Sankhyā : The Indian Journal of Statistics
1988, Volume 50, Series B, Pt. 2, pp. 245-257

DUAL PRICING AND DISTRIBUTION OF COMMODITIES: EXISTENCE AND EFFICIENCY

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SUMMARY. A good is said to be dual priced if it is sold in the free market at the competitive price and a fixed quota per consumer is sold in the ration shops at a controlled price less than the free market price. In this paper we describe the notion of a Dual Price Equilibrium and study its efficiency properties. We also prove the existence of such an equilibrium under certain very general conditions.

1. Introduction

It is well known that the use of the market mechanism along with a redistribution of income will lead to efficiency in the distribution of the goods and services. It is also known that the distributions resulting from very general classes of rationing schemes result in Pareto-inefficient allocations (see e.g. Nayak, 1980). Yet, in many economies including the industrially developed, prices of some goods and services vary with the types of consumers or producers. The purpose of this note is to study the existence and efficiency of equilibria, when there are two prices for a commodity.

2. The model

Consider an economy with finitely many commodities and consumers. Let 'A' denote the set of n consumers and 'l', the number of commodities. For $i \in A$, the consumption set, the preference preordering and the initial endowment of the i-th consumer are given by (X_i, u_i, e_i), where $X_i \subset \mathcal{R}^l_+$, $u_i : X_i \rightarrow \mathcal{R}_+$ and $e_i \in X_i$. We make the usual assumptions on the consumption set and the initial endowment ; we assume that u_i is differentiable and the preferred set is strictly convex. The set of prices is $Q = \mathcal{R}^l_+ \times \{1\}$, i.e. the l-th commodity is assumed to be the numeraire. We assume that the first commodity is subject to dual pricing, i.e., every consumer is entitled to buy a fixed amount, say 'D' units of this commodity, at a fixed price $\bar{p} > 0$. Note $\bar{p}$ and D are constants. Given any $q = (\bar{p}, q^1, \ldots, q^l) \in Q$, the i-th consumer's budget set, $\beta_i(q, D)$ is defined by

$$\{x \in X_i \mid \bar{p}(x^1 - e_i^1) + (q^1 - \bar{p})(\max[x^1 - D, 0] - \max[e_i^1 - D, 0]) + \sum_{2}^{l} q^j (x^j - e_i^j) \leqslant 0\}.$$

AMS (1980) subject classification : 90A14.

Key words and phrases : Rationing, Dual Price Equilibrium, Pareto efficiency, Existence of equilibrium.

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In this scheme, e_i lies on the boundary of the budget set. The consumer's demand correspondence is defined by

$$\phi_i(q, D) = \{x \in \beta_i(q, D) \mid x \succsim_i y, \ \forall\, y \in \beta_i(q, D)\}.$$

For $x_i \in \phi_i(q, D)$, excess demand is denoted by $z_i(q, D)$. An exchange economy is denoted by $\epsilon = (X_i, u_i, e_i)$.

This scheme may seem to be difficult to enforce since the government has to be informed on the consumption and resources of each agent. But if we want to implement this scheme for a necessity, say food, then we may assume that the consumers' initial endowments are nil.

Definition 1 : An exchange economy is said to have a dual price equilibrium (DPE) if there exists a ration quota D for the first commodity together with a price vector $q^* \in Q$ and a set of allocations $\{x_i^*, i \in A\}$, satisfying the following conditions :

(i) $\sum_1^n (x_i^* - e_i) = 0$

(ii) $x_i^* \in \phi_i(q^*, D)$

(iii) $q^{*1} > \bar{p}$ and $x_i^{1*} > D$ for at least one $i \in A$.

(For any vector $x \in R^l$, x^j denotes the j-th component).

The last condition means that the open market functions for the first commodity. Note that if either of the inequalities in this condition does not hold the DPE reduces to a competitive equilibrium.

A DPE is said to be trivial if the associated allocations can be obtained as a Walras equilibrium *without any redistribution of initial endowments*, i.e., if there is a $\bar{q} \in Q$, such that $\{x_i^*\}$ is a Walras equilibrium at price $\bar{q}$ and the initial endowments e_i, $i = 1, 2, \ldots, n$.

Dual price equilibrium, when it exists, will have some desirable properties at least with respect to the distribution of the good with two prices. For example, we can make sure that the total demand for the first good by *any* consumer is at least a prespecified minimum, say D, *irrespective* of the initial distribution of endowments. This will be the case, for instance, if $\bar{p}$ is set equal to zero, and there is no satiation of this good. We will now examine whether DPE allocations are Pareto efficient. For this purpose, we need the following result (For a proof and some characterizations of consumer and producer behaviour under dual pricing, see Chetty and Jha, 1986) which is an immediate consequence of Fenchel's duality theorem.

Let $x_i \in \phi_i(q, D)$. Then there exists a $\theta_i \in [0, 1]$ such that x_i^* maximizes u_i subject to the following budget constraint :

$$x_i^1[\theta_i q^1+(1-\theta_i)\bar{p}]+\sum_2^l x_i^j q^j \leq \bar{p}e_i^1+(q^1-\bar{p})\max(e_i^1-D, 0)+\theta_i(q^1-\bar{p})D+\sum_2^l e_i^j q^j.$$

This simply means that if the i-th consumer uses the price $\theta_i q^1+(1-\theta_i)\bar{p}$ to evaluate the demand for the first good, he will maximize his utility by the same vector x_i^*. It can also be shown that

$$x_i^{1*} > D \Longrightarrow \theta_i = 1,$$

and

$$x_i^{1*} < D \Longrightarrow \theta_i = 0.$$

If $x_i^{1*} = D$, Then θ_i may be anywhere from 0 to 1.

We can now prove the following

Theorem 1 : *Every non-tirvial dual price equilibrium is Pareto inefficient.*

Proof : Suppose not. Then there exists a price vector $\bar{q}$ and an allocation $\bar{x}_i$, $i = 1, \ldots, n$ which is Pareto efficient and is also a non-trivial DPE. We know that, under our assumptions any Pareto efficient allocation can be obtained as a Walras equilibrium with a suitable redistribution of income. Let q^* be a price vector associated with such a Walras equilibrium. Necessary conditions for Walras equilibrium imply that, for any pair of consumers (i, k)

$$\frac{u_{i1}}{u_{ij}} = \frac{q^{*,1}}{q^{*,j}} = \frac{u_{k1}}{u_{kj}} \;\forall j = 2, \ldots, l \qquad \ldots \quad (2.1)$$

[u^{ij} stands for the partial derivative of u_i with respect to the j-th good. The context will make clear the point at which the derivative is taken.]

We have proved earlier that there exists a $\theta_i \in [0, 1]$ such that the consumer demand is unaltered if the dual price $(\bar{p}, \bar{q}^1)$ is replaced by $\theta_i\bar{q}^1+(1-\theta_i)\bar{p}$ and the current expenditure by $\bar{p}e_i^1+(\bar{q}^1-\bar{p})\max(e_i^1-D, 0)+\sum_{j=2}^{l}\bar{q}_j e_i^j+\theta_i D(\bar{q}^1-\bar{p})$. Thus the necessary conditions for the consumer's maximization problem imply that

$$\frac{u_{i1}}{u_{ij}} = \frac{\theta_i\bar{q}^1+(1-\theta_i)\bar{p}}{\bar{q}^j}, \quad i = 1, 2, \ldots, n\,; \quad j = 2, \ldots, l \qquad \ldots \quad (2.2)$$

By definition $x_i^{1*} > D$ for at least one consumer, say that k-th. Again, we have shown earlier that $\theta_k = 1$ for consumers buying positive amounts of the first good in the open market. Hence we have, from (2.1) and (2.2)

$\frac{q^{*,1}}{q^{*,j}} = \frac{u_{k1}}{u_{kj}} = \frac{\bar{q}^1}{\bar{q}^j} \Rightarrow q^{*,j} = c\bar{q}^j$ for some positive scalar 'c'. In view of the homogeneity of the demand functions for DPE we could choose 'c' such that $q^{*,j} = \bar{q}^j, \forall j$. Now consider any consumer $i \neq k$. We know that,

$$\frac{\bar{q}^1}{\bar{q}^j} = \frac{q^{*,1}}{q^{*,j}} = \frac{u_{i1}}{u_{ij}} = \frac{\theta_i q^1+(1-\theta_i)\bar{p}}{\bar{q}^j} \rightarrow \theta_i = 1.$$

But $\theta_i = 1$ only if $\bar{x}_i^1 \geqslant D$. In view of this, the budget constraint of any consumer can be written as

$$\bar{p}\bar{x}_i^1+(q^{*,1}-\bar{p})(\bar{x}_i^1-D)+\sum_2^l q^{*,j}\,\bar{x}_i^j = \bar{p}e_i^1+(q^{*,1}-\bar{p})\max(e_i^1-D,0)+\sum_2^l q^{*,j}e_i^j$$

$$\Longrightarrow \bar{p}(\bar{x}_i^1-e_i^1)+(q^{*,1}-\bar{p})\,\{\bar{x}_i^1-D-\max(e_i^1-D,0)\}+\sum_2 q^{*,j}(\bar{x}_i^j-e_i^j) = 0.$$

Summing over i and using the fact that all markets clear, we get

$$\sum_{i=1}^n(\bar{x}_i^1-D)-\sum_{i=1}^n\max(e_i^1-D,0) = 0.$$

Notice that the above equation cannot be satisfied if $e_i^1-D < 0$ for any i. Hence $e_i^1-D \geqslant 0$. The budget constraint then becomes

$$q^{*,1}\bar{x}_i^1+\sum_{j=2}^l q^{*,j}\,\bar{x}_i^j = q^{*,1}e_i^1+\sum_{j=2}^l q^{*,j}e_i^j$$

which shows that $\bar{x}_i$ can be purchased at price $q^* = \bar{q}$ without any redistribution of endowments. Hence $\bar{x}_i$, $i = 1, \ldots, n$ is a trivial DPE, contrary to our assumption. Q.E.D.

We hasten to add that the Pareto inefficiency of an allocation is not so serious that we should discard DPE allocations, especially in view of its desirable properties of distributing goods. Hence it will be interesting to investigate the conditions under which dual price equilibrium will exist. For this purpose, we will now introduce an assumption.

Need for dual pricing will arise only when the initial distribution of the first good is not even and the demand for this good (in a Walrasian set up) is high when its price is low. For example, there is no need for dual pricing of this good, if every one initially has at least D units of this good. Hence it is reasonable to assume that (henceforth we shall use ϕ and x interchangeably),

$$\sum_{i=1}^n[\max(D-e_i^1,0)-\max(D-\phi_i^1,0)] > 0 \qquad \ldots \quad (2.3)$$

for all q with $q^1 > \bar{p}$. For example, this assumption holds if $e_i^1 < D$ for at least one i, and $\phi_i^1 > D$ for all i, which will be the case when $\bar{p}$ and D are small. We can now prove the following

[for our purpose, this assumption could be weakened to $\sum_1^n [\max(D-e_i^1, 0) - \max(D-\phi_i^1, 0)] \neq 0$.]

Theorem 2: *If assumption* (2. 3) *holds, there does not exist any dual price equilibrium.*

Proof: Suppose not. Then there exists a price vector $(\bar{p}, q^1, q^2, \ldots, q^l)$ such that

$$\sum_{i=1}^{n} (\phi_i^j(\bar{p}, q, D) - e_i^j) = 0 \ \forall \ j. \qquad \ldots \quad (2.4)$$

From the definition of the budget set, we have,

$$\bar{p}(\phi_i^1 - e_i^1) + (q^1 - \bar{p}) \{\max[\phi_i^1 - D, 0] - \max[e_i^1 - D, 0]\} + \sum_{j=2}^{l} q^j(\phi_i^j - e_i^j) = 0,$$

Summing over all i, we have

$$\bar{p} \sum_{i=1}^{n} (\phi_i^1 - e_i^1) + (q^1 - \bar{p}) \sum_i [\max(\phi_i^1 - D, 0) - \max(e_i^1 - D, 0)] + \sum_i \sum_j q^j(\phi_i^j - e_i^j) = 0$$

Using (2.4) we have, since $q^1 > \bar{p}$,

$$\sum_{i=1}^{n} [\max(\phi_i^1 - D, 0) - \max(e_i^1 - D, 0)] = 0 \qquad \ldots \quad (2.5)$$

But $\qquad \max(\phi_i^1 - D, 0) = (\phi_i^1 - D) + \max(D - \phi_i^1, 0)$

and $\qquad \max(e_i^1 - D, 0) = (e_i^1 - D) + \max(D - e_i^1, 0)$.

Hence

$$\sum_{i=1}^{n} [\max(\phi_i^1 - D) - \max(e_i^1 - D, 0)]$$

$$= \sum_{i=1}^{n} \{(\phi_i^1 - D) - (e_i^1 - D) - [\max(D - e_i^1, 0) - \max(D - \phi_i^1, 0)]\}$$

$$= -\sum_{i=1}^{n} [\max(D - e_i^1, 0) - \max(D - \phi_i^1, 0)] < 0 \text{ by } 2.3.$$

But this contradicts (2.5). Q.E.D.

Let us now find out the reason for the non-existence of an equilibrium. This is more easily understood in the case of two consumers and two goods. In Figure 1, the good on the horizontal axis is taken as the numeraire and the other good is subject to dual pricing. The two prices of the second good are shown by the budget lines KLM and K'L'M'. The ration quota D is given by the length $OD = O'F$. Suppose the endowment distribution is 'e', below the line DG. Then

$$\max(D-e_A, 0)+\max(D-e_B, 0) = D-e_A > 0. \qquad \ldots \quad (2.6)$$

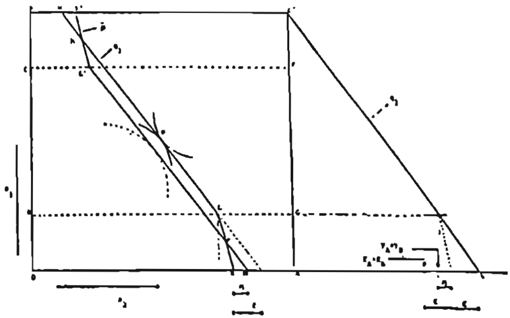

Figure 1. Dual Price Equilibrium : Non Existence

What is the total value, i.e. the total income of the two consumers ? With two prices for one good, there are several ways of defining the income. But we shall stick to our procedure. That is, for buying or selling, the second good is evaluated at price '$\bar{p}$' upto D units and then at price $q^1 > \bar{p}$. The total amount of the second good is $OB = O'A$. Its value at price q^1 is AQ. But we have to evaluate some units at price '$\bar{p}$'. For the consumer B (whose origin is O'), this involves a deduction of $(q^1-\bar{p})D = \epsilon$, since his initial endowment of this good is greater than D. For consumer 'A' who does not have D units of this good, we have to subtract $\eta < \epsilon$. Thus the total income is found to be the point shown in the figure as Y_A+Y_B.

Suppose the point at which both consumers maximize is 'P' (shown in the figure). First note that

$$\max(D-\phi_A^2, 0)+\max(D-\phi_B^2, 0) = 0. \qquad \ldots \quad (2.7)$$

From (2.6) and (2.7), we see that our assumption (2.3) is satisfied. In fact (2.3) is satisfied whenever the initial endowment point is outside the rectangle DCFG and the equilibrium point 'P' is inside this rectangle.

What is the expenditure at 'P' ? It is easy to check that this is obtained by subtracting $2\varepsilon = 2D(q^1-\bar{p})$ from AQ. This is shown as E_A+E_B in the figure. Thus $E_A+E_B < Y_A+Y_B$, contradicting Walras law.

Alternatively, we can prove the non-existence of an equilibrium as follows. Only those allocations which are on the indifference curves for both consumers having a common (tangent) budget line can be equilibrium allocations. For any price $q^1 > \bar{p}$, the only feasible allocation (i.e. a point common to both budget sets) is the initial endowment. But, since the budget lines are different this point cannot be on both indifference curves, having a common budget line.

The source of the difficulty is the following : The trader who has more than D units of the good subject to dual pricing values at price 'q^1' at the margin, while the buyer, who has less than D, values and pays '$\bar{p}$' $< q^1$. Hence there is a loss of $(q^1-\bar{p})D$ by this transaction. No individual will voluntarily provide a subsidy to another.

The problem, as we noted earlier, is due to different valuations by buyers and sellers at the margin.

If we allow for resale and if each consumer can always sell any amount (possibly more than the endowment) of good 1 and buys at most D at $\bar{p}$ and more at q^1 then each agent will always buy D at $\bar{p}$ and make a profit of $(q^1-\bar{p})D$. However, we shall assume that resale is not allowed since the dual price system does not work with this.

Suppose, therefore, that the government (which is the $(n+1)$-th agent) operates the public distribution system by buying the first good in the free market and selling it at a controlled price to the consumers. For this purpose it purchases nD units of this good and provides a quota of D units per head. But since some consumers may not utilize the full quota it effectively supplies an amount 'Q' $= \sum_{i=1}^{n} \min (\phi_i^1, D)$ through the ration shops. For simplicity assume that $(nD-Q)$ is kept as public stock.

Assume that the government finances its purchase of nD units of the rationed good by taxing free market consumption. That is, the owners of endowment receive prices $(p^1, \ldots, p^l)$ for their sales whereas the consumers pay $(q^1, \ldots, q^l)$ for their purchases in the free market and $\bar{p}$ for the purchase of the ration quota. The tax vector $t' = (t^1, \ldots, t^{l-1}, 0)$ is such that $q^j = p^j\ (1+t^j)\ \forall\, j = 1, \ldots, l$.

A typical consumer's budget constraint is then given by

$$\bar{p}x_i^1+(q^1-\bar{p})\max(x_i^1-D, 0)+\sum_{j=2}^{l} q^j x_i^j = \sum_{j=1}^{l} p^j e_i^j.$$

[The i-th consumer's budget and demand correspondences are modified accordingly.]

The government's budget restriction is

$$p^1 t^1 \sum_{i=1}^{n} \max(x_i^1-D, 0)+\sum_{j=2}^{l}\sum_{i=1}^{n} p^j t^j x_i^j+\bar{p}\sum_{i=1}^{n} \min(x_i^1, D) \geqslant p^1 nD.$$

Its demand for (stock of) good 1 is

$$x_{n+1}^1 = nD-\sum_{1}^{n} \min(x_i^1, D).$$

Summing the budget constraints of all the consumers and the government we have

$$p^1 \sum_{i=1}^{n+1} x_i^1+\sum_{j=2}^{l}\sum_{i=1}^{n} p^j x_i^j \leqslant \sum_{1}^{l}\sum_{1}^{n} p^j e_i^j$$

or

$$\sum_{1}^{l}\sum_{1}^{n+1} p^j(x_i^j-e_i^j) \leqslant 0 \text{ (for, } x_{n+1}^j = 0 \ \forall j \neq 1).$$

Given a vector $(\bar{p}, p^1, \ldots, p^l)$ of nominal sellers' prices, let us denote the vector of relative prices by $(\bar{P}, P^1, \ldots, P^l) \in \Delta^{l+1}$, the $(l+1)$ dimensional unit simplex. Note that with our specifications if demand functions are homogeneous of degree zero in $(\bar{p}, p^1, \ldots, p^l)$ then they are homogeneous of degree zero in $(\bar{p}, q^1, \ldots, q^l)$ also where $q^j = p^j(1+t^j) \ \forall j = 1, \ldots, l$. Hence, given the budget and demand correspondences in terms of $(\bar{p}, p^1, \ldots, p^l)$ they can be easily reduced to functions of $P' = (\bar{P}, P^1, \ldots, P^l) \in \Delta^{l+1}$. Sum of the budget constraints can therefore be stated in terms of P as

$$\sum_{1}^{l}\sum_{1}^{n+1} P^j(x_i^j-e_i^j) \leqslant 0 \text{ (if } p^j > 0 \text{ for at least one } j)$$

or

$$P'.z \leqslant 0$$

where $z'=(z^1, \ldots, z^l)$ is the vector of excess demands with $z^j=\sum_{i=1}^{n+1} z_i^j \ \forall j=1,\ldots,l$.

Definition 2 : An exchange economy is said to be in a dual price equilibrium with taxes (DPET) if, for given $t' = (t^1, \ldots, t^l) > 0$ with $t^l = 0$, [the notations $\geqslant$ and $\gg$ are used to denote weak and strong vector inequalities respectively. $>$ means the vector has at least one positive element.] $\exists$ a vector of prices $P' \in \Delta^{l+1}$ and a quota D per head of the first good together with a set of allocations $\{x_i^*, i \in A\}$ satisfying

(i) $z^{j*} = 0 \ \forall j$

(ii) $x_i^* \in \phi_i(p, D)$

(iii) $q^1 > \bar{p}$ and $x^{1*} > D$ for at least one i.

$[if\ x_i^1]^* < D\ \forall\ i$ then the economy is said to be in an equilibrium with rationing.]

To prove the existence of an equilibrium we make the following assumptions: $\forall\ i,\ X_i \subset \mathcal{R}^l_+$ is closed, convex and $0 \in X_i$. X_i has local nonsatiation property, i.e., for every $x \in X_i$ and for every neighbourhood V of x in $X_i \exists\ y \in V$ such that $y \succsim_i x$. Let X_i be such that $x+\mathcal{R}^l_+ \subset X_i$. ... (2.8)

$\succsim_i$ is continuous and strictly convex, i.e., $x \sim y \Longrightarrow \lambda x+(1-\lambda)y \succ x$ for $\lambda \in (0, 1)$. ... (2.9)

$e_i \in X_i$ and $e_i > 0\ \forall\ i$. (We do not consider goods 'j' for which $\sum_i e_i^j = 0$). Note that $0 \in X_i$ and $e_i > 0$ imply $\inf_{x \in X_i} E_{p,D}(x) < \sum_{j=1}^{l} p_j e_i^j$ for $p \gg 0$, where $E_{p,D}(x) = \bar{p}x^1+(q^1-\bar{p})\ \max(x^1-D, 0)+ \sum_{2}^{l} q^j x^j$. Note also that demand functions are homogeneous of degree zero in $p' = (\bar{p}, p^1, ..., p^l)$. ... (2.10)

For any $(P'_m, D_m) = (\bar{P}_m, P^1_m, ..., P^l_m, D_m) \in \Delta^{l+1} \times \mathcal{R}_+, P^h_m \to 0$ for some $h = 1, ..., l \Longrightarrow \lim_{m \to \infty} \inf \|\xi(P_m, D_m)\| = +\infty$. ... (2.11)

Lemma 1: *If $\beta_i(p, D)$ is compact then the set of maximal elements ϕ_i (p, D) in $\beta_i(p, D)$ is non-empty. If β_i is continuous at (p, D) then ϕ_i is upper hemi continuous[1] at (p, D).*

[See Theorem A. III. 3 in Hildenbrand and Kirman (1970) for a proof of Lemma 1.]

Lemma 2: *The correspondence β_i (p, D) is lower hemi continuous[2] at any (p, D), $P \in \Delta^{l+1}$ with $p = (1/P_l)P$ if $\inf_{x \in X_i} E_{p,D}(x) < \sum_{j=1}^{l} p^j e_i^j$. If $p \gg 0$ then β_i is continuous at (p, D).*

See the appendix for a proof of this lemma.

Theorem 3: *If assumptions (2.8) to (2.11) hold there exists an equilibrium with rationing.*

[1]Let S and T be subsets of metric spaces $\mathcal{R}^m$ and $\mathcal{R}^n$ respectively. Let ψ be a correspondence (a point-to-set mapping) from S into T. Then the correspondence ψ is said to be *upper hemi-continuous* (uhc) at $s \in S$ if, for every neighbourhood M of ψ (s), $\exists$ a neighbourhood O_s of s such that $x \in O_s \Longrightarrow \psi(x) \subset M$. ψ is said to be u.h.c if it is u.h.c at every $s \in S$.

[2]A correspondence $\psi : S \to T$ is said to be *lower hemi-continuous* (lhc) at $s \in S$ if, for every open set u with $\psi(s) \cap u \neq \phi$, $\exists$ a neighbourhood v of s such that $\psi(s') \cap u \neq \phi\ \forall\ s' \in v$. ψ is said to be lhc if it is lhc at every $s \in S$.

Proof: It is easy to prove (see Lemmas 1 and 2) that $\phi_i(p, D)$ is non-empty and continuous at any $(p', D) = (\bar{p}, p^1, \ldots, p^l, D)$ with $p' \gg 0$ given (2.8) to (2.10).

For calculating the equilibrium quota of the rationed good define the function

$$g(p, D) = \frac{1}{np^1}\left[p^1 t^1 \sum_1^n \max(x_i^1 - D, 0) + \sum_2^l \sum_1^n p^j t^j x_i^j + \bar{p}\sum_1^n \min(x_i^1, D)\right].$$

For obtaining the ration price $\bar{P}$ of the controlled commodity assume that it is a fraction $0 < \delta < 1$ of the corresponding free market price P^1 for the sellors.

Define

$$S_m = \left\{P \in R^{l+1} \,\middle|\, \bar{P} + \sum_1^l P^i = 1,\ \bar{P} = \delta P^1 \text{ and } P^h \geq \frac{1}{(1+\delta)m}\ \forall\ h,\ 0 < \delta < 1\right\}.$$

For $m > l$, define the map $(P_m, D_m) : S_m \times \left[0, \frac{e^1+\epsilon}{n}\right] \to S_m \times \left[0, \frac{e^1+\epsilon}{n}\right]$ by

$$P_m^h \to \frac{1}{m} + \left(1 - \frac{l}{m}\right)\frac{P_m^h + \max(z^h(P_m, D_m), 0)}{1 + \sum_1^l \max(z^j(P_m, D_m), 0)} \quad \forall\ h = 2, \ldots, l$$

$$P_m^1 \to \frac{1}{(1+\delta)m} + \left(1 - \frac{l}{m}\right)\frac{P_m^1 + \max(z^1(P_m, D_m), 0)/(1+\delta)}{1 + \sum_1^l \max(z^j(P_m, D_m), 0)}$$

$$\bar{P}_m = \delta P_m^1 \text{ and } D_m \to \min\left[g(P_m, D_m),\ \frac{e^1+\epsilon/2}{n}\right]$$

where $e^1 = \sum_1^n e_i^1$ and $0 < \epsilon < 1$. Note that $S_m \times \left[0, \frac{e^1+\epsilon}{n}\right]$ is non-empty, compact and convex and this map from $S_m \times \left[0, \frac{e^1+\epsilon}{n}\right]$ into itself is continuous. Hence, by Brouwer's fixed point theorem, we have a fixed point, say (P_m^*, D_m^*). Since (P_m^*, D_m^*) is a bounded sequence, let $(P_m^*, D_m^*) \to (P^*, D^*)$. We claim $(P^*, D^*) \gg 0$. We will first show that $P^* \gg 0$.

Suppose not. Let $P_m^{h*} \to 0$ for some $h = 1, \ldots, l$. Using (2.11) we then have $z^k(P_m^*, D_m^*) \to \infty$ for some $k = 1, \ldots, l$. However, since $P_m'. Z \leq 0$, we must have $z^r(P_m^*, D_m^*) \to -\infty$ with $P_m^{r*} > 0$ for some r, for large m. But this contradicts (2.8) since X_i is bounded below for all i.

Hence we must have $P^{*} \gg 0$. In fact, in the limit, a fixed point can occur only if $z^{h} \leqslant 0 \;\forall\; h$.

Now, if $z(P^{*}, D^{*})) \leqslant 0$, $(e^{1}+\varepsilon/2)/n^{*}$ cannot be fixed point for the map for ration quota. For, if $g(P^{*}, D^{*}) > (e^{1}+\varepsilon/2)/n$ then $D^{*} \rightarrow (e^{1}+\varepsilon/2)/n$ and demand in the first market becomes

$$\sum_{i=1}^{n+1} x_i^1 = nD^{*} + \sum_{i=1}^{n} \{x_i^1 - \min(x_i^1, D^{*})\}$$

$$= e^{1}+\varepsilon/2 + \sum_{i=1}^{n} \max(x_i^1 - D^{*}, 0) > e^{1},$$

the supply on the first market.

Note that $g(P, D) \geqslant 0 \;\forall\; P, D$ and hence for P^{*}, D^{*}. We have also already seen that $D^{*} \leqslant e^{1}/n$ which means $g(P^{*}, D^{*}) \leqslant e^{1}/n$. For if $e^{1}/n < g(P^{*}, D^{*}) \leqslant (e^{1}+\varepsilon/2)/n$ then $D^{*} \rightarrow g(P^{*}, D^{*})$ which violates $z(P^{*}, D^{*}) \leqslant 0$ using an earlier argument. Thus, $0 \leqslant g(P^{*}, D^{*}) \leqslant e^{1}/n$ which means the government's budget is balanced and hence Walras' law holds with equality.

By standard arguments, we can then show using $z(P^{*}, D^{*}) \leqslant 0$ together with Walras' law, $P^{*\prime}z = 0$ and $P^{*} \gg 0$ that $z^{h}(P^{*}, D^{*}) = 0 \;\forall\; h = 1, \ldots, l$.

Next note that zero cannot be fixed point for the map for D since it violates Walras' law by violating government's budget constraint. For, if $g(P^{*}, D^{*}) = 0$, then $D^{*} = 0$ and the government's expenditure is zero whereas revenue is positive since $P^{1*} \; nD = 0$ and $P^{1*}t^{1} \sum_{1}^{n} \max(x_i^1 - D, 0) + \sum_{2}^{l} \sum_{1}^{n} P^{j*}t^{j}x_i^j > 0$ because $t > 0$, $P^{*} \gg 0$ and $x_i^j > 0 \;\forall j$ for at least one i (since $z^{h}(p^{*}, D^{*}) = 0 \;\forall\; h$ and $e^{j} > 0 \;\forall\; j$). That is, $0 < D^{*} \leqslant e^{1}/n$. Q.E.D.

From the above theorem, it is clear that we cannot rule out D^{*} being equal to e^{1}/n, i.e., the ration quota may exhaust all the supplies and the open market may not function. Hence, it will be useful to find some conditions under which a DPET exists. In fact, it is interesting to note that there is a vector of taxes such that a DPET exists.

Since $\sum_{i} e_i^j > 0 \;\forall\; j$, we can show that at every convex exchange economy, $\varepsilon = ((X_i, \succ_i, e_i), t)$, with taxes the equilibrium price correspondence is u.h.c (see 2.2. Proposition 4 and B Theorem 1 in Hildenbrand, 1974). Also, since D is a continuous function of x, using Lemmas 1 and 2, and the Corollary to B Proposition 1 in Hildenbrand (1974) we know that D is u.h.c at every ε. Hence, if we take a sequence $t_m \rightarrow 0$ then we know that $D(t_m) \rightarrow 0$ with $D(0) = 0$. Thus, starting with a tax vector $t \gg 0$ if, in equilibrium, $D^{*} = e^{1}/n$ then by reducing t slightly we can reduce the equilibrium quota such that $D^{*} < e^{1}/n$ at the new equilibrium.

*In fact $(e^{1} + \varepsilon/2^{r})/n$ cannot be a fixed point $\forall\; r = 1, 2, \ldots$.

Appendix

Lemma 2: *The correspondence $\beta(p, D)$ is l.h.c at any $(P, D) \epsilon \Delta^{l+1} \times \mathcal{R}$, with $p = (1/P^1)P$ if $\inf_{x \epsilon X} E_{p,D}(x) < \sum_1^l p^j e^j$ where $E_{p,D}(x) = \bar{p}x^1 + (q^1 - \bar{p}) \max (x^1 - D, 0) + \sum_2^l q^j x^j$. If $p \gg 0$ then β is continuous at (p, D).*

Proof: Consider any sequence $(p_n, D_n) \rightarrow (p, D)$ and $y \epsilon \beta(p, D)$. We have to prove that $\exists\, y_n \epsilon \beta(p_n, D_n)$ with $y_n \rightarrow y$. (See theorem A III. 2 in Hildenbrand and Kirman (1976)). Note that $E_{p,D}(y) \leqslant \sum_1^l p^j e^j$.

Case (i): $E_{p,D}(y) < \sum_1^l p^j e^j$.

Since $(p_n, D_n) \rightarrow (p, D)$ and $E_{p,D}(y) - \sum_1^l p^j e^j$ is a continuous function of (p, D), for large n, say $n > N$, $E_{p_n, D_n}(y) - \sum_1^l p_n^j e^j < 0$. Choose the sequence y_n as

$$y_n = \begin{cases} y \;\forall\, n > N \\ \text{any point of } \beta(p_n, D_n) \text{ for } n < N. \end{cases}$$

Case (ii): $E_{p,D}(y) = \sum_1^l p^j e^j$.

Since $\inf_{x \epsilon X} E_{p,D}(x) < \sum_1^l p^j e^j \;\exists\, z \epsilon X$ such that $E_{p,D}(z) < \sum_1^l p^j e^j$. Choose $\lambda_n \epsilon [0, 1]$ such that $\lambda_n \downarrow 0$ and $E_{p_n, D_n}[\lambda_n z + (1 - \lambda_n)y] < \sum_1^l p_n^j e^j$ for large n. Hence construct the sequence y_n as follows:

$$y_n = \begin{cases} \lambda_n z + (1 - \lambda_n)y \;\forall\, n > N \\ \text{any point of } \beta(p_n, D_n) \;\forall\, n < N. \end{cases}$$

Note that N is chosen such that $E_{p_n, D_n}(z) < \sum_1^l p_n^j e^j \;\forall\, n > N$.

We now have to prove that β is l.h.c at (p, D) with $p \gg 0$. Note first that $\beta(p, D)$ is bounded. Also $\beta(p, D)$ is closed.

Consider the sequence $(p_n, D_n) \rightarrow (p, D)$ and $y_n \epsilon \beta(p_n, D_n)$. Since β is compact valued it is enough to prove that there is a subsequence $y_{n_q} \rightarrow y \epsilon \beta(p, D)$. (See Theorem A III. 1 in Hildenbrand and Kirman, 1976). Also since β is closed, it is enough to prove that the sequence $\{y_n\}$ is bounded. Suppose not. Since β is l.h.c and $y \epsilon \beta(p, D)$, $\exists\, z_n \epsilon \beta(p_n, D_n)$ with $z_n \rightarrow y$. Since $\beta(p, D)$ is compact, consider an ϵ-sphere around $\beta(p, D)$. Since $z_n \rightarrow y$, for large n, $z_n \epsilon \beta(p, D)$. Define

$$v_n = \mu_n z_n + (1 - \mu_n) y_n$$

by choosing μ_n such that the distance $d(v_n, \beta(p, D)) = \varepsilon \;\forall\; n$. Since β is convex valued, $v_n \in \beta(p_n, D_n)$. But β is closed and therefore there is a subsequence $v_{n_q} \to v \in \beta(p, D)$. That is, $\exists\; N$ such that $\forall\; n_q > N$, $v_{n_q} \in \beta(p, D)$, i.e., $d(v_{n_q}, \beta(p, D)) = 0$—a contradiction. Hence $\{y_n\}$ is bounded. Q.E.D.

Acknowledgment. The authors are very thankful to an anonymous referee for several fruitful suggestions and comments on earlier versions of the paper.

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Paper received: *February*, 1986.

Revised: *February*, 1988.