Stochastic field theory, holomorphic quantum mechanics, and supersymmetry

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Holomorphic quantum mechanics are studied from the point of view of stochastic quantization in Minkowski space which involves the introduction of two stochastic fields, one in the external space and the other in the internal space. The equilibrium condition is given by Z_2 symmetry between the external and internal fields. In the nonequilibrium case, $N=2$ Wess–Zumino quantum fields are arrived at giving rise to supersymmetry. This helps to define the supercharge operator Q when the Hamiltonian is given by $H=Q^2$ and an index theorem is derived for an interacting case when the superpotential is given by $V(\phi)=\lambda\phi^n$, ϕ being complex with $n>2$. It is found that the vacuum is degenerate and is in conformity with the result obtained by Jaffe, Lesniewski, and Lewenstein [Ann. Phys. **178**, 313 (1987)] in the two-dimensional $N=2$ Wess–Zumino quantum field model.

I. INTRODUCTION

Recently Jaffe, Lesniewski, and Lewenstein[1] have considered the ground state structure of the two-dimensional $N=1$ and $N=2$ Wess–Zumino quantum field models and have pointed out that the $N=2$ quantum mechanics has degenerate vacua. The space of vacuum states is found to be bosonic and its dimension is determined by the topological properties of the superpotential. The physical interpretation of $N=2$ Wess–Zumino quantum mechanics has been discussed and the feasibility of realizing holomorphic quantum mechanics has been pointed out with special reference to a spin $\frac{1}{2}$ particle in an external SU(2) gauge field and in the study of nuclear matter interacting with a pion condensate. Here we shall show that holomorphic quantum mechanics is realized in stochastic field theory also when stochastic quantization is achieved in Minkowski space, introducing a doublet of fields corresponding to the fields in the external and internal space. This can also be generalized to finite temperature when the formalism of thermofield dynamics is utilized identifying the internal field with the fictitious tilde field introduced by Takahashi and Umezawa.[2] We shall study here the supersymmetric properties of such fields when the equilibrium condition of Z_2 symmetry between the external and internal field is destroyed and shall show that we can uniquely define a supercharge for such a system. The index theorem for such a system representing holomorphic quantum mechanics is then discussed and it is found that the space of vacuum states has its dimension determined by the topological properties of the superpotential in conformity with the results obtained by Jaffe, Lesniewski, and Lewenstein[1] in the two-dimensional Wess–Zumino quantum field model.

In a recent paper,[3] it has been pointed out that stochastic quantization in Minkowski space as well as its generalization at finite temperature leading to the formalism of thermofield dynamics necessitates the introduction of a doublet of stochastic fields. This do[illegible] interpreted as comprising two fields, one corresponding to the field in the external space and the other corresponding to the field in the internal space. This internal field is also necessary to have a relativistic generalization of Nelson's formalism of stochastic quantization and the quantization of a Fermi field.[4] The equilibrium condition for such a doublet of stochastic fields is given by the Z_2 symmetry corresponding to the time reversal symmetry of the two fields. The nonequilibrium condition gives rise to supersymmetric quantum mechanics.

Here we shall point out that the doublet of stochastic fields may be taken to give rise to holmorphic quantum mechanics in four dimensions and the break down of Z_2 symmetry gives rise to the $N=2$ Wess–Zumino quantum field model. Moreover it is found that the two-dimensional result of Jaffe *et al.* regarding the degeneracy of the ground state except when the superpotential is quadratic is also valid here and the index $i(Q_t)$ is found to be identical with the degree of ∂V, where V is a polynomial of degree $n>2$.

In Sec. II we shall recapitulate the main features of stochastic quantization in Minkowski space and its generalization to finite temperature utilizing the formalism of thermofield dynamics. In Sec. III we shall formulate holomorphic quantum mechanics from stochastic field theory and shall derive supercharge for such a system. In Sec. IV we shall discuss the index theorem.

II. STOCHASTIC QUANTIZATION IN MINKOWSKI SPACE

Nelson's stochastic quantization procedure is based on the assumption that the configuration variable $q(t)$ is promoted to a Markov process $q(t)$.[5] The process $q(t)$ is determined by two conditions; the first is the hypothesis of universal Brownian motion and the second is the validity of the Euler–Lagrange equation. In a recent paper,[4] it has been shown that in Nelson's formalism, the relativistic generalization as well as the quantization of a Fermi field is achieved

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when an anisotropy in the internal space of a particle is introduced and it is taken into account that there are universal Brownian motions both in the external and internal space. In this formalism, the configuration variables are denoted as $Q(t,\xi_0)$, where ξ_0 is the fourth component (real) of the internal four-vector ξ_μ which is considered to be the attached vector to the space-time point x_μ. We assume that $Q(t,\xi_0)$ is a separable function and can be denoted as

$$Q(t,\xi_0) = q(t)q(\xi_0). \tag{1}$$

The process $Q(t,\xi_0)$ is assumed to satisfy the stochastic differential equations,

$$dQ_i(t,\xi_0) = b_i(Q(t,\xi_0),t,\xi_0)dt + dw_i(t), \tag{2}$$

$$dQ_i(t,\xi_0) = b'_i(Q(t,\xi_0),t,\xi_0)d\xi_0 + dw_i(\xi_0), \tag{3}$$

where $b_i(Q(t,\xi_0),t,\xi_0)$ and $b'_i(Q(t,\xi_0),t,\xi_0)$ correspond to certain velocity fields and dw_i are independent Brownian motions. It is assumed that $dw_i(t)(dw_i(\xi_0))$ does not depend on $Q(s,s')$ for $s<t$ $(s'<\xi_0)$ and the expectations have the following values at $T=0$ and $T\neq 0$:

$$\langle dw_i(t)\rangle_{T=0} = 0, \tag{4}$$

$$\langle dw_i(t)dw_j(t')\rangle_{T=0} = (\hbar/m)\delta_{ij}\delta(t-t')dt\,dt', \tag{5}$$

$$\langle dw_i(\xi_0)\rangle_{T=0} = 0, \tag{6}$$

$$\langle dw_i(\xi_0)dw_j(\xi'_0)\rangle_{T=0} = (\hbar/\pi^0)\delta_{ij}\delta(\xi_0-\xi'_0)d\xi_0\,d\xi'_0, \tag{7}$$

$$\langle dw_i(t)\rangle_{T\neq 0} = 0, \tag{8}$$

$$\langle dw_i(t)dw_j(t')\rangle_{T\neq 0} = \frac{\delta_{ij}}{\beta m}\sum_{n=\alpha}^{-\alpha} e^{iw_n(t-t')}dt\,dt', \tag{9}$$

$$\langle dw_i(\xi_0)\rangle_{T\neq 0} = 0, \tag{10}$$

$$\langle dw_i(\xi_0)dw_j(\xi'_0)\rangle_{T\neq 0} = \frac{\delta_{ij}}{\beta\pi^0}\sum_{n=-\alpha}^{\alpha} e^{iw_n(\xi_0-\xi'_0)}d\xi_0\,d\xi'_0, \tag{11}$$

with $w_n = 2\pi n/\beta\hbar$.

It is easily seen that in the limit $\beta\to\alpha$ Eqs. (9) and (11) give Eqs. (5) and (7), respectively. The form of Eqs. (9) and (11) is dictated by the KMS condition. To make the description time symmetrical in both "external" and "internal" time we also write

$$dQ_i(t,\xi_0) = b^*_i(Q(t,\xi_0),t,\xi_0)dt + dw^*_i(t), \tag{12}$$

$$dQ_i(t,\xi_0) = b'^*_i(Q(t,\xi_0),t,\xi_0)d\xi_0 + dw^*_i(\xi_0), \tag{13}$$

where dw^* has the same properties as dw except that $dw^*_i(t)(dw^*_i(\xi_0))$ are independent of $Q(s,s')$ with $s>t$ $(s'>\xi_0)$.

From the stochastic differential equations considered here, the following moments can be derived.

$$\langle Q_i(t,\xi_0)\rangle_{T=0} = \langle Q_i(t,\xi_0)\rangle_{T\neq 0} = 0, \tag{14}$$

$$\langle Q_i(t,\xi_0)Q_j(t',\xi'_0)\rangle_{T=0} = \frac{\hbar}{2mw}\frac{\hbar}{2\pi^0 w'}\delta_{ij}e^{-w(t-t')}e^{-w'(\xi_0-\xi'_0)} \quad (t>t',\xi_0>\xi'_0). \tag{15}$$

$$\langle Q_i(t,\xi_0)Q_j(t',\xi'_0)\rangle_{T\neq 0} = \frac{\delta_{ij}}{\beta m}\sum_{n=-\alpha}^{\alpha}\frac{e^{iw_n(t-t')}}{w^2+w_n^2}\times\frac{1}{\beta\pi^0}\sum_{n=-\alpha}^{\alpha}\frac{e^{iw_n(\xi_0-\xi'_0)}}{w'^2+w_n^2}, \tag{16}$$

with $w_n = 2\pi n/\beta\hbar$.

This follows from the fact that $Q_i(t,\xi_0)$ can be written in a separable way $q_i(t)q_i(\xi_0)$ and we can utilize the results for the moments of $q_i(t)$ as derived by Moore[6]

$$\langle q_i(t)\rangle_{T=0} = \langle q_i(t)\rangle_{T\neq 0} = 0, \tag{17}$$

$$\langle q_i(t)q_i(t')\rangle_{T=0} = \frac{\hbar}{2mw}\delta_{ij}e^{-w(t-t')} \quad (t>t'), \tag{18}$$

$$\langle q_i(t)q_i(t')\rangle_{T\neq 0} = \frac{\delta_{ij}}{\beta m}\sum_{n=-\alpha}^{\alpha}\frac{e^{iw_n(t-t')}}{w^2+w_n^2}. \tag{19}$$

These results can be extended to the variable $q_i(\xi_0)$ in an analogous way:

$$\langle q_i(\xi_0)\rangle_{T=0} = \langle q_i(\xi_0)\rangle_{T\neq 0} = 0, \tag{20}$$

$$\langle q_i(\xi_0)q_j(\xi'_0)\rangle_{T=0} = \frac{\hbar}{2\pi^0 w'}\delta_{ij}e^{-w'(\xi_0-\xi'_0)}(\xi_0>\xi'_0), \tag{21}$$

$$\langle q_i(\xi_0)q_j(\xi'_0)\rangle_{T\neq 0} = \frac{\delta_{ij}}{B\pi^0}\sum_{n=-\alpha}^{\alpha}\frac{e^{iw_n(t-t')}}{w'^2+w_n^2}. \tag{22}$$

Let $\{e_i(\mathbf{x})\}$ denote the complete orthonormal set of eigenfunctions of the three-dimensional Laplacian $-\Delta$:

$$\Delta e_i(\mathbf{x}) = -\mathbf{k}_i^2 e_i(\mathbf{x}). \tag{23}$$

Also we denote $\{e_j(\boldsymbol{\xi})\}$ as the set of complete orthonormal set of eigenfunctions of the three-dimensional Laplacian $-\Delta'$ in terms of the variables ξ_i

$$\left(\Delta' = \frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2} + \frac{\partial^2}{\partial\xi_3^2}\right)$$

so that

$$\Delta' e_j(\boldsymbol{\xi}) = -\boldsymbol{\pi}_j^2 e_j(\boldsymbol{\xi}). \tag{24}$$

Now we can construct a stochastic field ϕ which can be expressed as an orthonormal expansion in terms of $q_i(t), e_i(\mathbf{x}), q_j(\xi_0), e_j(\boldsymbol{\xi})$ and write

$$\phi(x,t,\xi) = \sum_{ij} q_i(t)e_i(\mathbf{x})q_j(\xi_0)e_j(\boldsymbol{\xi}). \tag{25}$$

Now from the moments of $q_i(t), q_j(\xi_0)$ we can determine the moments of $\phi(x,t,\xi)$,

$$\langle\phi(x,t,\xi)\rangle_{T=0} = \langle\phi(x,t,\xi)\rangle_{T=0} \neq 0, \tag{26}$$

$$\langle\phi(x,t,\xi)\phi(x',t',\xi')\rangle = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}g(t-t') \otimes \frac{1}{(2\pi)^3}\int d^3\boldsymbol{\pi}\, e^{i\boldsymbol{\pi}(\boldsymbol{\xi}-\boldsymbol{\xi}')}g(\xi_0-\xi'_0), \tag{27}$$

where $g(t-t')$ and $g(\xi_0-\xi'_0)$ are given by Eqs. (18) and (21) for $T=0$ and by Eqs. (19) and (22) for $T\neq 0$. Substituting these relations, we find

$$\langle \phi(x,t,\xi)\phi(x',t',\xi')\rangle_{T=0} = \frac{1}{(2\pi)^4}\int \frac{d^4x\, e^{i(k,(x-x'))}}{(k,k)+m^2} \times \frac{1}{(2\pi)^4}\int \frac{d^4\pi\, e^{i(\pi,(\xi-\xi'))}}{(\pi,\pi)+\pi'^2}. \tag{28}$$

Here (A,B) denotes a Euclidean product and the units have been chosen to be $\hbar = m = \pi^0 = 1$. In the case of $T \neq 0$, we find

$$\langle \phi(x,t,\xi)\phi(x',t',\xi')\rangle_{T\neq 0} = \frac{1}{(2\pi)^3}\frac{1}{\beta m}\int d^3\mathbf{k}\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \sum_{n=-\infty}^{\infty} \frac{e^{iw_n(t-t')}}{w^2+w_n^2} \times \frac{1}{(2\pi)^3}\frac{1}{\beta\pi^0}\int d^3\pi\, e^{i\pi(\xi-\xi')} \sum_{n=-\infty}^{\infty} \frac{e^{iw_n(\xi_0-\xi'_0)}}{w_1{}^2+w_n^2}. \tag{29}$$

Now for a particular mode $n=1$, we find that the expression becomes

$$\frac{1}{(2\pi)^3}\frac{1}{\beta m}\int d^3\mathbf{k}\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \frac{e^{ik_0(t-t')}}{k_0^2+w^2}\,\delta(k_0-w_1)\,dk_0 \times \frac{1}{(2\pi)^3}\frac{1}{\beta\pi^0}\int d^3\pi\, e^{i\pi(\xi-\xi')} \times \frac{e^{i\pi_0(\xi_0-\xi'_0)}}{\pi_0^2+w'^2}\,\delta(\pi_0-w_1)\,d\pi_0$$
$$= \frac{1}{\beta m}\frac{1}{(2\pi)^4}\int \frac{d^4k\, e^{i(k,(x-x'))}}{(k,k)+m^2}\, 2\pi\delta(k_0-w_1) \times \frac{1}{\beta\pi^0}\frac{1}{(2\pi)^4}\int \frac{d^4\pi\, e^{i(\pi,(\xi-\xi'))}}{(\pi,\pi)+\pi'^2}\, 2\pi\delta(\pi_0-w_1), \tag{30}$$

where π^0 corresponds to the quantity in the internal space analogous to the mass of the system.

Now from the relations (21) and (27) we note that for $\xi_0 = \xi'_0 = 0$ and integrating over the internal space variable ξ the correlation function just reduces to that of the scalar field in Euclidean space,

$$\langle \phi(x,t)\phi(x',t')\rangle_{T=0} = \frac{1}{(2\pi)^4}\int \frac{d^4k\, e^{i(k,(x-x'))}}{(k,k)+m^2}. \tag{31}$$

In a similar way for $T\neq 0$, we find from (22) and (27) considering one particular mode $n=1$,

$$\langle \phi(x,t)\phi(x',t')\rangle_{T=0} = \frac{1}{(2\pi)^4}\int \frac{d^4k\, e^{i(k,(x-x'))}}{(k,k)+m^2}\, 2\pi\delta(k_0-w_1) \tag{32}$$

normalizing $\beta = \pi^0 = m = 1$.

This is the Euclidean Markov field result which has been obtained from Nelson's real time formalism of Brownian motion and in this sense gives rise to the equivalence of these two formalisms as advocated by Guerra and Ruggiro.[7]

Now if we introduce an anisotropic feature of the internal space-time corresponding to the variable ξ_μ, we can obtain the fermionic propagator in Euclidean space-time. To this end, we introduce the anisotropy by having two opposite orientations of the internal variable ξ_μ (and hence of $\pi_\mu = i\delta/\delta\xi_\mu$) and take into account that each orientation denotes a separate field and the two opposite orientations depict two separate fields having two internal helicities corresponding to particle and antiparticle configurations. From Eq. (28), we note that it is effectively a correlation function in eight-dimensional space-time, four dimensional in the external space-time variable and four dimensional in the internal space-time variable. To make it an effective four-dimensional expression in the external space-time variable we may take into account that $k(x)$ is an implicit function of $\pi(\xi)$. For simplicity and dimensional reasons we take the form $k^2 = (k',\pi)$, $m^2 = m'\pi^0$, where (k',π) is the Euclidean product and each component of k is given by $k_i = \sqrt{k'_i\pi_i}$. So from the new field variable $\bar{\phi}(x,t,\xi)$ where this mapping is taken into account, we find from Eq. (31) the correlation function for $T=0$,

$$\langle \bar{\phi}(x,t,\xi)\bar{\phi}(x',t',\xi')\rangle_{T=0} = \int \frac{e^{i(\sqrt{k'}\pi)(x(\xi)-x'(\xi'))}}{(k',\pi)+m'\pi^0}\, d^4\sqrt{k'}\pi$$
$$= \int \frac{e^{i(\sqrt{k'}\pi)(x(\xi)-x'(\xi'))}}{(i\sqrt{k',\pi})+\sqrt{m'}\pi^0)(-i\sqrt{(k',\pi)}+\sqrt{m'\pi^0})} \times d^4\sqrt{k'}\pi. \tag{33}$$

Now taking into account that $i\sqrt{\pi}$ and $-i\sqrt{\pi}$ correspond to two different internal helicity states and denote two separate fields and particle and antiparticle states, for a single particle state with a specific internal helicity, we should take $-i\sqrt{\pi}$ (or $i\sqrt{\pi}$) as a vanishing term. Taking $-i\sqrt{\pi}=0$, we see that the expression (33) reduces to the form

$$\langle \bar{\phi}(x,t,\xi)\bar{\phi}(x',t',\xi')\rangle_{T=0} = \frac{1}{(2\pi)^4}\int \frac{e^{i(k,(x-x'))}}{i\sqrt{k^2}+m}\, d^4k, \tag{34}$$

where we have chosen the unit $m = \pi^0 = 1$.

Now we can choose a matrix $\gamma_\mu k_\mu + m = \not{k} + m$ with two degenerate eigenvalues $\pm i\sqrt{k^2}+m$, which can be diagonalized by a unitary matrix U:

$$\not{k} + m = U \begin{pmatrix} i\sqrt{k^2}+m & 0 & 0 & 0 \\ 0 & i\sqrt{k^2}+m & 0 & 0 \\ 0 & 0 & -i\sqrt{k^2}+m & 0 \\ 0 & 0 & 0 & -i\sqrt{k^2}+m \end{pmatrix} U \tag{35}$$

Thus we just get the fermionic propagator in Euclidean space-time

$$\langle\bar{\phi}(x,t,\xi)\bar{\phi}(x',t',\xi')\rangle_{T=0} = \frac{1}{(2\pi)^4}\int \frac{d^4k\, e^{i(k(x-x')}}{\not{k}+m}. \quad (36)$$

This shows that when a direction vector giving rise to an internal helicity in an anisotropic microlocal space-time is taken into account, we can have the quantized fermionic field from a Brownian motion process. This result will be valid for $T \neq 0$ also. Indeed, in a similar way we find from Eq. (32) for a particular mode $n = 1$,

$$\langle\bar{\phi}(x,t,\xi)\bar{\phi}(x',t',\xi')\rangle_{T=0} = \frac{1}{(2\pi)^4}\int \frac{d^4k\, e^{i(k_1(x-x'))}}{\not{k}+m}\, 2\pi\delta(k_0 - w_1) \quad (37)$$

normalizing $\beta = \pi^0 = m = 1$.

From this analysis, it is noted that the statistics of the particle depend on the internal space-time variable ξ_μ. That is, when ξ_μ appears as a direction vector with a fixed orientation in the structure of the particle so that it gives rise to two opposite internal helicities which correspond to particle and antiparticle states, we get a Fermi field. Indeed, the fermion number is associated with this internal helicity. Again when there is no anisotropy in the internal space so that there is no manifestation of ξ_μ in the external space we get a boson. Now the effect of temperature should definitely affect the internal motion and as such it may happen that at high temperature the anisotropic feature of the internal space will be destroyed and the fermion will be transformed into a boson. This is, a massive extended body depicting a fermion can have such a phase transition. However, this does not mean that fermion number conservation will be violated as Lorentz invariance and the equilibrium condition will not allow such a process to occur. The only effect of such a phase transition will be that a thermal pair of opposite statistics will emerge as zero energy modes at the critical temperature possibly leading to a nonequilibrium state corresponding to a supersymmetric phase.[8] Indeed the stochastic nonlocal field $\phi(x_\mu,\xi_\mu)$ which is assumed to satisfy the condition of separability $\phi(x_\mu,\xi_\mu) = \phi(x_\mu)\phi(\xi_\mu)$ can be written as a thermal doublet $\binom{\phi(x_\mu)}{\phi(\xi_\mu)}$ as the thermal effect on ξ_μ may be such that it may alter the statistics of the particle. However, though x and ξ represent two different spaces, yet as the external motion may be thought to be a manifestation of the internal motion, a mapping of x and ξ is possible. In that case x may be represented in the functional form $x(\xi)$ and the simplest form of the mapping can be taken to be $x = c\xi$, where c is a suitable constant. In view of this, there should be a mapping of $\phi(x)$ and $\phi(\xi)$ also. We can assume that $\phi(\xi) = \lambda\phi^\dagger(x) = \tilde{\phi}^\dagger(x)$, where λ is a suitable parameter. Thus the thermal doublet $\binom{\phi(x)}{\phi(\xi)}$ can be written as $\binom{\phi(x)}{\tilde{\phi}^\dagger(x)}$. This helps us to consider that there exists a conjugate Hilbert space $\tilde{H}$ associated with the Hilbert space H such that $\tilde{H}$ is the set H with the scalar multiplication $\lambda,\xi \to \bar{\lambda}\xi$, where $\lambda \in C$ and $\xi \in H$ and with scalar product $(\xi,\eta) \to (\overline{\xi,\eta})$ with $\xi,\eta \in H$ and $(\xi,\eta) \to (\xi,\eta)$ is the scalar product of H. In effect H is the Hilbert space associated with the external space and $\tilde{H}$ is the conjugate Hilbert space associated with the internal space.

Now we write the bosonic field function in terms of the thermal doublets

$$\phi(x) = \begin{pmatrix}\phi(x)\\ \phi(\xi)\end{pmatrix} = \begin{pmatrix}\phi(x)\\ \tilde{\phi}^\dagger(x)\end{pmatrix} = \int \frac{d^4p}{(2\pi)^3}\theta(p_0)\delta(p^2-m^2) \times\left\{\begin{pmatrix}a_+(\mathbf{p})\\ \tilde{a}^\dagger_+(\mathbf{p})\end{pmatrix}e^{-ipx} + \begin{pmatrix}a^\dagger_-(\mathbf{p})\\ \tilde{a}_-(\mathbf{p})\end{pmatrix}e^{ipx}\right\}, \quad (38)$$

$$\phi'(x) = \begin{pmatrix}\phi^\dagger(x)\\ \phi^\dagger(\xi)\end{pmatrix} = \begin{pmatrix}\phi^\dagger(x)\\ \tilde{\phi}(x)\end{pmatrix} = \int \frac{d^4p}{(2\pi)^3}\theta(p_0)\delta(p^2-m^2) \times\left\{\begin{pmatrix}a_-(\mathbf{p})\\ \tilde{a}^\dagger_-(\mathbf{p})\end{pmatrix}e^{-ipx} + \begin{pmatrix}a^\dagger_+(\mathbf{p})\\ \tilde{a}_+(\mathbf{p})\end{pmatrix}e^{ipx}\right\}. \quad (39)$$

In the case of fermions, we have to introduce the anisotropic feature in the internal space so that it can generate two internal helicities, corresponding to particle and antiparticle, and in view of this we can obtain the Dirac propagator when the external space-time variable x_μ is considered to be a function of the internal space-time variable ξ_μ. But it may be remarked that we may do the opposite also, i.e., the internal space-time variable ξ_μ may be taken to be a function of the external space-time variable x_μ, and we may obtain the Dirac propagator in the internal variable ξ_μ and conjugate π_μ when an anisotropy is introduced in its attached vector x_μ. That is, we can write Dirac functions $\psi(x)$ and $\psi(\xi)$ in Hilbert spaces H and $\tilde{H}$, respectively, in a symmetric way. It may be noted that when at high temperature the anisotropy of the internal ξ space is destroyed, the spinorial characteristic of the field $\psi(x_\mu)$ which is acquired through the anisotropy of the attached vector ξ_μ will be changed to a bosonic one, but the spinorial characteristic of the conjugate field $\psi(\xi_\mu)$ which is acquired through the anisotropy of the attached vector x_μ will not be altered and as such we will have a thermal doublet of opposite statistics. This indicates that at that critical temperature, we will have a nonequilibrium state corresponding to a supersymmetric phase and as such fermion number conservation will not be violated due to such a phase transition.[9] This is similar to the features of Z_2 symmetry which arises in the finite temperature formalism of quantum field theory in Minkowski space as proposed by Niemi and Semenoff.[8] Indeed, the field function in the internal space here corresponds to the ghost field introduced by these authors and the corresponding Z_2 symmetry is manifested in the anisotropic feature of the internal space leading to the generation of two opposite internal helicities. As argued by Niemi and Semenoff, the broken Z_2 symmetry leads to a nonelquilibrium state. Our present formalism also leads to a similar situation when at the critical temperature, the internal helicity is destroyed leading to a nonequilibrium state. Now introducing the mapping $\psi(\xi) = \lambda\psi^\dagger(x) = i\tilde{\psi}^\dagger(x)$ we can write the Fermi field $\psi(x)$ in terms of the thermal doublets as follows.

$$\psi(x)=\begin{pmatrix}\psi(x)\\ i\psi^{\dagger}(x)\end{pmatrix}=\int\frac{d^4p}{(2\pi)^3}\theta(p_0)\delta(p^2-m^2)\sum_{\lambda=1,2}\left\{\begin{pmatrix}b_{+}(\mathbf{p},\lambda)\\ i\tilde{b}^{\dagger}_{+}(\mathbf{p},\lambda)\end{pmatrix}V(\mathbf{p},\lambda)e^{-ipx}+\begin{pmatrix}b^{\dagger}_{-}(\mathbf{p},\lambda)\\ i\tilde{b}_{-}(\mathbf{p},\lambda)\end{pmatrix}V(\mathbf{p},\lambda)e^{ipx}\right\},\tag{40}$$

$$\bar{\psi}(x)=\begin{pmatrix}\bar{\psi}(x)\\ -i\tilde{\psi}(x)\gamma_0\end{pmatrix}=\int\frac{d^4p}{(2\pi)^3}\theta(p_0)\delta(p^2-m^2)\sum_{\lambda=1,2}\left\{\begin{pmatrix}b_{-}(\mathbf{p},\lambda)\\ -i\tilde{b}^{\dagger}_{-}(\mathbf{p},\lambda)\end{pmatrix}\bar{V}(\mathbf{p},\lambda)e^{-ipx}+\begin{pmatrix}b'_{+}(\mathbf{p},\lambda)\\ -i\tilde{b}_{+}(\mathbf{p},\lambda)\end{pmatrix}\bar{V}(\mathbf{p},\lambda)e^{ipx}\right\}.\tag{41}$$

This doubling of field may be suitably represented through the complexification of space-time variables. Indeed, if we write the doublet as

$$\begin{pmatrix}\phi(x)\\ \phi(\xi)\end{pmatrix}=\phi(x)+i\phi(\xi)\tag{42}$$

it implies that the space-time coordinate is given by $z=x+i\xi$. As we have constructed the stochastic fields from the stochastic variables $q(t)$ and $q(\xi_0)$ we can write for the configuration variable in complexified space-time

$$q(z_0)=q(t)+iq(\xi_0).\tag{43}$$

From this we have the correlations at $T=0$

$$\langle q(z_0)\rangle=\langle q(t)\rangle+i\langle q(\xi_0)\rangle=0,\tag{44}$$

$$\begin{aligned}\langle q(z_0)q(z_0')\rangle&=\langle q(t+i\xi_0)q(t'+i\xi_0')\rangle\\&=\langle q(t)q(t')\rangle-\langle q(\xi_0)q(\xi_0')\rangle\\&\quad+i[\langle q(t)q(\xi_0')\rangle+\langle q(\xi_0)q(t')\rangle].\end{aligned}\tag{45}$$

Now introducing the mapping $\lambda q(t)=q(\xi_0)$, we find

$$\langle q(z_0)q(z_0')\rangle=\langle q(t)q(t')\rangle[1-\lambda^2+2i\lambda].\tag{46}$$

Noting that

$$\langle q(t)q(t')\rangle=(\hbar/2mw)e^{-iw(t-t')},\tag{47}$$

we finally have

$$\langle q(z_0)q(z_0')\rangle=(\hbar/2mw)e^{-iw(t-t')}[1+\lambda^2-2i\lambda].\tag{48}$$

Now we can choose

$$\begin{aligned}(1-\lambda^2)e^{-w(t-t')}&=\cos w(t-t'),\\ 2\lambda e^{-w(t-t')}&=\sin w(t-t'),\end{aligned}\tag{49}$$

which implies that λ is a suitable function of the dimensionless variable $w(t-t')$ with the constraint

$$(1+\lambda^2)e^{-w(t-t')}=1.\tag{50}$$

So we can write

$$\langle q(z_0)q(z_0')\rangle=(\hbar/2mw)e^{iw(t-t')}.\tag{51}$$

As we have constructed the stochastic fields from the configuration variables through the relations

$$\begin{aligned}\phi(x)&=\sum_i e_i(\mathbf{x})q_i(t),\\ \phi(\xi)&=\sum_j e_i(\boldsymbol{\xi})q_j(\xi_0),\end{aligned}\tag{52}$$

where $e_i(\mathbf{x})\,[e_j(\boldsymbol{\xi})]$ are the set of orthonormal eigenfunctions of the Laplacian

$$\begin{aligned}-\Delta e_i(\mathbf{x})&=\mathbf{k}_i^2e_i(\mathbf{x}),\\ -\Delta' e_j(\boldsymbol{\xi})&=\pi_j^2e_i(\boldsymbol{\xi}).\end{aligned}\tag{53}$$

We can now derive the two-point correlation using the mapping $x=c\xi$, $k=(1/c)\pi$, and $\lambda q(t)=q(\xi_0)$ for the complex field

$$\begin{aligned}\phi(z)&=\phi(x)+i\phi(\xi)=\phi(x)+i\lambda\phi^{\dagger}(x)\\&=\phi_R(x)+i\phi_I(x).\end{aligned}$$

Indeed, we find

$$\begin{aligned}\langle\phi(z)\phi(z')\rangle&=\langle(\phi_R(x)+i\phi_g(x))(Q_R(x')+i\phi_I(x'))\rangle\\&=\langle\phi_R(x)\phi_R(x')\rangle-\langle\phi_I(x)\phi_I(x')\rangle\\&\quad+i[\langle\phi_R(x)\phi_I(x')\rangle+\langle\phi_I(x)\phi_R(x')\rangle].\end{aligned}\tag{54}$$

Now using relation $\phi_I(x)=\lambda\phi^{\dagger}(x)$, we find from Eq. (54)

$$\langle q(z)\phi(z')\rangle=\langle\phi(x)\phi'(x')\rangle[1-\lambda^2+2i\lambda].\tag{55}$$

At $T=0$, $\langle\phi(x)\phi^{\dagger}(x')\rangle$ is given by

$$\begin{aligned}&\langle\phi(x)\phi^{\dagger}(x')\rangle\\&=\frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\langle q(t)q(t')\rangle\\&=\frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\frac{1}{2w}e^{-iw(t-t')}\ (\hbar=m=1).\end{aligned}\tag{56}$$

Now utilizing the relations (49) and (51), we can finally write from Eq. (55)

$$\begin{aligned}&\langle\phi(z)\phi(z')\rangle\\&=\frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\frac{1}{2w}e^{iw(t-t')}\\&=\frac{i}{(2\pi)^4}\int d^4k\,\frac{e^{ik(x-x')}}{k_0^2-w^2+i\epsilon}.\end{aligned}\tag{57}$$

When we write $\phi(z)=\phi_R(x)+i\phi_I(x)$ as the doublet, we find from Eqs. (54) and (57)

$$\begin{aligned}\left\langle\begin{pmatrix}\phi_R(x)\\ \phi_I(x)\end{pmatrix}(\phi_R(x'),\phi_I(x'))\right\rangle&=\begin{bmatrix}\langle\phi_R(x)\phi_R(x')\rangle & \langle\phi_R(x)\phi_I(x')\rangle\\ \langle\phi_I(x)\phi_R(x')\rangle & \langle\phi_I(x)\phi_I(x')\rangle\end{bmatrix}\\&=\frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\begin{bmatrix}\cos w(t-t')/2w & \sin w(t-t')/2w\\ \sin w(t-t')/2w & -\cos w(t-t')/2w\end{bmatrix}\\&=\frac{i}{(2\pi)^4}\int\frac{d^4k\,e^{ik(x-x')}}{k_0^2-w^2+iE}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.\end{aligned}\tag{58}$$

Thus we get stochastic quantization in Minkowski space at $T=0$. This is identical with the result obtained in the path integral formulation at $T=0$ excepting the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The matrix corresponds to reflection invariance representing Z_2 symmetry, which is the criterion for equilibrium condition. When this Z_2 symmetry is destroyed, we have nonequilibrium statistical mechanics and it corresponds to supersymmetric quantum mechanics.[9] This is in conformity with the idea that the current velocity in the internal space is related to the osmotic velocity in the external space which helps us to interpret the Heisenberg uncertainty relation from the inherent stochastic nature of the internal space-time variable.[10]

This analysis can be generalized to the fermionic case also. Indeed writing

$$\psi(z) = \psi(x) + i\psi(\xi) = \psi(x) + i_\lambda \psi^\dagger(x) = \psi_R(x) + i\psi_I(x)$$

and taking the doublet

$$\psi(z) = \begin{pmatrix} \psi(x) \\ \psi(\xi) \end{pmatrix} = \begin{pmatrix} \psi_R(x) \\ \psi_I(x) \end{pmatrix}$$

we will have correlations at $T=0$

$$\langle \psi(z)\psi(z')\rangle = \left\langle \begin{pmatrix} \psi_R(x) \\ \psi_I(x) \end{pmatrix} (\psi_R(x'),\psi_I(x')) \right\rangle = \frac{i}{(2\pi)^4} \int d^4p\, e^{ip(x-x')} \frac{1}{\not p - m + iE} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (59)$$

These results can now be generalized to finite temperature using the formalism of thermofield dynamics when we identify $\phi(\xi) = \phi_I(x) = \tilde\phi^\dagger(x)$ and $\psi(\xi) = \psi_I(x) = i\tilde\psi^\dagger(x)$.
From these we can write

$$\langle T[\phi(x)\phi^t(y)]\rangle = \langle 0(\beta)| T\begin{pmatrix} \phi(x) \\ \tilde\phi^\dagger(x) \end{pmatrix} (\phi^\dagger(y),\tilde\phi(y)) |0(\beta)\rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} V_B(|\mathbf{p}|,\beta) \begin{pmatrix} 1/(p^2-m^2+i0) & 0 \\ 0 & -1/(p^2-m^2-i0) \end{pmatrix} V_B^\dagger(|\mathbf{p}|,\beta) \quad (60)$$

$$\langle \theta(x_0-y_0)[\phi(x),\phi^\dagger(y)]\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} i\theta(x^0-y^0)\Delta(x-y) = -i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Delta_{ret}(x-y), \quad (61)$$

$$\langle T\psi(x)\bar\psi(y)\rangle = \langle 0(\beta)\left[T\begin{pmatrix} \psi(x) \\ i\tilde\psi^\dagger(x) \end{pmatrix} (\bar\psi(y), -i\tilde\psi(y)\gamma_0)\right]|0(\beta)\rangle = i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} V_F(p,\beta)$$

$$\times \begin{pmatrix} 1/(\not p - m + i0) & 0 \\ 0 & 1/(\not p - m - i0) \end{pmatrix} V_F^\dagger(p,\beta), \quad (62)$$

$$\langle \theta(x^0-y^0)\{\psi(x),\bar\psi(y)\}\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (i\gamma_\mu\gamma_\mu + m) i\theta(x^0-y^0)\Delta(x-y) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} iS_{ret}(x-y). \quad (63)$$

The matrices V_B and V_F are the coefficients of Bogoliubov transformations given by

$$V_B(|\mathbf{p}|,\beta) = \begin{pmatrix} \cosh\theta(|\mathbf{p}|,\beta) & \sin\theta(|\mathbf{p}|,\beta) \\ \sinh\theta(|\mathbf{p}|,\beta) & \cosh\theta(|\mathbf{p}|,\beta) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{1-e^{-\beta\epsilon(\mathbf{p})}} & e^{-\beta\epsilon(\mathbf{p})/2}/\sqrt{1-e^{-\beta\epsilon(\mathbf{p})}} \\ e^{-\beta\epsilon(\mathbf{p})/2}/\sqrt{1-e^{-\beta\epsilon(\mathbf{p})}} & 1/\sqrt{1-e^{-\beta\epsilon(\mathbf{p})}} \end{pmatrix}, \quad (64)$$

$$V_F(p,\beta) = \begin{pmatrix} \cos\nu(|\mathbf{p}|),\beta) & -\epsilon(p^0)\sin\nu(|\mathbf{p}|,\beta) \\ \epsilon(p_0)\sin\nu(|\mathbf{p}|,\beta) & \cos\nu(|\mathbf{p}|,\beta) \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{1+e^{-\beta\epsilon(\mathbf{p})}} & -\epsilon(p^0)e^{-\beta\epsilon(\mathbf{p})/2}/\sqrt{1+e^{-\beta\epsilon(\mathbf{p})}} \\ \epsilon(p^0)e^{-\beta\epsilon(\mathbf{p})/2}/\sqrt{1+e^{-\beta\epsilon(\mathbf{p})}} & 1/\sqrt{1+e^{-\beta\epsilon(\mathbf{p})}} \end{pmatrix} \quad (65)$$

with

$$\epsilon(\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2}, \quad \epsilon(p_0) = \theta(p_0) - \theta(-p_0). \quad (66)$$

This suggests that in the case of free charged scalar field and free Dirac spinor field ψ, the total Lagrangians are given by

$$\bar L_\varphi = \alpha_\varphi - \tilde\alpha_\varphi$$

$$= \partial_\mu\phi^+\partial_\mu\phi - m^2\phi^\dagger\phi - \partial_\mu\tilde\phi^+\partial_\mu\tilde\phi + m^2\tilde\phi^2\phi^2$$

$$= \partial_\mu\phi^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_\mu\phi - m^2\phi^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\phi, \quad (67)$$

$$\bar L_\varphi = \alpha_\psi - \tilde\alpha_\psi$$

$$= \bar\psi((i/2)\gamma_\mu\overleftrightarrow{\partial}_\mu - m)\psi - \tilde\psi((-i/2)\gamma_\mu\overleftrightarrow{\partial}_\mu - m)\tilde\psi$$

$$= \bar\psi\left(\frac{i}{2}\gamma_\mu\overleftrightarrow{\partial}_\mu - m\right)\psi. \quad (68)$$

It is noted that the vacuum is now temperature dependent and satisfies the relation

$$(H-\tilde H)|0(\beta)\rangle = 0 \quad (69)$$

and the total Hamiltonian is given by

$$\bar H = H - \tilde H. \quad (70)$$

III. STOCHASTIC FIELD THEORY, HOLOMORPHIC QUANTUM MECHANICS, AND SUPERSYMMETRY

From our above analysis, we note that we can construct holomorphic quantum mechanics when for the configuration variable we take the complexified space-time $z_\mu = x_\mu + i\xi_\mu$. In fact, we can now depict the fields $\phi^\pm(z) = \phi(x) \pm i\phi(\xi)$ and can consider that $\phi^\pm$ is holomorphic in z. Now defining the operators

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial \xi}\right), \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \xi}\right)$$

we can write for a free field, the Hamiltonian

$$H = -2\partial\bar{\partial} + m^2\phi^-\phi^+ . \tag{71}$$

Identifying $\phi(\xi)$ as $\tilde{\phi}^+(x)$ as discussed in the previous section, we note that the Hamiltonian H corresponds to the system of free fields where the Lagrangian is given by $\bar{L}_\phi$ in Eq. (67). Now if we identify $\phi^\pm(z) = \mp i\sqrt{2}\,\partial V$ we can construct two operators Q_+ and Q_- such that

$$Q_- = \begin{pmatrix} \partial V & i\partial \\ i\bar{\partial} & -(\partial V)^* \end{pmatrix},$$
$$Q_+ = \begin{pmatrix} (\partial V) & i\partial \\ i\bar{\partial} & -\partial V \end{pmatrix}, \tag{72}$$

and the Hamiltonian H given by Eq. (71) can be expressed as

$$H = \mathrm{Tr}\, Q_+Q_- \quad (m=1)\,, \tag{73}$$

where

$$Q_+Q_- = (-\partial\bar{\partial} + |\partial V|^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{74}$$

Since Q_+Q_- is the sum of two positive operators, it has no zero mode. Besides, we note that this maintains the equilibrium condition of the Z_2 symmetry between the external and internal fields as the expression is invariant under the transformations, $\phi(x) \to -\phi(\xi)$, $x \to -\xi$.

However, from expression (72), we note that we can construct another operator Q_-Q_+ which is given by

$$Q_-Q_+ = Q_+Q_- + \begin{pmatrix} 0 & -i(\partial^2 V) \\ i(\partial^2 V)^* & 0 \end{pmatrix}. \tag{75}$$

This expression for Q_-Q_+ contains nondiagonal elements and the presence of the term $\partial^2 V$ breaks the reflection invariance $\phi(x) \to -\phi(\xi)$, $x \to -\xi$. Thus the system describes the nonequilibrium condition and corresponds to supersymmetric quantum mechanics. Indeed, we can now define an operator Q such that

$$Q = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix} \tag{76}$$

and we can construct the Hamiltonian H_S given by

$$H_S = Q^2 = \begin{pmatrix} Q_-Q_+ & 0 \\ 0 & Q_+Q_- \end{pmatrix}. \tag{77}$$

Evidently, the system given by the Hamiltonian H_S breaks down the Z_2 symmetry of reflection invariance of the external and internal fields. In fact, due to the presence of the operator Q_-Q_+ in H_S, it possesses zero modes as has been explicitly shown by Jaffe *et al.*[1] Thus we can define the supercharge Q such that the Hamiltonian is given by $H_S = Q^2$ when the stochastic field theory involving external and internal fields is described in terms of holomorphic quantum mechanics.

Moreover, following the procedure of Jaffe *et al.*,[1] we can show that this formalism of holomorphic quantum mechanics for stochastic field theory gives rise to $N=2$ Wess–Zumino quantum mechanics. In fact, we can also choose for the Hamiltonian H_S the following expression:

$$H_S = Q^2 = -\partial\bar{\partial} + |\partial V|^2 - \bar{\psi}_1\psi_1\,\partial^2 V - \bar{\psi}_2\psi_2(\partial^2 V)^*, \tag{78}$$

where

$$\psi_1 = \tfrac{1}{2}(\gamma_0 - i\gamma_3), \quad \bar{\psi}_1 = \tfrac{1}{2}(\gamma_1 + i\gamma_2),$$
$$\psi_2 = \tfrac{1}{2}(\gamma_1 - i\gamma_2), \quad \bar{\psi}_2 = \tfrac{1}{2}(\gamma_0 + i\gamma_3), \tag{79}$$

with

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & i\sigma_j \\ -i\sigma_j & 0 \end{pmatrix}, \quad j = 1,2,3. \tag{80}$$

These fermionic degrees of freedom satisfy the following anticommutation relations at equal time:

$$\{\bar{\psi}_1,\psi_2\} = \{\bar{\psi}_2,\psi_1\} = 1,$$
$$\{\psi_i,\psi_j\} = \{\bar{\psi}_i,\bar{\psi}_j\} = 0. \tag{81}$$

We can now define two conserved charges given by

$$Q_1 = i\bar{\psi}_1\partial - i\bar{\psi}_2(\partial V)^*,$$
$$Q_2 = i\bar{\psi}\bar{\partial} + i\psi_1\,\partial V, \tag{82}$$

so that the supercharge Q is given by

$$Q = Q_1 + Q_2. \tag{83}$$

The Lagrangian for such a system can be taken to be[1]

$$L = |\dot{z}|^2 + i(\bar{\psi}_1\dot{\psi}_2 + \bar{\psi}_2\dot{\psi}_1) + \bar{\psi}_1\psi_1\,\partial^2 V + \bar{\psi}_2\psi_2(\partial^2 V)^* - |\partial V|^2, \tag{84}$$

where $V = V(z)$ is a polynomial of degree n. The action $\int L\,dt$ is invariant under the following infinitesimal transformations:

$$\delta z = \bar{\psi}_1\epsilon, \quad \delta\bar{z} = \bar{\epsilon}\psi_2, \quad \delta\psi_1 = -(\partial V)^*\epsilon,$$
$$\delta\bar{\psi}_1 = i\dot{\bar{z}}\bar{\epsilon}, \quad \delta\psi_2 + i\dot{z}\epsilon, \quad \delta\bar{\psi}_2 = (\partial V)\epsilon. \tag{85}$$

Thus we find that we can derive $N=2$ Wess–Zumino quantum mechanics from stochastic fields in a complexified space-time. The supersymmetric quantum mechanics arises through the introduction of nondiagonal elements which breaks down the reflection invariance between the external and internal fields which is necessary for equilibrium condition. Also we note that through this formalism of holomorphic quantum mechanics we can derive a supercharge Q such that the supersymmetric Hamiltonian is given by $H_S = Q^2$. This links up the inherent supersymmetric feature in the stochastic quantization procedure as we first pointed out by Parisi and Sourlas[11] with the conventional formalism of supersymmetric quantum mechanics.

IV. STOCHASTIC FIELD THEORY AND INDEX THEOREM

Jaffe *et al.*[1] have shown that the $N=2$ Wess–Zumino quantum mechanics has degenerate vacua. The space of vacuum states is bosonic and its dimension is determined by the topological properties of the superpotential. The same result can be derived from the stochastic field theoretical formalism using the formalism of thermofield dynamics.

The index theorem for a free field theory has been derived from the stochastic field theory using the formalism of thermofield dynamics in a recent paper.[9] Here we shall generalize this result in the presence of a superpotential. In the free field case, we can define two Klein operators

$$\theta = (-1)^F, \tilde{\theta} = (-1)^{\tilde{F}}, \tag{86}$$

where $\tilde{\theta} = J\theta J$, J being an involution operator with the property $J^2 = 1$. As discussed in Sec. II, the tilde function is associated with the internal field and the total Hamiltonian of the system is given by

$$\bar{H} = H - \tilde{H},$$

where

$$\tilde{H} = JHJ. \tag{87}$$

In this formalism, the vacuum is temperature dependent and we have the relations

$$\bar{H}\,|0(\beta)\rangle = 0$$

$$\langle A\rangle = \langle 0(\beta)|A\,|0(\beta)\rangle. \tag{88}$$

Now it is noted that for the Klein operators θ and $\tilde{\theta}$ we have

$$\theta\,|0(\beta)\rangle = \tilde{\theta}\,|0(\beta)\rangle \neq |0(\beta)\rangle,$$

$$\bar{\theta}\,|0(\beta)\rangle = \theta\tilde{\theta}\,|0(\beta)\rangle = |0(\beta)\rangle. \tag{89}$$

We can define an index for the ground state given by

$$i(Q_+) = \mathrm{Tr}(-1)^{\bar{F}} e^{-\beta\bar{H}}|_{\beta-\alpha}, \tag{90}$$

where $\bar{F} = F + \tilde{F}$, $\bar{H} = H - \tilde{H}$. This can also written as

$$i(Q_+) = \langle 0(\beta)|\theta\tilde{\theta}\,|0(\beta)\rangle_{\beta-\alpha}. \tag{91}$$

As we have mentioned in Sec. II, the thermodynamic equilibrium is maintained as long as Z_2 symmetry (time reversal symmetry) is operative in nontilde and tilde objects corresponding to the external and internal space. That is, the orientation in the external space should be opposite to that in the internal space. However, this formalism of stochastic field theory suggests that there may exist a critical temperature T_c when the orientation of the internal space is changed leading to a nonequilibrium state. Indeed for the thermal doublet of a bosonic field $\phi(x) = \binom{\phi(x)}{\phi(\xi)} = \binom{\phi(x)}{\tilde{\phi}(x)}$ it may so happen that the isotropic feature of the bosonic field $\phi(\xi) = \tilde{\phi}^*(x)$ is lost at this critical temperture and an internal helicity is generated for this field giving rise to an anisotropic feature leading to the generation of a fermion. Thus beyond this temperature T_c we have the supersymmetric feature due to the breakdown of the Z_2 symmetry which will then give rise to thermal doublets of different statistics which will appear as zero energy modes as suggested by Matsumoto *et al.*[12]

Now to find out the index theorem in the supersymmetric phase we note that the equilibrium condition demands

$$\langle 0(\beta)|\theta\tilde{\theta}\,|0(\beta)\rangle = 1 = \int \delta(F + \tilde{F})dF. \tag{92}$$

In the nonsupersymmetric case, for a bosonic thermal doublet $F = \tilde{F} = 0$. However, for a supersymmetric phase, we may have $\tilde{F} = \pm 1$ depending on the orientation of the internal helicity developed leading to an anisotropy in the internal space. Thus we will have the index

$$i(Q_+) = \langle 0(\beta)|\tilde{\theta}\,|0(\beta)\rangle_{\beta-\alpha}$$

$$= \int \delta(F \pm 1)dF = 1. \tag{93}$$

However, in an interacting case with a superpotential given by

$$V(\phi)\tfrac{1}{2}m\phi^2 + \sum_{j=3}^{n} a_j\phi^j$$

with complex ϕ, $a_n \neq 0$ $n \geqslant 3$ as we have identified $\phi^\perp = \mp i\sqrt{2}\,\partial V$, we note that we will have $(n-1)$ images of such thermal doublets. So for an interacting case, with superpotential $V(\phi) = \lambda\phi^n$ we will have

$$i(Q_+) = (n-1)\langle 0(\beta)|\theta\tilde{\theta}\,|0(\beta)\rangle_{\beta-\alpha}$$

$$= (n-1)\int \delta(F \pm 1)dF = n - 1 = \deg \partial V.$$

this is identical with the result obtained by Jaffe *et al.*[1] in the two-dimensional $N = 2$ Wess–Zumino field model and we can conclude that the holomorphic quantum mechanics constructred from the stochastic field theory will also lead to a degenerate vacua in the interacing case when the superpotential is given by $V(\phi) = \lambda\phi^n$ $n > 2$.

V. DISCUSSION

We have shown here that the relativistic generalization of Nelson's stochastic mechanics as well as stochastic quantization in Minkowski space helps us to construct holomorphic quantum mechanics when in the nonequilibrium condition we can realize $N = 2$ Wess–Zumino quantum mechanics and supersymmetry. In the equilibrium condition, we get stochastic quantization in Minkowski space and we have Z_2 symmetry between the external and internal fields which form a doublet. When this reflection invariance is broken, we get supersymmetric quantum mechanics which imply that supersymmetry gets broken in a multiply connected space.

The inherent supersymmeric feature in stochastic quantization leading to Euclidean quantum field theory from a Langevin equation incorporating a fictitious time was first pointed out by Parisi and Sourlas.[11] However, it was not clear whether this supersymmeric feature which involves invariance of the action under certain supersymmetric transformations is equivalent to the conventional supersymmetric quantum mechanics which defines a supercharge Q such that the Hamiltonian is given by $H = Q^2$. Indeed in that case the action involves fermionic variables only through the determinant which arises in the averaging procedure and hence the invariance of the action under supersymmetric transformation in this case does not imply the existence of the supercharge Q as well as the existence of the grading operator γ such that $H = Q^2$ and $\gamma Q + Q\gamma = 0$. However, here we have pointed out that stochastic quantization in Minkowski space introduces two stochastic fields, one in the external space and the other in the internal space; in the nonequilibrium case we can construct holomorphic quantum mechanics out of these two fields, which becomes equivalent to $N=2$ Wess–Zumino quantum mechanics and gives rise to the supercharge operator.

Finally, it may be pointed out that this formalism helps us to study finite temperature field theory as well as finite temperature supersymmetry through the methodology of thermofield dynamics when we identify the tilde field with the stochastic field in the internal space. Indeed in a recent paper,[9] we have pointed out that there exists a critical temperature T_c below in which supersymmetry is broken and the zero energy mode is given by a thermal doublet of opposite statistics and in the case of a free field theory the index $i(Q_+) = n_+ - n_-$, where $n_\pm = \dim \operatorname{kernel} Q_\pm$ is found to be 1. It may take the value $\frac{1}{2}$ also when the zero energy mode is given by a thermal doublet of the same fermionic statistics.[9] Here we have pointed out that in the interacting case when the potential is given by $V(\phi) = \lambda\phi^n$, ϕ being a complex scalar field, and $n > 2$, the vacuum is degenerate and the index takes the value $n-1 = \deg \partial V$. This is identical with the result obtained by Jaffe *et al.* in the two-dimensional Wess–Zumino quantum field model.

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