \psi_n(g) \rangle = \langle u \otimes \psi(f), W(s, t) v \otimes \psi(g) \rangle$$

for all $0 \leqq s < t < \infty$, $u, v \in \mathfrak{h}_0$, $f, g \in C_c(\mathbb{R}_+, \mathfrak{k})$ where $\psi(f)$ denotes the coherent vector associated with f when f is considered an element of $L_2(\mathbb{R}_+) \otimes \mathfrak{k}$. Furthermore

$$\begin{aligned} &\frac{d}{dt} \langle u \otimes \psi(f), W(0, t) v \otimes \psi(g) \rangle \\ &= \langle u \otimes f(t) \otimes \psi(f), L(t) W(0, t) v \otimes \psi(g) \rangle \\ &\quad + \langle u \otimes f(t) \otimes \psi(f), \{Z(t) - 1\} W(0, t) v \otimes g(t) \otimes \psi(g) \rangle \\ &\quad - \langle u \otimes \psi(f), L^\dagger(t) W(0, t) v \otimes g(t) \otimes \psi(g) \rangle \\ &\quad - \langle u \otimes \psi(f), \{iH(t) + \tfrac{1}{2} L^\dagger(t) L(t)\} W(0, t) v \otimes \psi(g) \rangle \end{aligned} \tag{4.8}$$

where $L(t)$, $Z(t)$ and $W(0, t)$ denote also their ampliations to $\mathfrak{h}_0 \otimes \mathfrak{k} \otimes \Gamma(L_2(\mathbb{R}_+) \otimes \mathfrak{k})$.

Proof. We restrict ourselves to the case when $\mathfrak{k} = \mathbb{C}$ and L, Z, H are independent of t. The general case is proved exactly along the same lines except for the introduction of a more elaborate notation and using the results of [5] instead of [3] in boson stochastic calculus.

Fix $f, g \in C_c(\mathbb{R}_+)$ and define the operators M_{rn}, $r = 0, 1, 2, \cdots$ in $\mathfrak{h}_0$ through their bilinear forms by putting

$$\langle u, M_{rn} v \rangle = \langle u \otimes \psi_n(f), W_n(0, r/n) v \otimes \psi_n(g) \rangle \times \prod_{j=r+1}^{\infty} \left\{ 1 + \frac{1}{n} (\bar{f}g)(j/n) \right\}^{-1}. \tag{4.9}$$

For each fixed n we shall now write a difference equation for the sequence M_{rn}, $r = 0, 1, 2, \cdots$. To begin with we consider the case $n = 1$. Then

$M_{01} = 1$ and

$$
\begin{aligned}
&\langle u \otimes \psi_1(f), W_1(0, r)v \otimes \psi_1(g)\rangle \\
(4.10) \qquad &= \langle u \otimes \psi_1(f), V_{r1} W(0, r-1)v \otimes \psi_1(g)\rangle \\
&= \langle V_{r1}^\dagger u \otimes \psi_1(f), W_1(0, r-1)v \otimes \psi_1(g)\rangle
\end{aligned}
$$

where V_{r1} is the ampliation of

$$U = \begin{pmatrix} (1 - L^\dagger L)^{\frac{1}{2}} & -L^\dagger Z \\ L & (1 - L^\dagger L)^{\frac{1}{2}} Z \end{pmatrix} \begin{pmatrix} \exp(-iH) & 0 \\ 0 & \exp(-iH) \end{pmatrix}.$$

Considered as an operator in the tensor product of $\mathfrak{h}_0$ and the rth copy of $\mathbb{C} \oplus \mathbb{C}$ in $\mathscr{H} = \mathfrak{h}_0 \otimes (\mathbb{C} \oplus \mathbb{C}) \otimes \cdots \otimes (\mathbb{C} \oplus \mathbb{C}) \otimes \cdots$. We can express for any $u \in \mathfrak{h}_0$, $\alpha \in \mathbb{C}$

$$(4.11) \qquad U^\dagger u \otimes (1 \oplus \alpha) = R(\alpha)u \otimes (1 \oplus 0) + S(\alpha)u \otimes (0 \oplus 1)$$

where

$$
\begin{aligned}
(4.12) \qquad &R(\alpha) = \exp(iH)\{(1 - L^\dagger L)^{\frac{1}{2}} + \alpha L^\dagger\}, \\
&S(\alpha) = \exp(iH)Z^\dagger\{-L + \alpha(1 - LL^\dagger)^{\frac{1}{2}}\}.
\end{aligned}
$$

Substituting for $V_{r1}^\dagger$ in (4.10) from (4.11) and using the expressions for the toy coherent vectors $\psi_1(f)$, $\psi_1(g)$ we obtain

$$\langle u, M_{r1} v\rangle = \langle R(f(r))u, M_{r-1\,1} v\rangle + \langle \overline{g(r)} S(f(r))u, M_{r-1\,1} v\rangle$$

or equivalently, in view of (4.12)

$$
\begin{aligned}
(4.13) \qquad M_{r1} = &((1 - L^\dagger L)^{\frac{1}{2}} + \overline{f(r)} L + g(r)\{-L^\dagger + \overline{f(r)}(1 - LL^\dagger)^{\frac{1}{2}}\} Z) \\
&\times \exp(-iH) M_{r-1\,1}.
\end{aligned}
$$

To obtain the recurrence relations for M_{rn} we have only to change $L, f(r), g(r)$ and H to $n^{-\frac{1}{2}}L$, $n^{-\frac{1}{2}}f(r/n)$, $n^{-\frac{1}{2}}g(r/n)$ and $n^{-1}H$ in (4.13). We express it in the form of a difference equation:

$$(4.14) \qquad n\{M_{rn} - M_{r-1\,n}\} = D_{rn} M_{r-1\,n}$$

where

$$
\begin{aligned}
(4.15) \qquad D_{rn} = &\, n\{(1 - n^{-1}L^\dagger L)^{\frac{1}{2}} \exp(-in^{-1}H) - 1\} \\
&+ \{\bar{f}(r/n)L + (\bar{f}g)(r/n)(1 - n^{-1}LL^\dagger)^{\frac{1}{2}} Z - g(r/n)L^\dagger Z\} \exp(-in^{-1}H).
\end{aligned}
$$

It is significant that as $n \to \infty$ the difference equation (4.14) becomes the differential equation

$$(4.16) \qquad \frac{dM}{dt} = D(t)M(t), \qquad M(0) = 1$$

where

(4.17) $$D(t) = -iH - \tfrac{1}{2}L^{\dagger}L + \bar{f}(t)L + (\bar{f}g)(t)Z - g(t)L^{\dagger}Z.$$

Define the operator-valued step functions

$$D_n(t) = D_{rn} \quad \text{if} \quad \frac{r}{n} \leqq t < \frac{r+1}{n},$$

$$M_n(t) = M_{rn} \quad \text{if} \quad \frac{r}{n} \leqq t < \frac{r+1}{n}, \qquad r = 0, 1, 2, \cdots.$$

Since

$$\lim_{n\to\infty} \sup_{0\leqq s\leqq t} \|D_n(s) - D(s)\| = 0 \quad \text{for all} \quad t > 0$$

it follows from a routine but detailed analysis of (4.14) and (4.16) that

(4.18) $$\lim_{n\to\infty} \sup_{0\leqq s\leqq t} \|M_n(s) - M(s)\| = 0 \quad \text{for all} \quad t > 0$$

where M is the unique solution of (4.16). Now define

(4.19) $$N_n(t) = M_n(t) \prod_{j=r-1}^{\infty} \left\{1 + \frac{1}{n}\bar{f}g\left(\frac{j}{n}\right)\right\} \quad \text{if} \quad \frac{r}{n} \leqq t < \frac{r+1}{n}, \quad r = 0, 1, 2, \cdots$$

(4.20) $$N(t) = M(t) \exp \int_t^{\infty} \bar{f}g.$$

Then it follows from (4.18) that

(4.21) $$\lim_{n\to\infty} \sup_{0\leqq s\leqq t} \|N_n(s) - N(s)\| = 0.$$

From (4.16) and (4.20) we obtain

$$\frac{dN}{dt} = \{\bar{f}(t)L + (\bar{f}g)(t)(Z-1) - g(t)L^{\dagger}Z - iH - \tfrac{1}{2}L^{\dagger}L\}N(t).$$

Using (4.9), the definitions of $M_n(t)$, $N_n(t)$ and (4.21) we obtain

$$\lim_{n\to\infty} \langle u \otimes \psi_n(f), W_N(0, t)v \otimes \psi_n(g)\rangle$$

$$= \langle u, N(t)v\rangle$$

and the convergence is uniform in every bounded interval. From the boson stochastic calculus developed in [3] we know that there exists a unitary evolution $\{W(s, t)\}$ in $\mathfrak{h}_0 \otimes \Gamma(L_2(\mathbb{R}_+))$ such that $W(t) = W(0, t)$ obeys the quantum stochastic differential equation

$$dW = \{L\,dA^{\dagger} + (Z - 1)\,d\Lambda - L^{\dagger}Z\,dA - (iH + \tfrac{1}{2}L^{\dagger}L)\,dt\}W$$

with $W(0) = 1$, in terms of the triple $(A^{\dagger}, \Lambda, A)$ consisting of creation, gauge and annihilation processes. This is equivalent to saying that

$$\langle u \otimes \psi(f), W(0, t)v \otimes \psi(g)\rangle = \langle u, N(t)v\rangle$$

where $N(t)$ is the solution of (4.22) with initial condition $N(0) = \exp \int_0^{\infty} \bar{f}g$. Thus we have shown that

$$\lim_{n\to\infty} \langle u \otimes \psi_n(f), W_n(0, t)v \otimes \psi_n(g)\rangle = \langle u \otimes \psi(f), W(0, t)v \otimes \psi(g)\rangle$$

uniformly in t over every bounded interval. shifting the origin from 0 to s does not involve any change in the argument. Equation (4.22) is the same as (4.8) in the special case $\mathfrak{k} = \mathbb{C}$.

Remark. Using the Heisenberg dynamics induced by the evolution $\{W(s, t)\}$ in Theorem 4.1 one can define for any $X \in \mathcal{B}(\mathfrak{h}_0)$

$$X(t) = W(0, t)^{\dagger}(X \otimes 1)W(0, t), \qquad t \geqq 0.$$

Define the completely positive maps $T_t : \mathcal{B}(\mathfrak{h}_0) \to \mathcal{B}(\mathfrak{h}_0)$ by

$$\langle u, T_t(X)v\rangle = \langle u \otimes \Phi, X(t)v \otimes \Phi\rangle$$

where Φ is the Fock vacuum in $\Gamma(L_2(\mathbb{R}_+) \otimes \mathfrak{k})$. Then

$$\frac{d}{dt}T_t(X) = \mathcal{L}_t T_t(X)$$

where

$$\mathcal{L}_t(X) = i[H(t), X] - \tfrac{1}{2}\{L^{\dagger}(t)L(t)X + XL^{\dagger}(t)L(t) - 2L^{\dagger}(t)(X \otimes 1_t)L(t)\},$$

which is the well-known generator of a quantum Markov semigroup of operators on $\mathcal{B}(\mathfrak{h}_0)$ derived by Gorini et al. [2] as well as Lindblad [6], We can say that $\{W(s, t)\}$ is the Schrödinger picture of a quantum diffusion described by the generators $\mathcal{L}_t$. If $H(t)$, $L(t)$, $Z(t)$ are independent of t then $\{W(s, t)\}$ is covariant under time shifts and one obtains a homogeneous diffusion. Thus we have shown how quantum diffusions can be obtained as limits of discrete-time quantum random walks.

In Theorem 4.1 one would expect that for any $X \in \mathcal{B}(\mathfrak{h}_0)$

$$\lim_{n\to\infty} \langle u \otimes \psi_n(f), W_n(s, t)^{\dagger}(X \otimes 1)W_n(s, t)v \otimes \psi_n(g)\rangle$$
$$= \langle u \otimes \psi(f), W(s, t)^{\dagger}(X \otimes 1)W(s, t)v \otimes \psi(g)\rangle$$

for all $u, v \in \mathfrak{h}_0$, $f, g \in C_c(\mathbb{R}_+, \mathfrak{k})$. (This has now been established: see [10].)

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