Essays on Voting and Auction Theory

Mridu Prabal Goswami

Thesis submitted to the Indian statistical Institute in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy

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Contents

Chapter 1

Introduction

This thesis explores some issues in social choice and auction theory. The three chapters deal with three specific problems that have normative and strategic elements knitted together.

Consider the following examples. Suppose a fixed supply of various goods/commodities have to be distributed amongst a set of consumers/agents. Suppose that these agents draw preference orderings over commodity bundles from a known set of preferences. A reasonable requirement for the resulting allocation is that it should be *Pareto-efficient*. Otherwise, agents would have the incentive to reallocate the commodities amongst themselves. In order to allocate the resources in a Pareto-efficient manner knowledge of preferences of the agents is required. However, if agent preferences are private information (a standard and plausible assumption), an added requirement for the allocation procedure is that it should satisfy *incentive-compatibility*. This ensures that the allocation rule or social choice function (SCF) induces agents to reveal their preferences truthfully. In the second chapter of the thesis, we study allocation problems of this type, i.e we are concerned with SCFs that are incentive-compatible and Pareto-efficient in classical exchange economies.

Consider another variant of this allocation problem. Suppose that each agent is endowed with some amount of the goods. A reasonable requirement for an allocation rule in this setting is that it should leave every agent at least as well-off as consuming her endowment, i.e. it should be *individually rational*. If it is not, dissatisfied agents will presumably "opt out" of the process and consume their private endowment. In the third chapter we study SCFs that are incentive-compatible and individually rational in classical exchange economies.

In the fourth chapter of the thesis we consider another allocation problem, that of procurement. A buyer wants to buy a single unit of a good from one of several potential sellers of the good. Each seller incurs some cost to produce the good that is known only to the seller herself. The seller from whom the buyer buys must be paid a price. In this set-up, a SCF determines the identity of the seller from whom the buyer purchases the good and the payment made to the seller. This chapter explores a model where buyer cares about the quality of the good.

We now discuss the content of these chapters in greater detail.

1.1 Strategy-proofness and Pareto-Efficiency in Classical Exchange Economies

In this chapter we provide a characterization of SCFs that are strategy-proof and Pareto-effcient in classical exchange economies. We consider a classical exchange economy with $m \geq 2$ goods and $n \geq 2$ agents. The set of goods is denoted by M and that the set of agents by I. Total endowment of each good is given by Ω_j, for $j \in M$. The Edgeworth box is, $\Delta = \{(x_{i1}, ..., x_{im}) | x_{ij} \geq 0, \text{ for all } j \in M \text{ and } i \in I \text{ and } \sum_{i \in I} x_{ij} = \Omega_j \text{ for all } j \in M\}$.

A classical preference is one that is continuous, strictly monotonic and strictly convex. We study the behavior of SCFs defined on a quasi-linear domain, i.e domains where the preferences for any agent i can be represented by the utility functions of the form

$$u_i(x_{i1}, \ldots, x_{im}; \theta_i) = \theta_i\{\sqrt{x_{i1}} + \ldots + \sqrt{x_{im-1}}\} + x_{im}, \ \theta_i > 0. \tag{1.1}$$

For every profile of preferences, a SCF picks an allocation in Δ. We impose the requirement of *Pareto-efficiency* and the incentive-compatibility requirement of *strategy-proofness*. If a SCF is strategy-proof, then no agent can benefit by lying irrespective of her beliefs regarding the announcements of other agents. Strategy-proofness is a stringent requirement. According to the well-known Gibbard-Satterthwaite Theorem (Gibbard (1977) and Satterthwaite (1975)), a SCF defined over an unrestricted domain with a range of at least three alternatives, is strategy-proof only if it is *dictatorial*. A dictatorial SCF is a trivial SCF which always picks the best outcome for a given agent.

The Gibbard-Sattherthwaite Theorem cannot be directly applied to our model because preferences are *restricted*. Characterizations of strategy-proof SCFs on restricted domains such as those in exchange economies are typically quite difficult. We apply techniques developed in the context of auction design. Using them, we show that strategy-proofness and Pareto-efficiency in conjunction with *non-bossiness* and *continuity* implies dictatorship. The dictatorship result then extends in a straight forward way to all supersets of this domain including in particular to the domain of all classical preferences. Several results exist for restricted domains when there are exactly two agents - our result however is the only result of its kind for models where there are an arbitrary number of agents.

Our approach also allows us to provide extensions of some existing results. One of these is the result of Serizawa-Weymark according to which strategy-profness and the minimum consumption guarantee or MCG axiom cannot be satisfied simultaneously. We are able to prove this result for all domains that are supersets of the domain described above using simple algebraic arguments.

1.2 Non Fixed-Price Trading Rules in Single-Crossing Classical Exchange Economies

In this chapter, we consider a classical exchange economy where agents have endowments and investigate SCFs that are strategy-proof and individually rational. It is well-known that Pareto-efficiency and individual-rationality are incompatible in classical exchange economies - Hurwicz (1972) demonstrates this for the case of two-good and two-agent models and Serizawa (2002) extends it to arbitrary numbers of agents and goods.

Barberà and Jackson (1995) show that a strategy-proof and individually rational SCF defined on the domain of *all* classical preferences is a *Fixed-Price Trading* or FPT rule in a two-agent economy. We revisit this question in the context of domains of *single-crossing* preferences. These domains have been shown to be important in a wide variety of contexts in economic theory. We construct examples of non-FPT SCFs that are strategy-proof and individually rational. Single-crossing preferences do not admit *concavification* - a property whereby an arbitrary indifference curve can be "bent upwards" at an arbitrary consumption bundle while still remaining in the domain. The set of all classical preferences clearly admit concavification and the arguments in Barberà and Jackson (1995) rely heavily on the property.

We show that single-crossing domains can be linearly ordered. By imposing a *richness* condition we show that a rich single-crossing domain is a maximal single-crossing domain. We define an order topology on a rich single-crossing domain and a notion of continuity of a SCF on such domains. Using this notion of continuity, we characterize a set of trading rules which we call *Generalized Trading* or GT rules. We also assume the SCFs to have range that is closed and to lie in the interior of Δ. With the help of these assumptions we show that a SCF is strategy-proof and individually rational if and only if it is a GT rule. A FPT rule with connected range is a GT rule. With the help of examples, we show that the assumptions of continuity and interiority of the range are necessary for our characterization.

1.3 Procurement Auctions: Technical Bids, Subjective Evaluation and Corruption

In recent years in India, many public infrastructure projects have been contracted out to private organizations with the developer selected by an auction mechanism. An example is the recent expansion/refurbishment of the Delhi and Mumbai Airports. This auction mechanism involved bids on *two* dimensions, the first being the quality of the airport that a bidder would deliver and the second being the percentage of annual revenue that a bidder would share with the government upon winning the auction. Auctions of this nature have been widely used by the Government of India and this chapter analyzes their theoretical properties.

The auction rules were as follows. Bidders were asked to simultaneously submit a *technical* and a *financial* bid. The technical bid consists of a report of various bidder characteristics related to quality (such as experience and so on) which are then aggregated to form a *quality-score* according to a pre-determined weighting scheme. A minimum quality-score was also announced a priori and the financial bids of *qualified bidders*, i.e. those whose quality-scores exceeded the minimum quality-score, were opened. The winner was a qualified bidder whose financial bid was the highest amongst the qualified bidders.

We analyze a general class of auctions where the minimum quality-score is endogenous. Consider an integer λ such that $2 \leq \lambda \leq n-1$ that is announced by the buyer before the bids are submitted. A bidder wins the auction if the quality-score obtained by its technical bid is among the top λ quality-scores and its bid on price is the lowest among the bidders who qualify. We show that continuous and strictly monotonic Bayes-Nash equilibrium in pure strategies does not exist in such auctions, irrespective of the payment rules.

Procurement auctions where a minimum cut-off of quality-score is announced exogenously, are easy to analyze. It is a strictly dominant strategy to bid the minimum quality-score and competition takes place only in financial bids. The auction reduces to a standard first-price or second-price auction. We consider a variant of this auction where the evaluation procedure of technical bids is *subjective* i.e. the buyer cannot convey the quality-score function perfectly to the bidders. We consider a model where bidders receive a signal regarding a subjectivity parameter (a real number). The buyer also realizes such a parameter once the bids have been submitted. We analyze Subjective Nash Equilibrium (Kalai and Lehrer (1993), Kalai and Lehrer (1995)) in this model. We show that all bidders qualify for the financial stage of the auction in equilibrium. Therefore, inefficiencies of the kind where efficient bidders do not qualify, do not arise in equilibrium.

Subjectivity in the evaluation of technical bids creates special opportunities for corruption. In particular, the buyer may use an evaluation procedure that is tilted to favor a particular bidder. Since the evaluation is subjective it may be impossible to establish the existence of such collusion in a court of law. We consider an auction where the minimum cut-off level quality-score is fixed exogenously by the buyer. We assume that a bidder and a particular buyer collude and that this is common knowledge. This bidder/buyer player can choose the subjectivity parameter for its benefit. We demonstrate the existence of a particular Bayes-Nash equilibrium in pure strategies where both bidders qualify but the honest bidder has a cost disadvantage because she is held to a higher quality requirement. Inefficiency arises because the dishonest bidder may win without having the lowest cost parameter.

Chapter 2

Strategy-proofness and Pareto-Efficiency in Classical Exchange Economies

Introduction

Allocating available resources amongst a given set of agents who have preferences defined over these resources has been a well studied problem. A minimal and uncontroversial criterion for such allocations is *Pareto-efficiency*. However, in order to allocate the resources in a Pareto-efficient manner, the mechanism designer needs to know the true preferences of the agents concerned. If true preferences of the agents are private information then an added requirement criterion for the allocation procedure is *incentive-compatibility* that is, the allocation rule or Social Choice Function (SCF) must induce agents to reveal their preferences truthfully. The most attractive incentive-compatibility requirement to impose on a SCF is *strategy-proofness*; if a SCF is strategy-proof, then no agent can benefit by lying irrespective of her beliefs regarding the announcements of other agents. However, strategy-proofness is a stringent requirement. According to the well-known Gibbard-Satterthwaite Theorem (Gibbard (1977) and Satterthwaite (1975)), a SCF defined over an unrestricted domain with a range of at least three alternatives, is strategy-proof only if it is *dictatorial*. A dictatorial SCF is a trivial SCF which always picks the best outcome for a given agent.

In many contexts, it is natural to assume that agent preferences are *restricted*. In such cases the dictatorship result need not hold. A large literature has developed investigating the structure of strategy-proof SCFs in models such as single-peaked domains, quasi-linear domains with money as a numeraire good and so on. In this chapter, we consider a familiar restricted domain model that of a *classical exchange economy*. In this model there is a fixed amount of m goods, $m \geq 2$ which have to be distributed amongst n agents $n \geq 2$. Agent preferences defined over bundles of m goods that are assumed to be *strictly increasing, continuous* and *strictly convex*. Although a large literature exists on this problem, there is as yet no comprehensive characterization of strategy-proof and Pareto-efficient SCFs on this domain (see literature review below).

The main objective of this chapter is to establish the equivalence of strategy-proofness and Pareto-efficiency in the presence of certain mild regularity assumptions on the SCFs. Our methodological innovation is to consider a narrow class of *quasi-linear* preferences and use the techniques developed in the context of auction design and show that strategy-proofness and Pareto-efficiency in conjunction with *non-bossiness* and *continuity* implies dictatorship. The dictatorship result then extends in a straight forward way to all supersets of this domain including in particular to the domain of all classical preferences. Our approach also allows us to provide relatively simple extensions of existing results in the literature.

The classic paper on incentive-compatibility in exchange economies is Hurwicz (1972). It shows that there does not exist strategy-proof, efficient and individually rational SCFs when there are two-agents. A general characterization of strategy-proof and efficient SCFs however remains an open question and has been the focus of considerable research. Dasgupta et al. (1979) show that in the case of two agents, every efficient and strategy-proof SCF is dictatorial when the domain of agent preferences is the set of all (strictly) convex and monotone orderings. Zhou (1991) extends this result to the case where preferences are classical, i.e. (strictly) convex and monotone and continuous. There are several versions of this result for the two agent case on restricted domains, for example Schummer (1997) for linear preferences, Ju (2003) for classical quasi-linear and CES and Hashimoto (2008) for Cobb-Douglas preferences. There are significant difficulties involved in extending these results to the case of an arbitrary number of agents. Zhou (1991) conjectured that efficient dictatorial SCFs in the case of $n \geq 3$ must be *inversely dictatorial*. However Kato and Ohseto (2002) have shown by means of an example that the conjecture is false. If the domain is non-classical then it is possible to construct SCFs that are strategy-proof, Pareto-efficient and non-dictatorial. For instance, Nicolò (2004) shows that in the domain of Leontief preferences, fixed-price trading rules are strategy-proof and Pareto-efficient.

The only results that exist for general n for strategy-proof and Pareto-efficient social choice functions are Serizawa and Weymark (2003) and Serizawa (2006). The former shows that every strategy-proof and efficient SCF violates a *minimum consumption guarantee* or MCG assumption. In particular, for every $\epsilon > 0$ but arbitrarily small there exists a profile and an agent whose allocation is less than ϵ in terms of the Euclidean norm. Although this result is illuminating it is far from being a characterization. In particular it says nothing about the value of an efficient and strategy-proof SCF at an arbitrary profile.

Serizawa (2006) proves a dictatorship result by strengthening strategy-proofness to effective pairwise strategy-proofness. Effective pairwise strategy-proofness requires pairs of agents not to have a "self-enforcing manipulation" in addition to strategy-proofness. A manipulation by a pair of agents is self-enforcing if it does not decrease the utility of either agents in the pair, increases utility of at least one and neither of the agents has an incentive to betray his partner. Note that effective pairwise strategy-proofness like notions such as group strategy-proofness, requires coordination between agents. However if information is private, it is not clear how this coordination actually takes place and the notion is therefore somewhat problematic.

Two other papers that deal with the $n \geq 2$ agents case are Barberà and Jackson (1995) and Satterthwaite and Sonnenschein (1981). Neither consider Pareto-efficiency. Barberà and Jackson (1995) show that a strategy-proof, individually rational, anonymous and *non-*

bossy SCF must be a fixed-proportion trading rule. Satterthwaite and Sonnenschein (1981) show that a strategy-proof, *non-bossy*, *regular* and *everywhere total* SCF must be serially dictatorial.

We consider a domain of classical quasi-linear preferences of the following kind:

$$u_i(x_{i1}, \ldots, x_{im}; \theta_i) = \theta_i\{\sqrt{x_{i1}} + \ldots + \sqrt{x_{im-1}}\} + x_{im},\ \theta_i > 0.$$

We use methods in auction design developed by Myerson (1981) to characterize strategy-proof social choice functions. We prove three main results. First, we provide an elementary proof the MCG result of the Serizawa and Weymark (2003). Second, we show that in two-agent economies, every strategy-proof and Pareto-efficient social choice function, is dictatorial. This result is independent of the existing two-person dictatorship results in the literature because of the specificity of our domain. Finally we show that in the case of three or more agents, strategy-proofness and Pareto-efficiency together with non-bossiness and continuity, imply dictatorship. We believe that the $n \geq 3$ result is the first of its kind in the literature. Note that continuity in this problem is defined in the usual Euclidean sense. Note that all dictatorship results extend to all domains that include this domain.

Our chapter is organized as follows. In Section 2.1 we describe our model. In Section 2.2 we prove some critical results regarding strategy-proofness and Pareto-efficiency in our domain. In Section 2.3 we prove the MCG result; in Section 2.4 we prove dictatorship results first, for the two agents and then for the multi-agent case. Section 2.5 concludes.

2.1 Notation and Definitions

We consider an exchange economy with the set of agents $I = \{1, 2, ..., n\}$ and the set of goods $M = \{1, 2, ..., m\}$. We assume that $n \geq 2$ and $m \geq 2$. Let the fixed total endowment of good j be denoted by Ω_j and the total endowment vector to be $\Omega = (\Omega_1, \Omega_2, ..., \Omega_m)$. We assume $\Omega_j > 0$, for all $j \in M$. The set of feasible allocations denoted by Δ is the set $\Delta = \{(x_{i1}, ..., x_{im}) | x_{ij} \geq 0,$ for all $j \in M$ and $i \in I$ and $\sum_{i \in I} x_{ij} = \Omega_j$ for all $j \in M\}$.

A preference ordering for agent i, R_i is a complete, reflexive and transitive ordering of the elements of $\Re^m_+$. We say that R_i is *classical* if it is (a) continuous, (b) strictly monotonic in $\Re^m_{++}$ and (c) the upper contour sets are strictly convex in $\Re^m_{++}$.[1] The asymmetric component of R_i will be denoted by P_i. Let the set of classical orderings be denoted by $\mathbb{D}^c$. A preference profile R is an n-tuple $R \equiv (R_1, R_2, \ldots, R_n) \in [\mathbb{D}^c]^n$. We shall let R_{-i} denote the $(n-1)$-tuple $R_{-i} \equiv (R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, u_n) \in [\mathbb{D}^c]^{n-1}$.

An *admissible* domain $\mathbb{D}$ is a subset of $\mathbb{D}^c$. A *Social Choice Function (SCF)* is a map $F : [D]^n \rightarrow \Delta$. We will let $F_i(R_i, R_{-i})$ denote the allocation to agent i at the profile (R_i, R_{-i}) under the SCF F.

We now introduce some important but standard definitions.

Definition 2.1.1 *A SCF F is* **Manipulable** *by agent i at profile R via $R_i^{'} \in \mathbb{D}$ if $F(R_i^{'}, R_{-i}) P_i F(R)$. It is* **Strategy-Proof** *if it is not manipulable by any agent at any profile. Equivalently F is strategy-proof if $F_i(R)) R_i F_i(R_i^{'}, R_{-i}))$ for all $R_i, R_i^{'} \in \mathbb{D}$, for all $R_{-i} \in [\mathbb{D}]^{n-1}$ and for all $i \in I$.*

In the usual strategic voting model, an agent's preference ordering is private information and F represents the mechanism designer's objectives. If F is strategy-proof, all agent has dominant-strategy incentives to reveal their private information truthfully.

Definition 2.1.2 *An allocation $x \in \Delta$ is* **Pareto-Efficient** *at profile R if there does not exist another allocation $x' \in \Delta$ such that $x_i' R_i x_i$ for all $i \in I$ and $x_j' P_j x_j$ for some $j \in I$.*

Let $PE(R)$ denote the collection of Pareto-Efficient allocations at the profile R.

Definition 2.1.3 *A SCF F is* **Pareto-Efficient** *if $F(R) \in PE(R)$ for all $R \in [\mathbb{D}]^n$.*

[1] For a preference ordering R_i and a vector $x \in \Re^m_+$, the upper contour set of R_i at x is denoted by $UC(R_i, x)$ and is the set $\{z \in \Re^m_+ | z R_i x\}$. Similarly the lower contour set of R_i at x is denoted by $LC(R_i, x)$ and is the set $\{z \in \Re^m_+ | x R_i z\}$. A preference ordering R_i is continuous if $UC(R_i, x)$ and $LC(R_i, x)$ are both closed for all $x \in \Re^m_+$. A preference ordering R_i is strictly convex if $UC(R_i, x)$ is strictly convex for all $x \in \Re^m_{++}$. For $x, z \in \Re^m_+$ by $x > z$ we mean $x_k \geq z_k$ for all $k \in M$ and $x_k > z_k$ for some k. A preference ordering is strictly monotonic in $\Re^m_{++}$ if $x > z$ implies $x P_i z$.

Definition 2.1.4 *A SCF F is* **Non-Bossy** *if, for all $R_i, R_i' \in \mathbb{D}$, $R_{-i} \in [\mathbb{D}]^{n-1}$ and $i \in I$,*

$$[F_i(R_i, R_{-i}) = F_i(R_i', R_{-i})] \Rightarrow [F(R_i, R_{-i}) = F(R_i', R_{-i})]$$

The non-bossiness axiom was introduced by Satterthwaite and Sonnenschein (1981). In a non-bossy SCF, an agent who is unable to change her allocation by a unilateral deviation from a preference profile, is also unable to change the allocation of any other agent by the same deviation. The non-bossiness axiom is particularly useful in characterizing strategy-proof SCFs in environments where agent can be indifferent across allocations. It has been widely used in the literature.[2]

An important and familiar SCF is *dictatorship*.

Definition 2.1.5 *A SCF F is* **Dictatorial** *if there exists an agent i such that for all $R \in [\mathbb{D}]^n$*

$$F_i(R) = \Omega$$

.

The dictatorial SCF gives all resources to the same agent at all preference profiles. It is of course, both strategy-proof and Pareto-efficient but ethically unsatisfactory. Serizawa and Weymark (2003) introduce a condition that ensures that all agents receive a minimal bundle of goods. In the definition below $||\cdot||$ denotes the Euclidean norm.

Definition 2.1.6 *A SCF F satisfies the* **Minimum Consumption Guarantee (MCG)** *axiom if there exists an $\epsilon > 0$ such that for all profiles $R \in [\mathbb{D}]^n$ and all $i \in I$,*

$$||F_i(R)|| \geq \epsilon$$

[2]See for example S.Pápai (2000), Svensson (1999), Barberà and Jackson (1995). For a review see Barberà (2010).

2.2 Quasi-linear Domains

Quasi-linear preferences are preference orderings R_i that can be represented by utility functions of the form $u_i(x) = v_i(x_{i1}, \ldots, x_{im-1}) + x_m$.[3] The use of quasi-linear preferences is pervasive in economic theory. In this chapter, we restrict attention to a small sub-class of quasi-linear preferences. These preferences are represented by utility functions of the following form:

$$u_i(x_i, y_i; \theta_i) = \theta_i\{\sqrt{x_{i1}} + \ldots + \sqrt{x_{i,m-1}}\} + y_i \tag{2.1}$$

where $\theta_i > 0$. Note that these quasi-linear preferences are classical.

For notational convenience we denote the m^{th} good by y. We denote the set of preferences above by $\mathbb{D}^q$. Note that all preferences from $\mathbb{D}^q$ are represented by a parameter θ_i. Hence a preference profile in $[\mathbb{D}^q]^n$ can be represented by an n-tuple $\theta \equiv (\theta_1, \theta_2, \ldots, \theta_n)$. The definitions stated in the preceding section are applicable for $\mathbb{D}^q$ with the profiles being written as θ. An allocation at profile θ is denoted by $(x(\theta), y(\theta))$, where $(x(\theta), y(\theta)) = ((x_1(\theta), y_1(\theta)), (x_2(\theta), y_2(\theta)), \ldots, (x_n(\theta), y_n(\theta)))$ and $x_i(\theta) = (x_{i1}(\theta), \ldots, x_{im-1}(\theta))$ for all $i \in I$. The set of Pareto-efficient allocations at profile θ will now be denoted by $PE(\theta)$.

All the results presented in this chapter are first proved for the domain $\mathbb{D}^q$ and then extended to any domain that includes $\mathbb{D}^q$. In particular the results holds for $\mathbb{D}^c$.

In the next two subsections, we present some critical results relating to Pareto-efficiency and strategy-proofness of F over the domain $\mathbb{D}^q$.

2.2.1 Pareto-Efficiency in $\mathbb{D}^q$

We make an observation regarding Pareto-Efficient allocations in $\mathbb{D}^q$.

Proposition 2.2.1: If $(x^*(\theta), y^*(\theta)) \in PE(\theta)$ then it satisfies the following properties:

[3] A preference ordering R_i is represented by the utility function $u_i : \Re^m_+ \to \Re$ if, for all $x, x' \in \Re^m_+$, $xR_ix' \Leftrightarrow u_i(x) \geq u_i(x')$.

- If $x^*_{ij}(\theta) = 0$ for some $j \in \{1, \ldots, m-1\}$ then $x^*_{ij}(\theta) = 0$ for all $j \in \{1, \ldots, m-1\}$
- If $x^*_{ij}(\theta) > 0$ for some $j \in \{1, \ldots, m-1\}$ then $\frac{x^*_{ij}(\theta)}{x^*_{ij'}(\theta)} = \frac{\Omega_j}{\Omega_{j'}}$ for all $j' \in \{1, \ldots, m-1\}$.

The proof of the result is contained in the Appendix 2.6. According to it, every Pareto-efficient allocation has the feature that every agent i receives all goods from 1 through $m-1$ in fixed proportions independently of θ_i. This suggests a reduction of the problem from an m-good to a two-good model. The utility of agent i from a Pareto-efficient allocation $(x^*_1, \ldots, x^*_{m-1}, y)$ in the m-good model is

$$u_i((x^*_{i1}, \ldots, x^*_{im-1}, y^*_i; \theta_i)) = \theta_i \Big[1 + \sum_{j \in M \setminus \{1\}} \sqrt{\frac{\Omega_j}{\Omega_1}}\Big] \sqrt{x^*_{i1}} + y^*_i.$$

Now consider a two-good model with goods x_1 and y with endowments Ω_1 and Ω_m respectively. Since $\theta_i[1+\sum_{j \in M \setminus \{1\}} \sqrt{\frac{\Omega_j}{\Omega_1}}]$ is a positive real number, it follows that the allocation (x^*_1, y^*) is Pareto-efficient in the two-good economy in the domain $\mathbb{D}^q$ for the profile δ where $\delta_i = \theta_i[1+\sum_{j \in M \setminus \{1\}} \sqrt{\frac{\Omega_j}{\Omega_1}}]$. Now consider a SCF F which is strategy-proof and Pareto-efficient in the m-good economy. We can construct a two-good SCF $\bar{F}$ from F as follows: for every m-good profile θ,

$$[F(\theta) = (x_1, \ldots, x_{m-1}, y)] \Rightarrow [\bar{F}(\delta) = (x_1, y)]$$

where δ is defined as above. By our earlier arguments, $\bar{F}$ is Pareto-efficient. It is easily verified that $\bar{F}$ is strategy-proof. For every strategy-proof and Pareto-efficient SCF in the m-good model, there is an "equivalent" (in the sense above) strategy-proof and Pareto-efficient SCF in the two-good model. Henceforth, we restrict attention to the two-good model and the results generalize in an obvious way to the m-good case.

In what follows, we consider two goods x and y and utility functions of the form $u_i(x_i, y_i; \theta_i) = \theta_i \sqrt{x_i} + y_i$. For each profile $\theta \in \Re^n_{++}$, $(x^*(\theta), y^*(\theta)) \in \Re^{2n}_+$ represents an allocation in $PE(\theta)$. Without loss of generality we set total endowments of both the goods at 1. We note that the domain we consider is "narrow" in a specific technical sense. In particular, it is a single-crossing domain and therefore it does not admit concavification. We do not employ concavification arguments used extensively in this literature (Serizawa and Weymark (2003), Serizawa (2006), Zhou (1991), Hashimoto (2008), Barberà and Jackson (1995)). Thus when θ_i is changed indifference curves can only intersect. The following proposition provides necessary conditions for allocations to be Pareto-efficient in the two-good model described above.

Proposition 2.2.2: If $(x^*(\theta), y^*(\theta)) \in PE(\theta)$ then it satisfies the following conditions.

(P1) If $x_i^*(\theta) < \frac{\theta_i^2}{\sum\limits_{k \in I} \theta_k^2}$ for some $i \in I$ then $y_i^*(\theta) = 0$.

(P2) If $x_i^*(\theta) > \frac{\theta_i^2}{\theta_i^2 + \min_{k \neq i} \theta_k^2}$ for some $i \in I$ then $y_i^*(\theta) = 1$.

The proof of the Proposition is contained in the Appendix 2.6. It is well-known that in quasi-linear domains that if x^* solves $\max_{x_1,\ldots,x_n} \sum_{i \in I} \theta_i \sqrt{x_i}$ subject to the resource constraint on x, then any allocation of good y together with x^*, is a Pareto-efficient allocation. For instance, $\left(\frac{\theta_i^2}{\sum_{k \in I} \theta_k^2}, \ldots, \frac{\theta_n^2}{\sum_{k \in I} \theta_k^2} \right)$ solves $\max_{x_1,\ldots,x_n} \sum_{i \in I} \theta_i \sqrt{x_i}$ subject to $\sum_{i \in} x_i = 1$. We say that agent i is *constrained* at θ if $x_i(\theta) < \frac{\theta_i^2}{\sum_{k \in I} \theta_k^2}$. According to condition $P1$, a constrained agent must not get a positive amount of good y. According to $P2$, any agent i whose x_i exceeds a certain bound, must get the entire amount of good y.

2.2.2 Strategy-proofness in $\mathbb{D}^q$

In this subsection, we prove some preliminary results regarding strategy-proof and Pareto-efficient SCFs over the domain $\mathbb{D}^q$. The first two results characterize strategy-proofness in this domain and are counterparts of the results in Myerson (1981) in the context of auction design.

Lemma 2.2.1 *Consider a strategy-proof SCF, $F : [\mathbb{D}^q]^n \to \Delta$. Then, for each i and for all θ_i, θ_i' with $\theta_i' > \theta_i$ and for all θ_{-i},*

$$x_i(\theta_i', \theta_{-i}) \geq x_i(\theta_i, \theta_{-i})$$

where $F(\theta) = (x(\theta), y(\theta))$ *and* $F((\theta_i', \theta_{-i})) = (x(\theta_i', \theta_{-i}), y(\theta_i', \theta_{-i}))$.

Proof: By the strategy-proofness of F, we have the following: for all, θ_i, θ_i' and θ_{-i},

$$\theta_i x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} + y_i(\theta_i, \theta_{-i}) \geq \theta_i x_i(\theta_i', \theta_{-i})^{\frac{1}{2}} + y_i(\theta_i', \theta_{-i}) \text{ and}$$

$$\theta_i' x_i(\theta_i', \theta_{-i})^{\frac{1}{2}} + y_i(\theta_i', \theta_{-i}) \geq \theta_i' x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} + y_i(\theta_i, \theta_{-i}).$$

Adding the two inequalities above, we obtain

$$\left[x_i(\theta_i', \theta_{-i})^{\frac{1}{2}} - x_i(\theta_i, \theta_{-i})^{\frac{1}{2}}\right][\theta_i' - \theta_i] \geq 0.$$

Therefore, $\theta_i' > \theta_i$ implies $x_i(\theta_i', \theta_{-i})^{\frac{1}{2}} \geq x_i(\theta_i, \theta_{-i})^{\frac{1}{2}}$, for all θ_{-i} from which the Lemma follows. ■

Lemma 2.2.2 *Consider a strategy-proof SCF,* $F : [\mathbb{D}^q]^n \to \Delta$. *Then, for each* i *and for all* $\theta_i \in [a_i,\ b_i] \subset \Re_{++}$ *and* θ_{-i},

$$u_i(F_i(\theta_i, \theta_{-i}); \theta_i) = u_i(F_i(a_i, \theta_{-i}); a_i) + \int_{a_i}^{\theta_i} x_i(t_i, \theta_{-i})^{1/2} dt_i.$$

Proof: Using strategy-proofness of F it follows that for all $i \in I$ and for all $\theta_i \in \Re_{++}$,

$$\begin{aligned} u_i(F_i(\theta_i, \theta_{-i}); \theta_i) &= \max_{z \in \Re_{++}} \left\{\theta_i x_i(z, \theta_{-i})^{1/2} + y_i(z, \theta_{-i})\right\} \\ &= \max_{z \in [a_i,\ b_i]} \left\{\theta_i x_i(z, \theta_{-i})^{1/2} + y_i(z, \theta_{-i})\right\}. \end{aligned}$$

Since $u_i(F_i(\theta_i, \theta_{-i}); \theta_i)$ is a maximum of a family of affine functions $u_i(F_i(\theta_i, \theta_{-i}); \theta_i)$ is a convex function in θ_i for all θ_{-i}. Therefore $u_i(F_i(\theta_i, \theta_{-i}); \theta_i)$ is an absolutely continuous function in $[a_i,\ b_i]$. Therefore, $\frac{d}{d\theta_i} u_i(F_i(\theta_i, \theta_{-i}); \theta_i) = x_i(\theta_i, \theta_{-i})^{1/2}$ almost everywhere. The result follows by applying the Fundamental Theorem of Calculus. ■

In the next proposition, we show that if F is strategy-proof, then there cannot exist a $S \subset I$, $|S| \geq 2$ and a neighborhood of profiles where F picks allocations such that no agent from S is constrained and agents not in S are allocated zero of both the goods in that neighborhood. We let $N_\epsilon(\theta)$ denote a open neighborhood of θ with radius ϵ.

Proposition 2.2.3: Consider a strategy-proof SCF $F : [\mathbb{D}^q]^n \to \Delta$. Then there does not exist $S \subset I$, $|S| \geq 2$, and a neighborhood $N_\epsilon(\theta')$ such that, $F(\theta) = (x(\theta), y(\theta))$ satisfies the following properties: for all $\theta \in N_\epsilon(\theta)$,

$$\begin{aligned} x_i(\theta) &= \frac{\theta_i^2}{\sum_{k\in S}\theta_k^2}\ \forall i \in S \\ \sum_{i\in S} y_i(\theta) &= 1 \end{aligned}$$

Proof: Suppose that the claim is false, i.e. there exists $S \subset I$, $|S| \geq 2$ and a neighborhood $N_\epsilon(\theta')$ such that for all $\theta \in N_\epsilon(\theta)$, $x_i(\theta) = \frac{\theta_i^2}{\sum_{k\in S}\theta_k^2}$ for all $i \in S$ and $\sum_{i\in S} y_i(\theta) = 1$.

Applying Lemma 2.2.2, it follows that for each $\theta \in N_\epsilon(\theta')$ and each $i \in S$,

$$u_i(F_i(\theta_i, \theta_{-i}); \theta_i) = u_i(F_i(a_i, \theta_{-i}); a_i) + \int_{a_i}^{\theta_i} \left[\frac{t_i^2}{\sum\limits_{k\in S\setminus\{i\}} \theta_k^2 + t_i^2}\right]^{\frac{1}{2}} dt_i \tag{2.2}$$

Substituting $u_i(F_i(\theta_i, \theta_{-i}); \theta_i)$ with $\theta_i \left[\frac{\theta_i^2}{\sum\limits_{k\in S}\theta_k^2}\right]^{\frac{1}{2}} + y_i(\theta_i, \theta_{-i})$ on the LHS and letting $h_i(\theta_{-i}) \equiv a_i \left[\frac{a_i^2}{a_i^2 + \sum\limits_{k\in S\setminus i}\theta_k^2}\right]^{\frac{1}{2}} + y_i(a_i, \theta_{-i})$ on the RHS of Equation 2.2, we obtain,

$$\theta_i \left[\frac{\theta_i^2}{\sum\limits_{k\in S}\theta_k^2}\right]^{\frac{1}{2}} + y_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) + \int_{a_i}^{\theta_i} \left[\frac{t_i^2}{\sum\limits_{k\in S\setminus\{i\}} \theta_k^2 + t_i^2}\right]^{\frac{1}{2}} dt_i \tag{2.3}$$

Now summing Equation 2.3 across i and noting that $\sum\limits_{i\in S} y_i(\theta) = 1$ we obtain,

$$\left[\sum_{i\in S}\theta_i^2\right]^{\frac{1}{2}} + 1 - \sum_{i\in S}\int_{a_i}^{\theta_i} \frac{t_i}{\left[\sum\limits_{k\in S\setminus\{i\}} \theta_k^2 + t_i^2\right]^{\frac{1}{2}}} dt_i = \sum_{i\in S} h_i(\theta_{-i}). \tag{2.4}$$

Solving for the integrals in the LHS of Equation 2.4 and simplifying further, we get

$$(1-n)\left[\sum_{i\in S}\theta_i^2\right]^{\frac{1}{2}}+1+\sum_{i\in S}\left[\sum_{k\in S\setminus\{i\}}\theta_k^2+a_i^2\right]^{\frac{1}{2}}=\sum_{i\in S}h_i(\theta_{-i}). \tag{2.5}$$

The LHS of Equation 2.5 is an infinitely differentiable function in $\Re_{++}^{|S|}$. Notice that it's $|S|^{\text{th}}$ order cross-partial derivative is $c(|S|)\,(-1)^{(|S|)}\left(\prod_{i\in S}\theta_i\right)\left(\sum_{i\in S}\theta_i^2\right)^{\frac{-(2|S|-1)}{2}}$ where $c(|S|)$ is a constant not equal to zero for any value of $|S|$. However, the $|S|^{\text{th}}$ order cross-partial derivative of the right hand side of vanishes at all θ. We have a contradiction. ■

Proposition 2.2.3 rules out the existence of strategy-proof and Pareto-efficient SCFs and $S \subset I$ and a neighborhood that give all agents strictly positive amounts of both goods to all agents is S and zero of both the goods to all agents outside S. We formalize this below.

Definition 2.2.1 *The SCF $F:\mathbb{D}^n \to \Delta$ satisfies* **S-interiority** *for $S \subset I$, $|S| \geq 2$, if there exists a neighborhood of profiles $N_\epsilon(\theta')$ such that for all $\theta \in N_\epsilon(\theta')$, we have $x_i(\theta), y_i(\theta) > 0$ for all $i \in S$ and $(x_i(\theta), y_i(\theta)) = (0,0)$ for all $i \notin S$, where $F(\theta) = (x(\theta), y(\theta))$.*

Proposition 2.2.4: Let $F:[\mathbb{D}^q]^n \to \Delta$ be a strategy-proof and Pareto-efficient SCF. Then F does not satisfy S-interiority for any S.

Proof: Let F be strategy-proof, Pareto-efficient. Suppose F satisfies S-interiority, i.e. there exists a neighborhood of profiles $N_\epsilon(\theta')$ such that for all $\theta \in N_\epsilon(\theta')$, we have $x_i(\theta), y_i(\theta) > 0$ for all $i \in S$ where $F(\theta) = (x(\theta), y(\theta))$. According to $P1$ in Proposition 2.2.2, we must have $x_i(\theta) = \frac{\theta_i^2}{\sum_{k\in S}\theta_k^2}$ for all $i \in S$ and all $\theta \in N_\epsilon(\theta')$. But now we have a contradiction to Proposition 2.2.3. ■

Remark 2.2.1: Hurwicz and Walker (1990) prove a result (their Theorem 3) similar to our Proposition 2.2.4. They consider a more general class of quasi-linear utility functions.

2.3 Minimum Consumption Guarantees

In this section we provide a simple proof of a logically independent variant of the main result of Serizawa and Weymark (2003). In particular we show that any strategy-proof and Pareto-efficient SCF defined on a domain that is a superset of $\mathbb{D}^q$ violates the MCG axiom. Thus a strategy-proof and Pareto-efficient SCF defined on the domain of classical preferences violates the MCG axiom.

Theorem 2.3.1 *Let $\mathbb{D}$ be an arbitrary domain such that $\mathbb{D}^q \subset \mathbb{D}$. Let $F : \mathbb{D}^n \to \Delta$ be a strategy proof and Pareto-efficient SCF. Then F does not satisfy MCG.*

Proof: It suffices to prove the result for a strategy-proof and Pareto-efficient SCF F : $[\mathbb{D}^q]^n \to \Delta$. Let F be such a SCF. We first establish the following result.

Lemma 2.3.1 *Let θ be a profile such that $x_i(\theta) < \frac{\theta_i^2}{\sum\limits_{k \in I} \theta_k^2}$, i.e agent i is constrained. Then $y_i(\theta_i', \theta_{-i}) < \theta_i$ whenever $\theta_i' < \theta_i$.*

Proof: Suppose not, i.e. let for some $\theta_i' < \theta_i$, $y_i(\theta_i', \theta_{-i}) \geq \theta_i$. Now, by strategy-proofness and the fact that $y_i(\theta_i, \theta_{-i}) = 0$, we have

$$\theta_i x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} \geq \theta_i x_i(\theta_i', \theta_{-i})^{\frac{1}{2}} + y_i(\theta_i', \theta_{-i}),$$

Hence

$$\theta_i[x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} - x_i(\theta_i', \theta_{-i})^{\frac{1}{2}}] \geq y_i(\theta_i', \theta_{-i})$$

Since $\theta_i > 0$,

$$[x_i(\theta_i, \theta_{-i})^{\frac{1}{2}} - x_i(\theta_i', \theta_{-i})^{\frac{1}{2}}] \geq \frac{y_i(\theta_i', \theta_{-i})}{\theta_i} \geq 1 \tag{2.6}$$

Since $x_i(\theta_i, \theta_{-i}), x_i(\theta_i', \theta_{-i}) \leq 1$, inequality 2.6 can be satisfied only if $x_i(\theta_i, \theta_{-i}) = 1$. However $x_i(\theta) < \frac{\theta_i^2}{\sum\limits_{k \in I} \theta_k^2} < 1$ leading to a contradiction. ■

Returning to the proof of the Theorem, pick $0 < \epsilon < \sqrt{2}$. We show existence of an agent i and a profile $(\theta_i^{'}, \theta_{-i})$ such that

$$||(x_i(\theta_i^{'}, \theta_{-i}), y_i(\theta_i^{'}, \theta_{-i}))|| < \epsilon.$$

Consider the open set $O = \prod_{j=1}^{N}(0,\ \frac{\epsilon}{\sqrt{2}})$. By Proposition 2.2.3 we know that there is a profile $(\theta_i, \theta_{-i}) \in O$ and an agent i such that $y_i(\theta_i, \theta_{-i}) = 0$ and $x_i(\theta_i, \theta_{-i}) < \frac{\theta_i^2}{\sum_{k \in I} \theta_k^2}$. By Lemma 2.3.1 and the choice of ϵ, we have the following: for any $\theta_i^{'} < \theta_i$, $y_i(\theta_i^{'}) < \theta_i < \frac{\epsilon}{\sqrt{2}} < 1$. Applying $P2$ in Proposition 2.2.2, we infer that $x_i(\theta_i^{'}, \theta_{-i}) \leq \frac{\theta_i^{'2}}{\theta_i^{'2} + \min_{j \neq i} \theta_j^2}$ for all $\theta_i^{'} < \theta_i$. Observe that the RHS of the inequality above converges to zero as $\theta_i^{'}$ converges to zero. Hence, $\lim_{\theta_i \to 0}\ x_i(\theta_i, \theta_{-i}) = 0$. Therefore, there exists $\theta_i^{'} < \theta_i$ such that $x_i(\theta_i^{'}, \theta_{-i}) < \frac{\epsilon}{\sqrt{2}}$ and $y_i(\theta_i^{'}, \theta_{-i}) < \frac{\epsilon}{\sqrt{2}}$.

Hence,

$$\sqrt{(x_i(\theta_i^{'}, \theta_{-i}))^2 + (y_i(\theta_i^{'}, \theta_{-i}))^2} < \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2} = \epsilon$$

i.e.

$$||(x_i(\theta_i^{'}, \theta_{-i}), y_i(\theta_i^{'}, \theta_{-i}))|| < \epsilon.$$

■

Remark 2.3.1: Our result is different from its counterpart in Serizawa and Weymark (2003) because of the differences in the domains considered. They use the homothetic preference domain while we use a sub-domain of quasi-linear preferences. The arguments in Serizawa and Weymark (2003) are geometric while ours are analytical. Moreover, we are able to explicitly construct a set of profiles where minimum consumption guarantee fails.

2.4 Dictatorship in Classical Exchange Economies

In this section, we prove dictatorship results. In Section 2.4.1, we show dictatorship in the case of two agents. In Section 2.4.2 we extend the result to the case of more than two agents with additional hypotheses on the social choice function.

2.4.1 The $n = 2$ case

Our goal in this subsection is to prove that strategy-proofness and Pareto-efficiency implies dictatorship. In fact, we establish a more precise result which establishes dictatorship for *sub-domains* of $\mathbb{D}^q$.

Let $\alpha > 0$. We let the domain $\overline{\mathbb{D}}^q_\alpha$ consist of all preference orderings that can be represented by utility functions of the form $u_i(x_i, y_i; \theta_i) = \theta_i\{\sqrt{x_{i1}} + \ldots + \sqrt{x_{i,m-1}}\} + y_i$ where $\theta_i \geq \alpha$. Similarly $\underline{\mathbb{D}}^q_\alpha$ consists of all preference orderings that can be represented by utility functions of the form $u_i(x_i, y_i; \theta_i) = \theta_i\{\sqrt{x_{i1}} + \ldots + \sqrt{x_{i,m-1}}\} + y_i$ where $0 < \theta_i \leq \alpha$.

Theorem 2.4.1 *Let $\mathbb{D}$ be an arbitrary domain such that either $\overline{\mathbb{D}}^q_\alpha \subset \mathbb{D}$ or $\underline{\mathbb{D}}^q_\alpha \subset \mathbb{D}$ for some α. Let $F : \mathbb{D}^2 \to \Delta$ be a strategy-proof and Pareto-efficient SCF. Then F is dictatorial.*

Proof: We first consider the case of a strategy-proof and Pareto-efficient SCF $F : [\overline{\mathbb{D}}^q_\alpha]^2 \to \Delta$ for some α.

Let $I = \{i, j\}$, $\alpha > 0$ and let $F : [\overline{\mathbb{D}}^q_\alpha]^2 \to \Delta$ be a strategy-proof and Pareto-efficient SCF defined over this society. We will show that either agent i or agent j is a dictator.

It follows from Proposition 2.2.3 that there exists a profile, say $\theta^* \in [\overline{\mathbb{D}}^q_\alpha]^2$ and an agent, say j such that j is constrained at θ^*, i.e. $x_j(\theta^*) < \frac{\theta_j^{*2}}{\theta_i^{*2} + \theta_j^{*2}}$ and $y_j(\theta^*) = 0$. Suppose $x_j(\theta^*) = 0$. We claim that in this case i is a dictator. To see this consider an arbitrary profile θ. By strategy-proofness of F, $F_j(\theta_i^*, \theta_j) = (0, 0)$; otherwise j manipulates at θ^* via θ_j. Again by strategy-proofness of F, $F_j(\theta_i, \theta_j) = (0, 0)$; otherwise i manipulates at (θ_i, θ_j) via θ_i^*. Hence i is a dictator.

Assume therefore that $x_j(\theta^*) > 0$. We know that $x_i(\theta^*) > \frac{\theta_i^{*2}}{\theta_i^{*2}+\theta_j^{*2}}$ and $y_i(\theta^*) = 1$. Now consider θ_i such that $x_i(\theta^*) = \frac{\theta_i{}^2}{\theta_i{}^2+\theta_j^{*2}}$. Note that $\theta_i > \theta_i^*$. It follows from Lemma 2.2.1 that $x_i(\theta_i, \theta_j^*) \geq x_i(\theta^*)$. Suppose $x_i(\theta_i, \theta_j^*) > x_i(\theta^*)$. In order to prevent i from manipulating F at θ^* via θ_i, we must have $y_i(\theta_i, \theta_j^*) < 1$. Therefore $x_i(\theta_i, \theta_j^*) > \frac{\theta_i{}^2}{\theta_i{}^2+\theta_j^{*2}}$ and $y_i(\theta_i, \theta_j^*) < 1$. Since $F(\theta_i, \theta_j^*)$ is Pareto-efficient at (θ_i, θ_j^*), we have a contradiction to $P2$ in Proposition 2.2.2. Therefore $x_i(\theta_i, \theta_j^*) = \frac{\theta_i{}^2}{\theta_i{}^2+\theta_j^{*2}}$ and $y_i(\theta_i, \theta_j^*) = 1$. In fact, we will assume without loss of generality that $x_i(\theta^*) = \frac{\theta_i^{*2}}{\theta_i^{*2}+\theta_j^{*2}}$ and $y_i(\theta^*) = 1$.

Since $x_j(\theta^*) > 0$, we can pick $\bar{x}_j$ such that $0 < \bar{x}_j < x_j(\theta^*)$. Moreover, by the Intermediate Value Theorem, we can find θ_j' such that $\theta_j' x_j(\theta^*)^{\frac{1}{2}} = \theta_j' \bar{x}_j^{\frac{1}{2}} + 1$. Clearly $\theta_j' > \theta_j^*$. Let θ_i' be such that $\bar{x}_i = \frac{\theta_i'^2}{\theta_i'^2+\theta_j'^2}$. (Note that $\bar{x}_i + \bar{x}_j = 1$).

Let $F(\theta') = z$. We shall argue all choices for z lead to contradiction.

Case 1: Suppose $z = ((x_i, y_i), (x_j, y_j))$ such that $x_j \leq \frac{\theta_j'^2}{\theta_i'^2+\theta_j'^2} = \bar{x}_j$ and $y_j < 1$. By our choice of θ_j', we have $\theta_j' x_j(\theta^*)^{\frac{1}{2}} = \theta_j' \bar{x}_j^{\frac{1}{2}} + 1 > \theta_j' x_j^{\frac{1}{2}} + y_j$. Hence $u_j(F_j(\theta^*); \theta_j') > u_j(F_j(\theta'); \theta_j')$. Now choose $\theta_j'' > \theta_j'$ such that $\frac{\theta_j''^2}{\theta_i'^2+\theta_j''^2} > x_j(\theta^*) = \frac{\theta_j^{*2}}{\theta_i^{*2}+\theta_j^{*2}}$. Note that $\theta_j'' > \theta_j' > \theta_j^*$. Now Lemma 2.2.1 implies that $x_j(\theta_i^*, \theta_j'') \geq x_j(\theta^*)$. If this inequality is strict, then j will manipulate at θ^* via θ_j'' since $y_j(\theta^*) = 0$ by assumption. Hence $x_j(\theta_i^*, \theta_j'') = x_j(\theta^*)$. Now $\theta_i' > \theta_i^*$, so that $x_i(\theta_i', \theta_j'') \geq x_i(\theta_i^*, \theta_j'')$ (Lemma 2.2.1). But $x_i(\theta_i^*, \theta_j'') = x_i(\theta^*) = \frac{\theta_i^{*2}}{\theta_i^{*2}+\theta_j^{*2}} > \frac{\theta_i'^2}{\theta_i'^2+\theta_j''^2}$. Now suppose $x_i(\theta_i', \theta_j'') > x_i(\theta_i^*, \theta_j'')$. By $P2$ in Proposition 2.2.2, $y_i(\theta_i', \theta_j'') = 1$. But then agent i manipulates F at (θ_i^*, θ_j'') via θ_i'. Hence $x_i(\theta_i', \theta_j'') = x_i(\theta_i^*, \theta_j'') = x_i(\theta^*)$ and $x_j(\theta_i', \theta_j'') = x_j(\theta_i^*, \theta_j'') = x_j(\theta^*)$. Moreover strategy-proofness also implies $y_j(\theta_i', \theta_j'') = y_j(\theta_i^*, \theta_j'') = y_j(\theta^*) = 0$.

Truth-telling at θ' gives player j (x_j, y_j). Lying gives her $F_j(\theta^*)$. Since $u_j(F_j(\theta^*); \theta_j') > u_j(F_j(\theta'); \theta_j')$ by construction, j will manipulate.

Case 2: $z = ((x_i, y_i), (x_j, y_j))$ such that $x_j(\theta^*) > x_j \geq \bar{x}_j$ and $y_j = 1$.

Pick $\tilde{x}_j$ such that $x_j(\theta^*) > \tilde{x}_j > x_j$. Choose $\theta_j'' > \theta_j'$ such that $\theta_j'' x_j(\theta^*)^{\frac{1}{2}} = \theta'' \tilde{x}_j^{\frac{1}{2}} + 1$. Once again, the existence of θ_j'' follows from the Intermediate Value Theorem. Now pick θ_i'' such that $x_i < \frac{\theta_i''^2}{\theta_i''^2+\theta_j''^2}$. It follows from earlier arguments involving Lemma 2.2.1 and

Proposition 2.2.2 that $F(\theta'') = z$. Now observe that by considering θ^* as before, replacing θ' by θ'' and $\bar{x}_j$ by $\tilde{x}_j$, we can apply the arguments of Case 1 to conclude that $F(\theta'') \neq z$. But this contradicts our assumption that $F(\theta') = z$.

Case 3: $z = ((x_i, y_i), (x_j, y_j))$ such that $x_j \geq x_j(\theta^*)$ and $y_j = 1$.

We claim that $F(\theta_i', \theta_j^*) = z$. Since $\theta_j^* < \theta_j'$, Lemma 2.2.1 implies that $x_j(\theta_i', \theta_j^*) \leq x_j(\theta')$. Suppose the inequality is strict. Since $y_j(\theta') = 1$, agent will manipulate at (θ_i', θ_j^*) via θ_j'. Hence $F(\theta_i', \theta_j^*) = z$. Now observe that $x_i(\theta_i', \theta_j^*) \leq x_i(\theta^*)$ while $0 = y_i(\theta_i', \theta_j^*) < y_i(\theta^*) = 1$. Hence agent i manipulates at (θ_i', θ_j^*) via θ_i^*.

Cases 1-3 exhaust all possibilities. Therefore F is dictatorial.

We now consider the case where $F : [\mathbb{D}_\alpha^q]^2 \to \Delta$ is a strategy-proof and Pareto-efficient SCF. The following Claim is an important intermediate step.

Claim 1: Let $F : [\mathbb{D}_\alpha^q]^2 \to \Delta$ be a strategy-proof and Pareto-efficient SCF. Consider a profile $(\theta_i^*, \theta_j^*) \in [\mathbb{D}_\alpha^q]^2$ such that $x_i(\theta_i^*, \theta_j^*) = \frac{\theta_i^{*2}}{\theta_i^{*2}+\theta_j^{*2}}$ and $y_i(\theta_i^*, \theta_j^*) = 0$. Then there exists $[a_i, b_i] \times [a_j, b_j] \subset [\mathbb{D}_\alpha^q]^2$ such that $x_j(\theta_i, \theta_j) = \frac{\theta_j^2}{\theta_i^2+\theta_j^2}$ for all $(\theta_i, \theta_j) \in [a_i, b_i] \times [a_j, b_j]$ and for all $j \in I$.

Proof of Claim 1: We proceed in two steps.

Step 1: We have $x_i(\theta_i^*, \theta_j^*) = \frac{(\theta_i^*)^2}{(\theta_i^*)^2+(\theta_j^*)^2}$ and $x_j(\theta_i^*, \theta_j^*) = \frac{(\theta_j^*)^2}{(\theta_i^*)^2+(\theta_j^*)^2}$. By Lemma 2.2.1 and Pareto-efficiency we can choose $a_i < b_i < \theta_i^*$ such that $1 > y_i(a_i, \theta_j^*) > y_i(b_i, \theta_j^*) > 0$ and $x_i(\theta_i, \theta_j^*) = \frac{\theta_i^2}{\theta_i^2+\theta_j^*}$ for all $(\theta_i, \theta_j^*) \in [a_i, b_i] \times \{\theta_j^*\}$.

Step 2: Consider θ_i', θ_i'' in $[a_i, b_i]$ such that $\theta_i' < \theta_i''$. Let $\theta_j^* - \epsilon(\theta_i^{'}) \equiv \sup\{\theta_j | y_j(\theta_i^{'}, \theta_j) = 1\}$ and $\theta_j^* - \epsilon(\theta_i^{''}) \equiv \sup\{\theta_j | y_j(\theta_i^{''}, \theta_j) = 1\}$. Also assume that suprema are attained. Then $\theta_j^* - \epsilon(\theta_i^{'}) \leq \theta_j^* - \epsilon(\theta_i^{''})$.

Suppose Step 2 is false, i.e. the hypothesis of Claim 1 holds but $\theta_j^* - \epsilon(\theta_i^{'}) > \theta_j^* - \epsilon(\theta_i^{''})$. Note that $x_j(\theta_i', \theta_j^* - \epsilon(\theta_i^{'})) = \frac{(\theta_j^* - \epsilon(\theta_i^{'}))^2}{(\theta_i^{'})^2 + (\theta_j^* - \epsilon(\theta_i^{'}))^2}$ and $x_j(\theta_i'', \theta_j^* - \epsilon(\theta_i^{''})) = \frac{(\theta_j^* - \epsilon(\theta_i^{''}))^2}{(\theta_i^{''})^2 + (\theta_j^* - \epsilon(\theta_i^{''}))^2}$ and $y_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{'})) = 1$. Therefore, $F_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{'})) = F_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''}))$. To see this note that $\theta_j^* - \epsilon(\theta_i^{'}) > \theta_j^* - \epsilon(\theta_i^{''})$ implies $\frac{(\theta_j^* - \epsilon(\theta_i^{'}))^2}{(\theta_i^{'})^2 + (\theta_j^* - \epsilon(\theta_i^{'}))^2} > \frac{(\theta_j^* - \epsilon(\theta_i^{''}))^2}{(\theta_i^{'})^2 + (\theta_j^* - \epsilon(\theta_i^{''}))^2}$. Now by Lemma 2.2.1

$y_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''})) \geq y_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{'}))$ and $x_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''})) \leq x_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{'}))$. By Pareto-efficiency $y_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''})) = 1$. Now if $x_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''})) < x_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{'}))$ then agent j can manipulate F at $(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''}))$ via $\theta_j^* - \epsilon(\theta_i^{'})$. Therefore, $F_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{'})) = F_j(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''}))$. Hence, $x_i(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{'})) = \frac{(\theta_i^{'})^2}{(\theta_i^{'})^2 + (\theta_j^* - \epsilon(\theta_i^{'}))^2} = x_i(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''}))$.

Again by Lemma 2.2.1 $x_i(\theta_i^{''}, \theta_j^* - \epsilon(\theta_i'')) \geq x_i(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i''))$ and $y_i(\theta_i^{''}, \theta_j^* - \epsilon(\theta_i^{''})) \leq y_i(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i''))$ since $\theta_i^{''} > \theta_i^{'}$. By Pareto-efficiency $y_i(\theta_i^{''}, \theta_j^* - \epsilon(\theta_i^{''})) = 0 = y_i(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''}))$. However, $x_i(\theta_i^{''}, \theta_j^* - \epsilon(\theta_i^{''})) = \frac{(\theta_i^{''})^2}{(\theta_i^{''})^2 + (\theta_j^* - \epsilon(\theta_i^{''}))^2} > \frac{(\theta_i^{'})^2}{(\theta_i^{'})^2 + (\theta_j^* - \epsilon(\theta_i^{'}))^2} = x_i(\theta_i^{'}, \theta_j^* - \epsilon(\theta_i^{''}))$. This is because, $\frac{(\theta_j^* - \epsilon(\theta_i^{'}))^2}{(\theta_i^{'})^2 + (\theta_j^* - \epsilon(\theta_i^{'}))^2} > \frac{(\theta_j^* - \epsilon(\theta_i^{''}))^2}{(\theta_i^{'})^2 + (\theta_j^* - \epsilon(\theta_i^{''}))^2}$ and $\theta_i' < \theta_i''$. Therefore, agent i will manipulate the SCF at $(\theta_i^{'}, \theta_j - \epsilon(\theta_i^{''}))$ via $\theta_i^{''}$. Hence Step 2 follows.

From Step 1 and Step 2 it follows that $x_j(\theta_i, \theta_j) = \frac{\theta_j^2}{\theta_i^2 + \theta_j^2}$ for all $(\theta_i, \theta_j) \in [a_i, b_i] \times [\theta_j^* - \epsilon(b_i), \theta_j^*]$. Set $a_j \equiv \theta_j^* - \epsilon(b_i)$ and $b_j \equiv \theta_j^*$. Note that if for some $\theta_i \in [a_i, b_i]$, $\sup\{\theta_j | y_j(\theta_i, \theta_j) = 1\}$ is not attained then this means that $x_i(\theta_i, \theta_j) = \frac{\theta_i^2}{\theta_i^2 + \theta_j^2}$ and $x_j(\theta_i, \theta_j) = \frac{\theta_j^2}{\theta_i^2 + \theta_j^2}$ for all $\theta_j < \theta_j^*$. It follows that the set $[a_i, b_i] \times [a_j, b_j]$ specified in Claim 1 can be constructed.

In order for Claim 1 not to contradict Proposition 2.2.3, it must be the case that agent i, if constrained, must get zero amounts of both goods. By our earlier argument, agent j must be a dictator.

Now let $F : \mathbb{D}^2 \rightarrow \Delta$ be strategy-proof and Pareto-efficient SCF where $\overline{\mathbb{D}}_\alpha^q \subset \mathbb{D}$ or $\underline{\mathbb{D}}_\alpha^q \subset \mathbb{D}$ We know from our earlier arguments that F restricted to the domains $\overline{\mathbb{D}}_\alpha^q$ and $\underline{\mathbb{D}}_\alpha^q$ is dictatorial. Let i be the dictator in $\overline{\mathbb{D}}_\alpha^q$, i.e. for all $\theta \in [\overline{\mathbb{D}}_\alpha^q]^2$, we have $F_i(\theta) = (1, 1)$. Pick an arbitrary profile $R \in \mathbb{D}^2$. If $F_i(R_i, \theta_j) \neq (1, 1)$, i will manipulate F at (R_i, θ_j) via θ_i. If $F_j(R) \neq (0, 0)$, agent j will manipulate F at (R_i, θ_j) via R_j. Therefore i is a dictator in F.

■

Remark 2.4.1: In the Theorem above we assumed that $\sup_\alpha\{\overline{\mathbb{D}}_\alpha^q\} = \infty$ and $\inf_\alpha\{\underline{\mathbb{D}}_\alpha^q\} = 0$. However if $\mathbb{D}^* \subset \mathbb{D}^q$ is such that $\sup_\alpha\{\mathbb{D}^*\} < \infty$ and $\inf_\alpha\{\mathbb{D}^*\} > 0$, we can construct a non-dictatorial, strategy-proof and Pareto-efficient SCF. Let $\inf_\alpha\{\mathbb{D}^*\} = \gamma > 0$ and $\inf_\alpha\{\mathbb{D}^*\} = \beta < \infty$. Now note that for all i, $\sup_\alpha\{\frac{\theta_i^2}{\theta_i^2 + \theta_j^2} | (\theta_i, \theta_j) \in [\mathbb{D}^*]^2\} = \frac{\beta^2}{\gamma^2 + \beta^2} < \infty$

and $\inf_\alpha\{\frac{\theta_i^2}{\theta_i^2+\theta_j^2}|(\theta_i,\theta_j)\in[\mathbb{D}^*]^2\} = \frac{\gamma^2}{\gamma^2+\beta^2} > 0$. Define $F(\theta) = ((\frac{\gamma^2}{\gamma^2+\beta^2},0),(\frac{\beta^2}{\gamma^2+\beta^2},1))$ for all $\theta\in[\mathbb{D}^*]^2$. This SCF is strategy-proof because its range is singleton. Pareto-efficiency follows from Proposition 2.2.1.

Remark 2.4.2: When preferences are non-classical, it is possible to construct a non-dictatorial, strategy-proof and Pareto-efficient SCF - see Nicolò (2004).

2.4.2 The $n \geq 3$ Case

In this section we consider the case of more than two agents. This case is different from the two agent case because strategy-proof and Pareto-efficient SCFs need not be dictatorial as shown in Kato and Ohseto (2002). We have shown that every strategy-proof and Pareto-efficient SCF for an arbitrary number of agents defined over a quasi-linear domain, must satisfy a highly restrictive property: in every neighborhood of preference profiles, at least one agent must receive a zero amount of good y. However when there are at least three, agents the identity of the agent who does not receive good y, may depend on the announcements of the other agents. This increases the possible complexity in the behavior of a strategy-proof SCF very dramatically. However, by imposing certain familiar regularity assumptions on SCFs, are able to recover the dictatorship result.

Observe that a SCF $F:[\mathbb{D}^q]^n\rightarrow\Delta$ is a map $F:\Re^n_{++}\rightarrow\Re^{2n}_+$. Therefore **continuity** of F can be defined in a standard way.

Definition 2.4.1 *Let $\mathbb{D}$ be an arbitrary domain such that $\mathbb{D}^q\subset\mathbb{D}$. We say a SCF $F:\mathbb{D}^n\rightarrow\Delta$ is* **q-Continuous** *if the restriction of F to $[\mathbb{D}^q]^n$ is continuous.*

We have already introduced the non-bossiness axiom earlier. Our main result is the following:

Theorem 2.4.2 *Let $\mathbb{D}$ be an arbitrary domain with $\mathbb{D}^q\subset\mathbb{D}$. Let $F:\mathbb{D}^n\rightarrow\Delta$ be a strategy-proof, Pareto-efficient, non-bossy and q-continuous SCF. Then F is dictatorial.*

Proof: Let $F : [\mathbb{D}^q]^n \to \Delta$ be a strategy-proof, Pareto-efficient, non-bossy and continuous SCF. We will show that F is dictatorial. We will first establish two lemmas.

Lemma 2.4.1 *Let θ be an arbitrary profile and let $S = \{j \in I | y_j(\theta) > 0\}$. Let $i \notin S$ be such that $x_i(\theta) > 0$. Then there exists θ_i^* and a neighborhood $N_\epsilon(\theta_i^*, \theta_{-i})$ such that, for all $\theta' \in N_\epsilon(\theta_i^*, \theta_{-i})$, we have $y_k(\theta') > 0$ for all $k \in S \cup \{i\}$.*

Proof: Let θ, i and S be as specified in the statement of the Lemma. By Proposition 2.2.2 we know that $x_i(\theta) \leq \frac{\theta_i^2}{\theta_i^2 + \min_{k \neq i} \theta_k^2}$. Consider a decreasing sequence $\theta_i^r \to 0$ as $r \to \infty$. By Lemma 2.2.1, $x_i(\theta_i^r, \theta_{-i})) \leq x_i(\theta_i, \theta_{-i})$. Suppose $x_i(\theta_i^r, \theta_{-i}) = x_i(\theta_i, \theta_{-i})$ for all r. Clearly $y_i(\theta_i^r, \theta_{-i}) = y_i(\theta_i, \theta_{-i}) = 0$, otherwise i will manipulate. Observe that $\frac{(\theta_i^r)^2}{(\theta_i^r)^2 + \min_{k \neq i} \theta_k^2} \to 0$ as $r \to \infty$. Therefore, $x_i(\theta_i^r, \theta_{-i}) > \frac{(\theta_i^r)^2}{(\theta_i^r)^2 + \min_{k \neq i} \theta_k^2}$ while $y_i(\theta_i^r, \theta_{-i}) = 0$ for r large enough. This contradicts $P2$ in Proposition 2.2.2. Hence $x_i(\theta_i^r, \theta_{-i}) < x_i(\theta_i, \theta_{-i})$ for r large enough which also implies $y_i(\theta_i^r, \theta_{-i}) > 0$ for r large enough. Let $\bar{\theta}_i = \inf_r \{\theta_i^r : y_i(\theta_i^r, \theta_{-i}) = 0\}$. Since $F_i(\bar{\theta}_i, \theta_{-i}) = F_i(\theta_i, \theta_{-i})$, the non-bossiness of F implies that $F(\bar{\theta}_i, \theta_{-i}) = F(\theta_i, \theta_{-i})$. By the continuity of F, there exists $\theta_i^* < \bar{\theta}_i$ and a neighborhood $N_\epsilon(\theta_i^*, \theta_{-i})$ such that for all θ' in the neighborhood, $y_k(\theta') > 0$ for all $k \in S \cup \{i\}$.

■

Lemma 2.4.2 *Let θ be an arbitrary profile and let $S = \{j \in I | y_j(\theta) > 0\}$. Then there exists a neighborhood $N_\epsilon(\theta')$ and $S' \subset I$ with $S \subset S'$ such that for all $\tilde{\theta}$ in the neighborhood, we have $x_i(\tilde{\theta}), y_i(\tilde{\theta}) > 0$ for all $i \in S'$ and $\sum_{i \in S'} x_i(\tilde{\theta}) = \sum_{i \in S'} y_i(\tilde{\theta}) = 1$.*

Proof: Let θ be an arbitrary profile and let $S = \{j \in I | y_j(\theta) > 0\}$. Suppose $\sum_{i \in S} x_i(\theta) = 1$, then the Lemma follows by the continuity of F. Suppose there exists an $i \notin S$ and $x_i(\theta) > 0$ but $y_i(\theta) = 0$. Applying Lemma 2.4.1, it follows that there exists a profile θ' such that for all θ'' in this neighborhood $y_k(\theta'') > 0$ for all $k \in S \cup \{i\}$. Now suppose there exists an agent i' with $i' \notin S \cup \{i\}$ such that $x_{i'}(\theta'') > 0$ and $y_{i'}(\theta'') = 0$ for some θ'' in this neighborhood. Now applying Lemma 2.4.1 again, we can find another neighborhood such that for all profiles θ in this neighborhood $x_k(\theta), y_k(\theta) > 0$ for all $k \in S \cup \{i, i'\}$. Proceeding in this way and noting that the number of agents is finite the desired conclusion follows.

■

We now show that $F : [\mathbb{D}^q]^n \to \Delta$ is dictatorial. In order to see this, suppose that there exists a profile θ and a set S with $|S| \geq 2$ such that $y_i(\theta) > 0$ for all $i \in S$. Then by Lemma 2.4.2, there exists a neighborhood and a set of agents S' with $S \subset S'$ with $x_i(\theta), y_i(\theta) > 0$ for all $i \in S'$ and $\sum_{i \in S'} x_i(\theta) = \sum_{i \in S'} y_i(\theta) = 1$. However, this implies that F satisfies S'-interiority contradicting Proposition 2.2.4. Therefore $|S| = 1$ for all profiles. By Pareto-efficiency this implies that for all profiles θ there exists an agent i such that $F_i(\theta) = (1, 1)$. A simple argument using non-bossiness establishes that F is dictatorial.

Now let $F : \mathbb{D}^n \to \Delta$ be strategy-proof and Pareto-efficient SCF where $\mathbb{D}^q \subset \mathbb{D}$. We know from our earlier arguments that F restricted to the domain $\mathbb{D}^q$ is dictatorial. Let i be the dictator, i.e. for all $\theta \in [\mathbb{D}^q]^n$, we have $F_i(\theta) = (1, 1)$. Pick an arbitrary profile $R \in \mathbb{D}^n$. If $F_i(R_i, \theta_{-i}) \neq (1, 1)$, i will manipulate F at (R_i, θ_{-i}) via θ_i. Note also that for all $j \neq i$ strategy-proofness implies $Fj(R_i, R_j, \theta_{i,j}) = (0, 0)$. By non-bossiness $F_i(R_i, R_j, \theta_{i,j}) = (1, 1)$. By repeating this argument it follows that $F_i(R) = (1, 1)$ so that i is the dictator in F.

■

Remark 2.4.3: Observe that q-continuity is a relatively mild condition because it imposes continuity only on the quasi-linear sub-domain $\mathbb{D}^q$.

Remark 2.4.4: An open question relates to the role that non-bossiness and q-continuity assumptions play in our result. A reasonable conjecture is that strategy-proofness and Pareto-efficiency imply the *extreme-valuedness* of F, i.e. at all profiles, there exists an agent who receives the entire allocation of all goods.

2.5 Conclusion

In this chapter, we have analyzed the structure of strategy-proof and Pareto-efficient social choice functions in classical exchange economies. Our methodological contribution is to focus on a small class of quasi-linear domains and use techniques developed in the context of auction design. This approach yields sharper results on minimum consumption guarantees. It also allows for a dictatorship characterization for arbitrary numbers of agents provided that social choice functions satisfy mild regularity assumptions. These results immediately extend to supersets of quasi-linear domains and therefore apply to the domain of all classical preferences. There are no existing results for strategy-proof and Pareto-efficient social choice functions in the case of more than two agents and our results are therefore the first of their kind.

2.6 Appendix

We provide a proof of Proposition 2.2.1 below.

Proof: Let $(x^*(\theta), y^*(\theta))$ be a Pareto-efficient allocation. Fix an agent i. We first show that if $x^*_{ij}(\theta) = 0$ for some $j \in \{1, \dots, m-1\}$, then $x^*_{ij'}(\theta) = 0$ for all $j' \in \{1, \dots, m-1\}$. Suppose this false, i.e. $x^*_{ij}(\theta) = 0$ but $x^*_{ij'}(\theta) > 0$ for some $j' \in \{1, \dots, m-1\}$. We argue that this allocation is not Pareto-efficient. There must exist an agent i' with an allocation $(x^*_{i'}(\theta), y^*_{i'}(\theta))$ and $x^*_{i'j}(\theta) > 0$. For agents i and i', define $\Omega^{(i,i')}_j \equiv x^*_{ij}(\theta) + x^*_{i'j}(\theta) > 0$ and $\Omega^{(i,i')}_{j'} \equiv x^*_{ij'}(\theta) + x^*_{i'j'}(\theta) > 0$. Fix the the allocation of other agents and other goods and consider the set of Pareto-efficient allocations in the Edgeworth box of agents i and i' with total endowments of j and j' being $\Omega^{(i,i')}_j$ and $\Omega^{(i,i')}_{j'}$ respectively. In this box Pareto-efficient points lie on the diagonal, i.e. by fixing agent i''s *utility level* at $\theta_{i'}[x^{*1/2}_{i'j}(\theta) + x^{*1/2}_{i'j'}(\theta)]$ agent i can be made better off than at $x^*_{ij}(\theta) = 0, x^*_{ij'}(\theta) > 0$. Hence the initial allocation cannot be Pareto-efficient.

To prove the Proposition consider the following optimization problem for agent i

$$Max_{\{x_i,y_i\}_{i=1}^N}\left[\theta_i\sum_{j=1}^{m-1}x_{ij}^{1/2}+y_i\right]$$

$$subject\ \ to\ \ \left[\theta_k\sum_{j=1}^{m-1}x_{jk}^{1/2}+y_k\right]\geq\bar{u}_k,\forall\ \ k\in N\setminus\{i\}, \qquad (\mathbf{P})$$

$$\sum_{i\in N}x_{ij}=\Omega_j\ \ \forall j\in\{1,\ldots,m-1\},\sum_{i\in N}y_i=\Omega_m,$$
$$x_{ij}\geq 0\ \ \forall i\in N,\ \ \forall j\in\{1,\ldots,m-1\}\ \ and\ \ y_i\geq 0\ \ \forall i\in N.$$

If agent i is the only agent who obtains positive amounts of the first $(m-1)$ goods then we are done. So let $T\subseteq N$ (with $|T|>1$ and $i\in T$) be the set of agents who obtain positive amount of the first $(m-1)$ goods. Since for any pair of agents in T the marginal rate of substitution between any two goods j and j' (from the first $(m-1)$ goods) must be equal we get $\frac{(x^*_{ij}(\theta))^{1/2}}{(x^*_{ij'}(\theta))^{1/2}}=\frac{(x^*_{i'j}(\theta))^{1/2}}{(x^*_{i'j'}(\theta))^{1/2}}$. Hence

(A) $\frac{x^*_{ij}(\theta)}{x^*_{ij'}(\theta)}=\frac{x^*_{i'j}(\theta)}{x^*_{i'j'}(\theta)}$ for all $i'\in T\setminus\{i\}$.

(B) $\sum_{i'\in T}x^*_{i'j}(\theta)=\Omega_j$ for all $j\in\{1,\ldots,m-1\}$.

Using (A) and (B) we get

$$\frac{\sum\limits_{i'\in T}x^*_{i'j}(\theta)}{\sum\limits_{i'\in T}x^*_{i'j'}(\theta)}=\frac{\Omega_j}{\Omega_{j'}}\Rightarrow\frac{x^*_{ij}(\theta)+x^*_{ij}(\theta)\left(\sum\limits_{i'\in T\setminus\{i\}}\frac{x^*_{i'j'}(\theta)}{x^*_{ij'}(\theta)}\right)}{\sum\limits_{i'\in T}x^*_{i'j'}(\theta)}=\frac{\Omega_j}{\Omega_{j'}}\Rightarrow\frac{x^*_{ij}(\theta)}{x^*_{ij'}(\theta)}=\frac{\Omega_j}{\Omega_{j'}}.$$

■

We now prove Proposition 2.2.2.

Proof: We proceed in four steps.

Step 1: Consider a two agent economy with agents i and j and an arbitrary total endowment. We prove the following result Fix a profile $\theta\in[\mathbb{D}^q]^2$. If $y^*_i(\theta)>0$ then $x^*_i(\theta)>0$.

Note that a Pareto-efficient allocation is a solution to the following optimization problem:

$$\max_{x_i,y_i} \theta_i x_i^{\frac{1}{2}} + y_i$$

$$\text{s.t. } \theta_j(\Omega_x - x_i)^{\frac{1}{2}} + \Omega_y - y_i \geq \bar{u}_j \text{and}$$

$$x_i \geq 0 \text{ and } y_i \geq 0$$

where $\bar{u}_j$ is a positive number. Now note that by strict monotonicity of the objective function maximum would take place at allocations (x_i^*, y_i^*) such that $\theta_j(\Omega_x - x_i^*)^{\frac{1}{2}} + \Omega_y - y_i^* = \bar{u}_j$. The constraint can be rewritten as, $y_i = \Omega_y - \bar{u}_j + \theta_j(\Omega_x - x_i)^{\frac{1}{2}}$. This is a strictly decreasing function of x_i. Also the level sets of the objective function are strictly decreasing with $\lim_{x_i \to 0} \frac{dy_i}{dx_i} = -\infty$. But the derivative of the function $y_i = \Omega_y - \bar{u}_j + \theta_j(\Omega_x - x_i)^{\frac{1}{2}}$ exists for all $x_i < \Omega_x$. From this it can be argued that the level set of the objective function that meets the constraint at $x_i = 0$ must cut the constraint from below. Thus $y_i^*(\theta) > 0$ and $x_i^*(\theta) = 0$ cannot be a Pareto-efficient allocation. Hence the result follows.

Step 2: Consider the n-agent economy and suppose $(x^*(\theta), y^*(\theta)) \in PE(\theta)$. Fix an agent i. If $y_i^*(\theta) > 0$, then $x_i^*(\theta) > 0$.

Suppose not, that is let a Pareto-efficient allocation be such that $y_i^*(\theta) > 0$ and $x_i^*(\theta) = 0$. Let $x_{i'}^*(\theta) > 0$. Let agent i and i' share $\Omega_1^{(i,i')}$ and $\Omega_2^{(i,i')}$ of good x and good y respectively. Fix the the allocation of other agents. The utility functions of agent i and i' are now of the form $\theta_i x_i(\theta)^{1/2} + y_i(\theta)$ and $\theta_{i'} x_{i'}(\theta)^{1/2} + y_{i'}(\theta)$ respectively. However, from Step 1, we know that Pareto-efficient allocations in the two-agent, two-good model are such that if $x_i^*(\theta) = 0$ then $y_i^*(\theta) = 0$. Therefore by keeping agent i''s utility level fixed at $\theta_{i'}(x_{i'}^*(\theta))^{1/2} + y_{i'}^*(\theta)$ agent i can be made better off with a positive amount of good x. This contradicts our assumption that $(x^*(\theta), y^*(\theta))$ is Pareto-efficient. This proves Step 2.

Step 3: If $(x^*(\theta), y^*(\theta)) \in PE(\theta)$ and for agent $i \in I$, $y_i^*(\theta) > 0$, then $x_i^*(\theta) \geq \frac{\theta_i^2}{\sum\limits_{k \in S} \theta_k^2}$ where $S = \{k \in I \mid x_k^*(\theta) > 0\}$.

Let $(x^*(\theta), y^*(\theta)) \in PE(\theta)$ be such that all agents in the set $S (\subseteq I)$ are allocated a positive amount of good x and all agents in the set $S' (\subseteq I)$ are allocated a positive amount of good y. By Step 2, $S' \subseteq S$. Let $i \in S'$. The Lagrangian for agent i's optimization problem (P) is

$$\begin{aligned} L &= u_i(x_i, y_i; \theta_i) + \sum_{k \in I \setminus \{i\}} \alpha_k[-\bar{u}_k + u_k(x_k, y_k; \theta_k)] \\ &+ \sum_{k \in I} (\beta_{k1} x_i + \beta_{k2} y_i) + \gamma_1 (1 - \sum_{k \in I} x_k) + \gamma_2 (1 - \sum_{k \in I} y_k) \end{aligned}$$

where α_ks',$\beta_{k1} s'$, β_{k2}s', γ_1 and γ_2 are all Lagrange multipliers. The first order conditions and complementary slackness conditions are

$$\frac{\partial L}{\partial x_i} = \frac{\theta_i}{2x_i^{1/2}} + \beta_{i1} - \gamma_1 = 0, \tag{2.7}$$

$$\frac{\partial L}{\partial x_k} = \frac{\theta_k \alpha_k}{2x_k^{1/2}} + \beta_{k1} - \gamma_1 = 0, \quad \forall k \in S \setminus \{i\}, \tag{2.8}$$

$$\frac{\partial L}{\partial y_i} = 1 + \beta_{i2} - \gamma_2 = 0, \tag{2.9}$$

$$\frac{\partial L}{\partial y_k} = \alpha_k + \beta_{k2} - \gamma_2 = 0, \quad \forall k \in S \setminus \{i\}, \tag{2.10}$$

$$\alpha_k \frac{\partial L}{\partial \alpha_k} = \alpha_k \left[-\bar{u}_k + u_k(x_k, y_k; \theta_k)\right] = 0, \quad \forall k \in S \setminus \{i\}, \tag{2.11}$$

$$\sum_{i \in I} x_i = 1, \sum_{i \in I} y_i = 1, \tag{2.12}$$

$$\beta_{k1} x_k = 0, \beta_{k2} y_k = 0, \quad \forall k \in I, \tag{2.13}$$

$$\alpha_k \geq 0, \quad \forall k \in S \setminus \{i\}, \tag{2.14}$$

$$\beta_{ij} \geq 0, \quad \forall i \in I, \quad \forall j \in \{1, 2\}. \tag{2.15}$$

From (2.13) and $y_i(\theta) > 0$ it follows that $\beta_{i2} = 0$ and hence using (2.9) we get $\gamma_2 = 1$. Since $\gamma_2 = 1$ from (2.10) we get $\alpha_k + \beta_{k2} = 1 \; \forall \; k \in S \setminus \{i\}$. By (2.14) and (2.15) we obtain $0 \leq \alpha_k \leq 1 \; \forall \; k \in S \setminus \{i\}$. Since by assumption $x_k > 0$ for all $k \in S$, we have $\beta_{k1} = 0$ for all $k \in S$. Now from (2.7) and (2.8) we have,

$$\frac{\theta_i}{2x_i^{1/2}} = \frac{\alpha_k \theta_k}{2x_k^{1/2}}, \forall k \in S \setminus \{i\}.$$

By squaring both sides and simplifying we obtain,

$$\frac{\theta_i^2}{x_i} = \frac{\alpha_k^2 \theta_k^2}{x_k}, \forall k \in S \setminus \{i\}$$

Hence,

$$x_k(\theta) = x_i(\theta) \frac{\alpha_k^2 \theta_k^2}{\theta_i^2}, \forall k \in S \setminus \{i\},$$

Now from (2.12) we obtain,

$$x_i(\theta) + x_i(\theta) \sum_{k \in S \setminus \{i\}} \frac{\alpha_k^2 \theta_k^2}{\theta_i^2} = 1,$$

since, $\alpha_k \leq 1$, $\forall k \in S \setminus \{i\}$,

$$x_i^*(\theta) = \frac{\theta_i^2}{\theta_i^2 + \sum\limits_{k \in S \setminus \{i\}} \alpha_k^2 \theta_k^2} \geq \frac{\theta_i^2}{\sum\limits_{k \in S} \theta_k^2}. \tag{2.16}$$

This proves Step 3.

Let $(x^*(\theta), y^*(\theta))$ be a Pareto-efficient allocation at θ. If $S' = \{k \in I \mid y_k^*(\theta) > 0\}$ and $S = \{k \in I \mid x_k^*(\theta) > 0\}$ then Step 2 implies $S' \subseteq S$. Also Step 3 implies $x_i^*(\theta) \geq$

$\frac{\theta_i^2}{\sum\limits_{k\in S}\theta_k^2} \geq \frac{\theta_i^2}{\sum\limits_{k\in I}\theta_k^2}$ for all $i \in S'$. Therefore, $y_i^*(\theta) > 0$ implies $x_i^*(\theta) \geq \frac{\theta_i^2}{\sum\limits_{k\in I}\theta_k^2}$ which is equivalent to condition $P1$ of this Proposition.

Step 4: Let $(x^*(\theta), y^*(\theta)) \in PE(\theta)$. If $x_i^*(\theta) > \frac{\theta_i^2}{\theta_i^2+\min_{k\neq i}\theta_k^2}$ for some $i \in I$, then $y_i^*(\theta) = 1$.

Suppose not, i.e. $x_i^*(\theta) > \frac{\theta_i^2}{\theta_i^2+\min_{k\neq i}\theta_k^2}$ and $y_i^*(\theta) < 1$. Therefore there is at least one agent $i'(\neq i)$ such that $y_{i'}^*(\theta) > 0$. By Step 2, $x_{i'}^*(\theta) > 0$. Solving the optimization problem in Step 3 for agent i', we obtain $\alpha_i \leq 1$. Suppose agent i' is the only agent other than i who obtains a positive allocation of good x. Then, $x_i^*(\theta) = \frac{\alpha_i^2\theta_i^2}{\alpha_i^2\theta_i^2+\theta_{i'}^2} \leq \frac{\theta_i^2}{\theta_i^2+\theta_{i'}^2} \leq \frac{\theta_i^2}{\theta_i^2+\min_{k\neq i}\theta_k^2}$. Note that if we allow more agents to obtain positive allocations of x, then the denominator in the fraction $\frac{\alpha_i^2\theta_i^2}{\alpha_i^2\theta_i^2+\theta_{i'}^2}$ will increase. As a result, the allocation of agent i of good x will decrease further. Hence we have a contradiction to our assumption that $x_i^*(\theta) > \frac{\theta_i^2}{\theta_i^2+\min_{k\neq i}\theta_k^2}$. This proves Step 4 and condition $P2$ of the Proposition. ∎

Chapter 3

Non Fixed-Price Trading Rules in Single-Crossing Classical Exchange Economies

Introduction

The set of rules for allocating available resources to a given set of agents will typically vary with the axioms that these rules are required to satisfy. In the previous chapter we considered SCFs that satisfy Pareto-efficiency in addition to the requirement of strategy-proofness. In the present chapter we replace the notion of Pareto-efficiency with *individual-rationality*. In other words, every agent has an endowment of goods and the SCF is required to generate allocations that make agents at least as well off as consuming their endowment. If a SCF is not individually rational, agents will presumably "opt out" of the mechanism and consume their private endowment. Thus it is a minimal requirement that ensures the voluntary participation of agents in the mechanism.

It is well-known that Pareto-efficiency and individual-rationality are incompatible in our model. Hurwicz (1972) demonstrates this for the case of two-good and two-agent models. Serizawa (2002) extends this result to an arbitrary numbers of agents and goods. Moreover

Serizawa (2002)'s result only requires a domain of preferences that includes all homothetic preferences.

In an important paper Barberà and Jackson (1995) show that a strategy-proof and individually rational SCF defined on the domain of all classical preferences is a *Fixed-Price Trading* or FPT rule. These rules have also been shown to be salient in other mechanism design problems. For instance, Hagerty and Rogerson (1987) consider a bilateral trading model with quasi-linear utility functions for the agents and show that a strategy-proof, individually rational and budget-balanced SCF is a FPT rule.

Our goal in this chapter is to consider the Barberà and Jackson (1995) problem over an important restricted domain of preferences. A critical element of the arguments in Barberà and Jackson (1995) is the use of *concavified* preferences. Concavification requires the preference domain to be rich enough in order to permit indifference curves to be "bent upwards" at any bundle in the commodity space. We restrict attention to preferences that satisfy the *single-crossing* property. This rules out concavification. We show that there are non-FPT rules that are strategy-proof and individually rational in such domains. Furthermore, we provide a complete characterization of such SCFs in two-agent economies that satisfy an additional continuity requirement. Our characterization can be briefly described as follows. An FPT rule requires the range of the SCFs to be piecewise linear with a kink (possibly) at the endowment. We show that for single-crossing domains, the range need not be piecewise linear though it must contain the endowment and satisfy additional properties. We call these rules *Generalized Trading* rules. A FPT rule, with connected range, is an example of a Generalized Trading rule.

Single-crossing domains have been extensively used in mechanism design and contract theory. Some classic papers in this regard are Spence (1973), Mirrless (1971) and Rothschild and Stiglitz (1976).

Saporiti (2009) considers a single-crossing domain with a finite number of alternatives and strict preferences and provides a characterization of strategy-proof SCFs. Our results are independent of his because our models are completely different. A more detailed discussion of the relationship between our models can be found in Section 3.2. Barberà and Jackson (2004) consider a model where society's preferences over voting rules satisfy single-crossingness. However, their objective is to analyze self-stable rather than strategy-proof voting rules. Gans and Smart (1996) study an Arrovian aggregation problem with single-crossing preferences for voters. They show that median voters are decisive in all majority elections between pairs of alternatives. Corchón and Rueda-Llano (2008) analyze a public-good-private-good production economy where agents' preferences satisfy single-crossingness. They show the non-existence of smooth strategy-proof, Pareto-efficient SCFs that give strictly positive amount of both goods to the agents.

We remark that the concavification property of various preference domains has been used extensively in the characterization of strategy-proof SCFs in economic environments. See for instance, Zhou (1991), Hashimoto (2008), Serizawa and Weymark (2003), Serizawa (2006) and Ju (2003). In the previous chapter we have shown that results in the literature on strategy-proofness and Pareto-efficiency carry over to domains where concavification is not permitted. However, the present chapter demonstrates that a similar conclusion does not obtain when Pareto-efficiency is replaced by individual-rationality. In particular, a strictly larger class than FPT rules will satisfy strategy-proofness and individual-rationality.

This chapter is organized as follows. Section 3.1 sets up the notation, the definition of an FPT rule and the fundamental Barberà and Jackson (1995) characterization result. Section 3.2 introduces concavification, single-crossing domains and some preliminary but useful results pertaining to this domain. A subsection of this section discusses our notion of single-crossingness with that of Saporiti (2009). Section 3.3 provides examples of non-FPT rules that are strategy-proof and individually rational and Section 3.4 contains the main characterization result. Section 3.5 concludes.

3.1 Notation and Definitions

Throughout the chapter we will restrict attention to a two-agent, two-good model. We denote the set of agents by $I = \{1, 2\}$ and the two goods by x and y. The set of goods is denoted by M. Each agent i has an endowment ω_i^x and ω_i^y of goods x and y respectively. Let $\omega = ((\omega_1^x, \omega_1^y), (\omega_2^x, \omega_2^y))$ denote the endowment vector. Let $\Omega_j = \sum_{i=1}^{2} \omega_i^j$, where $j \in M$ is the total endowment of the good j. Let $\Omega = (\Omega_x, \Omega_y)$ denote the total endowment in the economy. Then define the set of feasible allocations to be, $\Delta = \{((x_1, y_1), (x_2, y_2))| \; x_1 + x_2 = \Omega_x \text{ and } y_1 + y_2 = \Omega_y; x_i \geq 0, \; y_i \geq 0, \text{for all } i \in I\}$.

A preference ordering of agent i will be denoted by R_i. The asymmetric component of R_i will be denoted by P_i and the symmetric component by I_i. Upper and lower contour sets have been defined in the previous chapter. Thus $UC(R_i, (x_i, y_i))$ is the set of commodity bundles that are at least as good as (x_i, y_i) according to R_i and $LC(R_i, (x_i, y_i))$ is the set of commodity bundles that are no better than (x_i, y_i) according to R_i. An indifference curve for preference R_i through a bundle (x_i, y_i) denoted by $IC(R_i, (x_i, y_i))$ is defined as follows: $IC(R_i, (x_i, y_i)) = UC(R_i, (x_i, y_i)) \cap LC(R_i, (x_i, y_i))$.

As in the previous chapter, we consider only classical preferences and denote the set of such preferences by $\mathbb{D}^c$.

As before, a SCF F is a mapping $F : [\mathbb{D}]^2 \to \Delta$ where $\mathbb{D} \subset \mathbb{D}^c$. The range of a SCF F will be denoted by $\Re_F$.

The notions of strategy-proofness and Pareto-efficiency have been defined earlier and are not repeated. Individual-rationality is defined below: it ensures that an agent is not be made worse-off relative to his endowment by F.

Definition 3.1.1 *A SCF $F : [\mathbb{D}]^2 \to \Delta$ satisfies* **Individual-Rationality (IR)** *with respect to ω, if $F_i(R) R_i (\omega_i^x, \omega_i^y))$ for all i and for all $R \in [\mathbb{D}]^2$.*

Our next goal is to introduce FPT rules. We closely follow notation and definitions in Barberà and Jackson (1995). Consider $a \in \Delta$ and let $a_i = (x_i, y_i)$.

Definition 3.1.2 *A set $B \subset \Delta$ is* **Diagonal** *if for each agent i and for all distinct a and b in B, $a_i \ngeq b_i$ and $b_i \ngeq a_i$.* [1]

For any $a, b \in \Delta$ let $\overline{ab} = \{x | \exists \gamma \in [0,1] \text{s.t. } x = \gamma a + (1-\gamma)b\}$ i.e. $\overline{ab}$ is the straight line segment that connects a and b. If $c_i \geq \gamma a_i + (1-\gamma)b_i$ for some $\gamma \in [0,1]$, we write $c \otimes_i b$ i.e. c lies on or above $\overline{ab}$ from agent i's origin.

For any $B \subset \Delta$ and $R \in [\mathbb{D}^c]^2$, let $Top(R_i; B, R_j)$ denote the set of allocations in B that maximize R_i given R_j. [2] A function t_i which is a selection from $Top(R_i; B, R_j)$ is called a *tie-breaking rule.* A tie-breaking rule t_i is $j-$ favorable at $B \subset \Re_F$ if for any R, $t_i(R_i; B, R_j) \neq t_i(R_i; B, R_j^{'})$ only if $t_i(R_i; B, R_j^{'})R_j^{'}t_i(R_i; B, R_j)$. If agent i has multiple tops in a set under R_i and if the choice of the tops varies when agent j changes his announcement of preference ordering, then it should vary in such a way that agent j cannot manipulate.

Definition 3.1.3 *(***Fixed-Price Trading Rules***) A SCF $F : [\mathbb{D}]^2 \rightarrow \Delta$ is an FPT rule if $\Re_F$ is closed, diagonal and contains ω and there exists an agent i such that the following hold:*

1. *For all distinct a and b in $\Re_F$ either $a \in \overline{\omega b}$, $b \in \overline{\omega a}$ or $\omega \otimes_i \overline{ab}$.*

2. *There exist tie-breaking rules t_i and t_j such that t_i is j favorable at $\Re_F$ and t_j is $i-$favorable at $\overline{\omega a} \cap \Re_F$ for all $a \in \Re_F$.*

3. *$F(R) = t_j(R_j; \overline{\omega a} \cap \Re_F, R_i)$ where $a = t_i(R_i; \Re_F, R_j)$.*

[1]Recall the following notation: for all $p, q \in \Re^2$, $p \geq q$ if $p_j \geq q_j$ for all $j \in \{1,2\}$ and $p \gg q$ if $p_j > q_j$ for all $j \in \{1,2\}$.

[2]Of course, $Top(R_i; B, R_j)$ does not depend on R_j; however as we shall see, this notation helps in defining the tie-breaking rule for FPT rules. Later in the chapter while describing our results, we will drop R_j from the notation because tie-breaking is not required in our characterization.

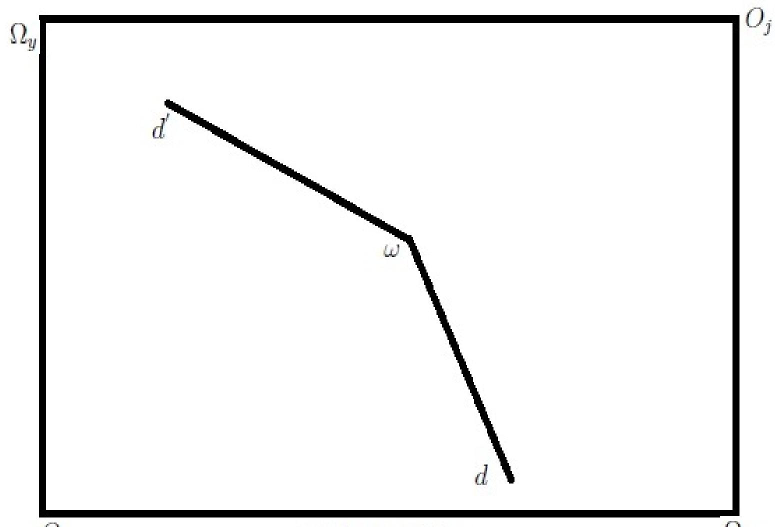

Diagram 3.1

The first condition ensures that $\Re_F$ contains ω and is piecewise linear with a possible kink at ω, as depicted on Diagram 3.1. If any two feasible allocations a and b lie on the same side of the endowment then ω, a and b are collinear. Since the indifference curves of classical preferences are strictly convex, both the agents' preferences are single-peaked on both the line segments if $\Re_F$ is connected.

Theorem 3.1.1 *Barberà and Jackson (1995) Let $F : [\mathbb{D}^c]^2 \to \Delta$ be a SCF. The SCF F is strategy-proof and individually rational if and only if it is a FPT rule.* [3]

We end this section by introducing further notation that will be used subsequently.

Consider an allocation (x', y'). We let $FIQ_i(x', y')$, $SEQ_i(x', y')$ $THQ_i(x', y')$ and $FOQ_i(x', y')$ denote the first, second, third and fourth quadrants of (x', y') from i's perspective. Specifically, $FIQ_i(x', y') = \{(x, y)|x_i \geq x'_i \text{ and } y'_i \geq y_i)\}$; $SEQ_i(x', y') = \{(x, y)|x_i \leq x'_i \text{ and } y'_i \geq y_i)\}$, $THQ_i(x', y') = \{(x, y)|x_i \leq x'_i \text{ and } y'_i \leq y_i)\}$ and $FOQ_i(x', y') = \{(x, y)|x_i \geq x'_i \text{ and } y'_i \geq y_i)\}$.

Let $a, b \in \Delta$, then $[ab] = \{B \subset \Delta | B \text{ is diagonal, connected and } a, b \in B\}$. Note that $\overline{ab} \in [ab]$. A typical element of $[ab]$ will be denoted by $\widetilde{ab}$ if $\widetilde{ab}$ is not a straight line segment.

[3] Barberà and Jackson (1995) prove a FPT result for an arbitrary number of agents by imposing further assumptions on the SCF.

3.2 Concavification and Single-Crossing Domains

We first introduce the notion of concavification.

Definition 3.2.1 *Let R_i be a preference ordering and let $(x_i, y_i) \in \Re^2_+$. The preference ordering $R_i^{''}$ is a* **concavification** *of $R_i^{'}$ at (x_i, y_i) if*

(i) $UC(R_i^{''}, (x_i, y_i)) \subset UC(R_i^{'}, (x_i, y_i))$ *and*

(ii) $(x_i^{'}, y_i^{'}) \in UC(R_i^{''}, (x_i, y_i))$ *and* $(x_i^{'}, y_i^{'}) \neq (x_i, y_i) \Rightarrow (x_i^{'}, y_i^{'}) P_i (x_i, y_i)$.

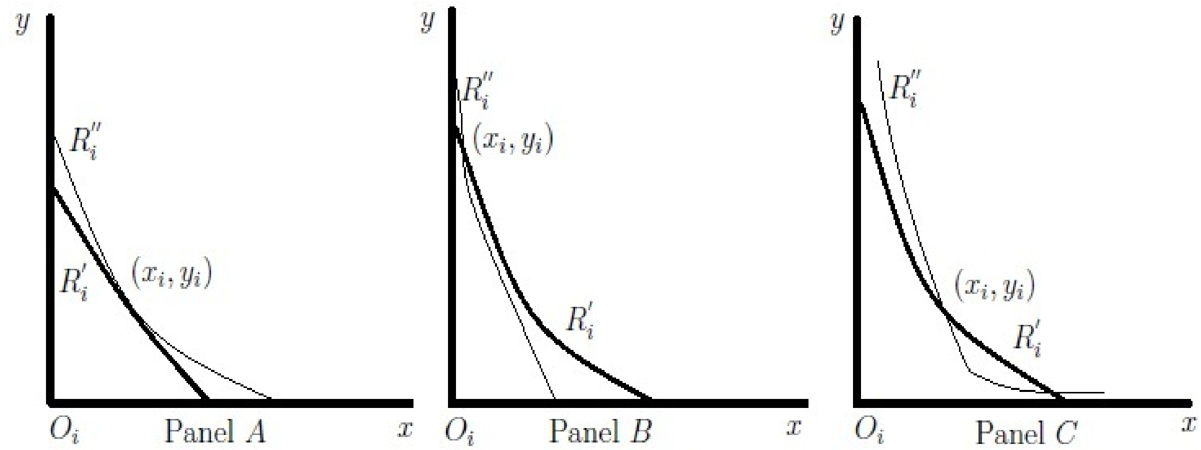

Diagram 3.2

In Diagram 3.2 Panel A, $R_i^{''}$ is a concavification of $R_i^{'}$ at (x_i, y_i). The indifference curve of $R_i^{''}$ touches the indifference curve of $R_i^{'}$ at (x_i, y_i) and lies strictly above it at all other bundles. In other words, the indifference curve of $R_i^{'}$ at (x_i, y_i) can be "bent upwards" at (x_i, y_i). In Panel B neither $R_i^{''}$ is a concavification of $R_i^{'}$ nor $R_i^{'}$ a concavification of $R_i^{''}$ at (x_i, y_i).

Definition 3.2.2 *The domain $\mathbb{D}$* **Admits Concavification** *if for all $R_i \in \mathbb{D}$ and $a_i \in \Re^2_+$, there exists $\tilde{R}_i \in \mathbb{D}$ that is a concavification of R_i at a_i.*

The domain $\mathbb{D}^c$ is an example of a domain that admits concavification. An example of a "smaller" domain that also admits concavification is provided below.

Example 3.2.1: Consider the domain represented by the utility function

$$u_i(x_i, y_i; \theta_i, \alpha_i) = \theta_i x_i^{\alpha_i} + y_i, \quad \theta_i > 0 \text{ and } 0 < \alpha_i < 1.$$

Note that this domain consists of classical preferences; moreover all preferences in the domain are smooth. Fix an utility function $(\theta_i^{'}, \alpha_i^{'})$ and a consumption bundle (x_i^*, y_i^*). If $x_i^* = 0$ then $(\theta_i^{''}, \alpha_i^{'})$ is a concavification of $(\theta_i^{'}, \alpha_i^{'})$ at (x_i^*, y_i^*), where $\theta_i^{''} < \theta_i^{'}$. Now consider the case where $x_i^* > 0$. Then the absolute value of the slope of an indifference curve at (x_i^*, y_i^*) is $\theta_i \alpha_i (x_i^*)^{\alpha_i - 1}$.

Note that, $\lim_{\alpha_i \to 0} \theta_i^{'} \alpha_i (x_i^*)^{(\alpha_i - 1)} = 0$. Therefore, we can choose $\alpha_i^{''} < \alpha_i^{'}$ such that $\theta_i^{'} \alpha_i^{''} (x_i^*)^{(\alpha_i^{''} - 1)} < \theta_i^{'} \alpha_i^{'} (x_i^*)^{(\alpha_i^{'} - 1)}$. Then choose $\theta_i^{''} > \theta_i^{'}$ such that $\theta_i^{''} \alpha_i^{''} (x_i^*)^{(\alpha_i^{''} - 1)} = \theta_i^{'} \alpha_i^{'} (x_i^*)^{(\alpha_i^{'} - 1)}$. We claim that $(\theta_i^{''}, \alpha_i^{''})$ is a concavification of $(\theta_i^{'}, \alpha_i^{'})$ at (x_i^*, y_i^*). Since $\theta_i^{'} \alpha_i^{'} (x_i^*)^{(\alpha_i^{'} - 1)} = \theta_i^{''} \alpha_i^{''} (x_i^*)^{(\alpha_i^{''} - 1)}$, slopes of the two utility functions at (x_i^*, y_i^*) are equal. We can write the equality as $\frac{\theta_i^{'} \alpha_i^{'}}{\theta_i^{''} \alpha_i^{''}} = \frac{1}{(x_i^*)^{(\alpha_i^{'} - \alpha_i^{''})}}$. Since $(\alpha_i^{'} - \alpha_i^{''}) > 0$, if $x_i < x_i^*$ then $\theta_i^{'} \alpha_i^{'} x_i^{(\alpha_i^{'} - 1)} < \theta_i^{''} \alpha_i^{''} x_i^{(\alpha_i^{''} - 1)}$ and if $x_i > x_i^*$ then $\theta_i^{'} \alpha_i^{'} x_i^{(\alpha_i^{'} - 1)} > \theta_i^{''} \alpha_i^{''} x_i^{(\alpha_i^{''} - 1)}$. This establishes our claim.

The results in Barberà and Jackson (1995) depend heavily on the concavification property of the domain $\mathbb{D}^c$. Our objective in this chapter is to analyze the structure of strategy-proof and individually rational SCFs over a class of domains that do not admit concavification.

The domain that we focus on satisfy the *single-crossing* property. This property requires that two different indifference curves never "touch" each other i.e. either they cut each other from above or from below. Thus, the situation in Panel A never occurs and we always have the situation in Panel B occurring. This is defined formally below.[4]

Definition 3.2.3 *The domain $\mathbb{D}^s$ is single-crossing if all $R_i^{'}$, $R_i^{''}$ and all $(x_i, y_i) \in \Re_+^2$, either* 1 *or* 2 *below hold,*

1. $IC(R_i^{''}, (x_i, y_i)) \setminus \{(x_i, y_i)\} \subset [int\ UC(R_i^{'}, (x_i, y_i)) \cap int\ SEQ_i((x_i, y_i))]$ *and*

   $IC(R_i^{''}, (x_i, y_i)) \setminus \{(x_i, y_i)\} \subset [int\ LC(R_i^{'}, (x_i, y_i)) \cap int\ FOQ_i((x_i, y_i))]$.

[4]The notation $int\ B$ refers to the interior of the set B.

2. $IC(R_i^{'}, (x_i, y_i)) \setminus \{(x_i, y_i)\} \subset [int\ UC(R_i^{''}, (x_i, y_i)) \cap int\ SEQ_i((x_i, y_i))]$ *and*

$IC(R_i^{'}, (x_i, y_i)) \setminus \{(x_i, y_i)\} \subset [int\ LC(R_i^{''}, (x_i, y_i)) \cap int\ FOQ_i((x_i, y_i))]$.

If (1) *above holds then we say* $R_i^{''}$ *cuts* $R_i^{'}$ *from above at* (x_i, y_i)*; if* (2) *holds then we say* $R_i^{''}$ *cuts* $R_i^{'}$ *from below at* (x_i, y_i)*.*

Consider two indifference curves $IC(R_i^{'}, (x_i, y_i))$ and $IC(R_i^{''}, (x_i, y_i))$ i.e. they both pass through (x_i, y_i). If $IC(R_i^{''}, (x_i, y_i))$ is above $IC(R_i^{'}, (x_i, y_i))$ for $x_i^{'} < x_i$ and $IC(R_i^{'}, (x_i, y_i))$ is above $IC(R_i^{''}, (x_i, y_i))$ for $x_i^{'} > x_i$ then $R_i^{''}$ cuts $R_i^{'}$ from above at (x_i, y_i). On other hand, if $IC(R_i^{'}, (x_i, y_i))$ is above $IC(R_i^{''}, (x_i, y_i))$ for $x_i^{'} < x_i$ and $IC(R_i^{''}, (x_i, y_i))$ is above $IC(R_i^{'}, (x_i, y_i))$ for $x_i^{'} > x_i$ then $R_i^{'}$ cuts $R_i^{''}$ from above at (x_i, y_i).

The definition of single-crossing rules out the situation in Panel A because $IC(\tilde{R}_i, (x_i, y_i))$ is above the indifference curve $IC(\bar{R}_i, (x_i, y_i))$ both to the left and right of (x_i, y_i). Thus, neither (1) nor (2) in Definition 3.2.3 hold. Importantly, the situation in Panel C cannot hold either. This is because, to the right of (x_i, y_i), $IC(R_i^{''}, (x_i, y_i))$ lies both below and above $IC(R_i^{'}, (x_i, y_i))$, i.e. the second part of (1) in Definition 3.2.3 fails to hold. We record this below as an observation for future reference.

Observation 3.2.1 Let $R_i^{'}, R_i^{''} \in \mathbb{D}^s$. Then there does not exist $(x_i, y_i), (x_i^{'}, y_i^{'}) \in \Re_+^2$ such that

$$IC(R_i^{'}, (x_i, y_i)) = IC(R_i^{'}, (x_i^{'}, y_i^{'})) \text{ and } IC(R_i^{''}, (x_i, y_i)) = IC(R_i^{''}, (x_i^{'}, y_i^{'}))$$

We provide some examples of single-crossing domains.

Example 3.2.2: Consider preferences represented by utility functions of the form

$$u_i(x_i, y_i; \theta_i) = \theta_i \sqrt{x_i} + y_i, \quad \theta_i > 0$$

We claim that the this domain is single-crossing. Note that preferences are classical and smooth. The slope of an indifference curve is given by $\frac{-\theta_i}{2\sqrt{x_i}}$ for all $x_i > 0$. This means that at a given bundle (x_i, y_i), the slope of an indifference curve is described uniquely by the parameter θ_i. Suppose, $\theta_i^{'} > \theta_i^{''}$. Consider, the indifference curves $IC(\theta_i^{'}, (x_i, y_i))$ and $IC(\theta_i^{''}, (x_i, y_i))$. Notice that the slope of $IC(\theta_i^{'}, (x_i, y_i))$ is higher than $IC(\theta_i^{''}, (x_i, y_i))$ at (x_i, y_i). If either Condition 1 or 2 in Definition 3.2.3 is violated, then smoothness would imply that $\theta_i^{'} < \theta_i^{''}$ which is a contradiction. Hence, the domain is single-crossing.

In Example 3.2.2, preferences are quasi-linear. We also provide an example of a domain consisting of homothetic preferences that is single-crossing.

Example 3.2.3: Consider preferences represented by utility functions of the form

$$u_i(x_i, y_i; \alpha_i) = x_i^{\alpha_i} y_i, \quad \alpha_i > 0.$$

Note that we do not have to consider consumption bundles where either $x_i = 0$ or $y_i = 0$. For all other consumption bundles slope of an indifference curve is given by $\frac{\alpha_i y_i}{x_i}$. If $\alpha_i^{'} \neq \alpha_i^{''}$ then the slopes are different at any (x_i, y_i). The domain is single-crossing for the same reasons outlined in the previous example.

In the next section we will describe some important properties of single-crossing domains that will be useful in our characterization.

3.2.1 Some Properties of Single-Crossing Domains

We first obtain some restrictions on preferences over triples implied by the single-crossing property.

For any $a, b, c \in \Delta$ define the following sets of preference orderings over a, b, c. [5]

$$\mathbb{D}^1(\{a, b, c\}) = \{cP_ibP_ia,\ bP_icP_ia,\ bP_iaP_ic,\ aP_ibP_ic,\ bI_iaP_ic,\ bP_icI_ia,\ cI_ibP_ia\}$$

$$\mathbb{D}^2(\{a, b, c\}) = \{cP_ibP_ia,\ aP_ibP_ic,\ cI_ibI_ia\}$$

[5] For any $a, b \in \Delta$, aP_ib is written for a_i is preferred to b_i by agent i with preferences R_i while aI_ib signifies that agent i is indifferent between a_i and b_i under R_i

We will show that if a domain is single-crossing, then three diagonal allocations can be chosen in such a way that all possible preference orderings over these allocations are either from $\mathbb{D}^1(\{a, b, c\})$ for one agent and from $\mathbb{D}^2(\{a, b, c\})$ for the other.

Proposition 3.2.1: Consider an arbitrary domain $\mathbb{D}^s$. There exists three diagonal allocations a, b, c in Δ such that $R_i|_{\{a,b,c\}} \in \mathbb{D}^1(\{a, b, c\})$ and $R_j|_{\{a,b,c\}} \in \mathbb{D}^2(\{a, b, c\})$, $i \neq j$ for all $(R_i, R_j) \in [\mathbb{D}^s]^2$. [6]

Proof: We follow the proof in Diagram 3.3. [7]

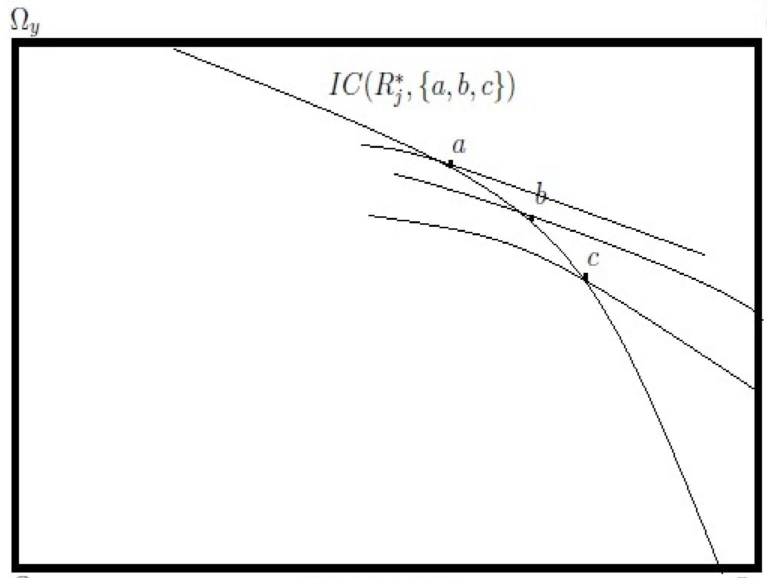

Diagram 3.3

Pick an agent j and a preference ordering $R_j^* \in \mathbb{D}^s$. Pick $a, b, c \in \Delta$ lying on an indifference curve under R_j^*. Label this indifference curve $IC(R_j^*, \{a, b, c\})$. Note that a, b and c must be diagonal. Furthermore since preferences are classical cP_iaP_ib, aP_icP_ib, cI_ibI_ia, cP_ibI_ia, cI_iaP_ib, aP_icI_ib and aI_ibI_ic are ruled out for any $R_i \in \mathbb{D}^s$.

Other preference orderings are not permissible because indifference curves cannot intersect. Therefore, $R_i|_{\{a,b,c\}} \in \mathbb{D}^1(\{a, b, c\})$ for all $R_i \in \mathbb{D}^s$.

[6] The notation $R_i|_{\{a,b,c\}}$ refers to R_i restricted to $\{a, b, c\}$.

[7] The indifference curves in Diagram 3.3 appear to be smooth. However, smoothness is not required for the proof of this Proposition or any of the results that follow.

The single-crossing property rules out indifference between any pair in $\{a, b, c\}$ for any $R_j \in \mathbb{D}^s$. If cP_jaP_jb then the indifference curve passing through a under R_j must intersect $IC(R_j^*, \{a, b, c\})$ twice. However, this would contradict Observation 3.2.1. Finally note that bP_jaP_jc and bP_jcP_ja are ruled out. This can happen only if an indifference curve of these two orderings cuts $IC(R_j^*, \{a, b, c\})$ between a and b and between b and c. This would again violate Observation 3.2.1. Therefore, $R_j|_{\{a,b,c\}} \in \mathbb{D}^2(\{a, b, c\})$ for all $R_j \in \mathbb{D}^s$ as required. ■

Proposition 3.2.1 specifies the restriction on preferences arising due to the single-crossing property. However, all preferences in $(\mathbb{D}^1(\{a, b, c\}), \mathbb{D}^2(\{a, b, c\}))$ pair need not be present in an arbitrary single-crossing domain. We will impose a *richness* requirement on the single-crossing domains that ensures that we can pick the a, b, c triple in such a way that all preferences in $(\mathbb{D}^1(\{a, b, c\}), \mathbb{D}^2(\{a, b, c\}))$ pair are present in the domain. The definition of richness is given below.

Definition 3.2.4 *The single-crossing domain $\mathbb{D}^s$ is* **Rich** *if for all diagonal allocations $(x_i^{'}, y_i^{'}), (x_i^{''}, y_i^{''}) \in \Re_+^2$, there exists $R_i \in \mathbb{D}^s$ such that $IC(R_i, (x_i^{'}, y_i^{'})) = IC(R_i, (x_i^{''}, y_i^{''}))$.*

A single-crossing domain is rich if any two diagonal bundles can be joined by an indifference curve. We show that the domains specified in Examples 3.2.2 and 3.2.3 earlier are rich.

Example 3.2.2 (**Continued**) Consider the domain introduced in Example 3.2.2. We have already seen that this domain is a single-crossing domain. Let $(x_i^{'}, y_i^{'}), (x_i^{''}, y_i^{''}) \in \Re_+^2$ be such that $x_i^{'} > x_i^{''}$ and $y_i^{''} > y_i^{'}$. Set $\theta_i = \frac{y_i^{''} - y_i^{'}}{\sqrt{x_i^{'}} - \sqrt{x_i^{''}}}$. Note that $\theta_i > 0$ and $\theta_i\sqrt{x_i^{'}} + y_i^{'} = \theta_i\sqrt{x_i^{''}} + y_i^{''}$. Hence $(x_i^{'}, y_i^{'})$ and $(x_i^{''}, y_i^{''})$ lie on the indifference curve corresponding to θ_i. Therefore the domain is rich.

Example 3.2.3 (**Continued**) Consider the domain introduced in Example 3.2.3. We have already seen that this is a single-crossing domain. Let $(x_i^{'}, y_i^{'}), (x_i^{''}, y_i^{''}) \in \Re_+^2$ be such that $x_i^{'} > x_i^{''}$ and $y_i^{''} > y_i^{'}$. Set $\alpha_i = \frac{\ln(y_i^{''}) - \ln(y_i^{'})}{\ln(x_i^{'}) - \ln(x_i^{''})}$. Since ln is a strictly increasing function, $\alpha_i > 0$. Therefore, $\alpha_i \ln(x_i^{'}) + \ln(y_i^{'}) = \alpha_i \ln(x_i^{''}) + \ln(y_i^{''})$ i.e. $(x_i^{'})^{\alpha_i} y_i^{'} = (x_i^{''})^{\alpha_i} y_i^{''}$. Hence, $(x_i^{'}, y_i^{'})$ and

(x_i'', y_i'') are on the same indifference curve corresponding to α_i. Therefore the domain is rich.

An important implication of richness is that a rich single-crossing domain is also a maximal single-crossing domain. We formalize this below.

Proposition 3.2.2: Let $\mathbb{D}^s$ be rich. Let $\mathbb{D}^*$ be another single-crossing domain. Then either $\mathbb{D}^* \cup \mathbb{D}^s$ is not a single-crossing domain or $\mathbb{D}^* \subset \mathbb{D}^s$.

Proof: Pick an arbitrary $R_i \in \mathbb{D}^* \setminus \mathbb{D}^s$. Pick an arbitrary indifference curve under R_i and $(x_i, y_i), (x_i', y_i') \in \Re^2_+$ such that $IC(R_i, (x_i, y_i)) = IC(R_i, (x_i', y_i'))$. Since $\mathbb{D}^s$ is rich, there exists $R_i' \in \mathbb{D}^s$, such that $IC(R_i', (x_i, y_i)) = IC(R_i', (x_i', y_i'))$. If $IC(R_i, (x_i, y_i)) \neq IC(R_i', (x_i, y_i))$, then we contradict Observation 3.2.1. If on the other hand, $IC(R_i, (x_i, y_i)) = IC(R_i', (x_i, y_i)) = IC(R_i, (x_i', y_i')) = IC(R_i', (x_i', y_i'))$, R_i and R_i' cannot be part of a single-crossing domain (Definition 3.2.3), i.e. $\mathbb{D}^* \cup \mathbb{D}^s$ is not single-crossing. ■

In the next Proposition we show that if a domain of single-crossing preferences is rich, we can find allocations a, b, c with $c > b > a$ (as in Diagram 3.3)[8], such that all preferences ordering from the pair $(\mathbb{D}^1(\{a, b, c\}), \mathbb{D}^2(\{a, b, c\})$ are induced.

Lemma 3.2.1 *Let $\mathbb{D}^s$ be rich. Then there exists $a, b, c \in \Delta$ with $c > b > a$ such that $\{R_i|_{\{a,b,c\}} | R_i \in \mathbb{D}^s\} = \mathbb{D}^1(\{a, b, c\})$ and $\{R_j|_{\{a,b,c\}} | R_j \in \mathbb{D}^s\} = \mathbb{D}^2(\{a, b, c\})$.*

Proof: Choose R_j^* and fix an indifference curve. Now pick R_i and an indifference curve of R_i such that it is tangent to the chosen indifference curve of R_j^* in the interior of Δ. Label

[8] Let a and b be two diagonal allocations. Fix an agent i. We say $a > b$ if $a_i^x > b_i^x$ (or $a_i^y < b_i^y$). Let $\{a^1, a^2, a^3\}$ be a set of diagonal allocations. We denote median$\{a^1, a^2, a^3\} \in \{a^1, a^2, a^3\}$ to be the median of $\{a^1, a^2, a^3\}$ if $\left|\{a^j | a^j \geq \text{median}\{a^1, a^2, a^3\}\}\right| \geq 2$ and $\left|\{a^j | \text{median}\{a^1, a^2, a^3\} \geq a^j\}\right| \geq 2$. The usual definition of single-peaked preferences apply on a diagonal set according to this order. For instance, if a diagonal set is a straight line segment and lies in the interior of Δ then all the allocations on it can be sustained as tops for some preference ordering from $\mathbb{D}^s$ (follows from the arguments similar to Proposition 3.2.4). Also, on both sides of the top, the allocation nearer (note that under the order, distance between two allocations can be defined in the Euclidean sense) to it are preferred to the one which is further.

the tangency point as b. Then choose a and c on the indifference curve of R_j^* such that bP_icI_ia. Also choose $R_j^{'}$ such that $IC(R_j^{'}, b)$ cuts $IC(R_j^*, b)$ from above. By richness, such an $R_j^{'}$ exists. Since the domain is single-crossing we have $aP_j^{'}bP_j^{'}c$. Also choose $R_j^{''}$ such that $IC(R_j^{''}, b)$ cuts $IC(R_j^*, b)$ from below. This will result in $cP_j^{''}bP_j^{''}a$. Hence, $\{R_j|_{\{a,b,c\}}|R_j \in \mathbb{D}^s\} = \mathbb{D}^2(\{a, b, c\})$.

Using the richness of $\mathbb{D}^s$ we can show that $\{R_i|_{\{a,b,c\}}|R_i \in \mathbb{D}^s\} = \mathbb{D}^1(\{a, b, c\})$. For instance, choose $R_i^{'}$ whose indifference curve passes through a and some allocation between b and c. This results in $bP_i^{'}aP_i^{'}c$. Choose $R_i^{''}$ to be such that agent i is indifferent between a and b. This gives $bI_i^{''}aP_i^{''}c$. Similarly all other preference orderings in $\mathbb{D}^1(\{a, b, c\})$ can be constructed.

■

We will use the sets $\mathbb{D}^1(\{a, b, c\})$ and $\mathbb{D}^2(\{a, b, c\})$ to construct SCF that are not FPT rules.

Our next goal is to show that an order relation can be defined on an arbitrary single-crossing domain $\mathbb{D}^s$. The next Proposition is crucial for that purpose.

Proposition 3.2.3: Consider an arbitrary domain $\mathbb{D}^s$. Let $\bar{R}_i, \tilde{R}_i \in \mathbb{D}^s$ and $(x_i^*, y_i^*) \in \Re_+^2$. If $IC(\tilde{R}_i, (x_i^*, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^*, y_i^*))$ from above, then $IC(\tilde{R}_i, (x_i, y_i))$ cuts $IC(\bar{R}_i, (x_i, y_i))$ from above at all $(x_i, y_i) \in \Re_+^2$.

Proof: We prove the Proposition in four steps.

Step 1: We show that an indifference curve of $\bar{R}_i$ that cuts the indifference curve of $\tilde{R}_i$ that contains (x_i^*, y_i^*) must cut from below. For contradiction suppose an indifference curve of $\bar{R}_i$ cuts the indifference curve of $\tilde{R}_i$ that contains (x_i^*, y_i^*) from above at $(x_i^{'}, y_i^{'})$. Suppose $(x_i^{'}, y_i^{'})$ is in the $SEQ_i((x_i^*, y_i^*))$. Hence, $IC(\bar{R}_i, (x_i^{'}, y_i^{'})) \cap UC(\bar{R}_i, (x_i^*, y_i^*)) \cap LC(\tilde{R}_i, (x_i^*, y_i^*)) \neq \emptyset$. Since, indifference curves of an ordering cannot intersect $IC(\bar{R}_i, (x_i^{'}, y_i^{'}))$ and $IC(\tilde{R}_i, (x_i^*, y_i^*))$ must cut twice contradicting Observation 3.2.1. We reach the same contradiction if $(x_i^{'}, y_i^{'})$ belongs to $FOQ_i((x_i^*, y_i^*))$.

Step 2: Let $\mathbb{E} = \{(x_i, y_i)|y_i = y_i^*, x_i \geq 0\}$. We will show that the indifference curves of

$\tilde{R}_i$ must cut the indifference curves of $\bar{R}_i$ at all the bundles in $\mathbb{E}$ from above. For the purpose of contradiction, consider $(x_i^{**}, y_i^*) \in \mathbb{E}$ such that $x_i^{**} > x_i^*$ and $IC(\tilde{R}_i, (x_i^{**}, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^{**}, y_i^*))$ from below.

Consider the set $\mathbb{E}' = \{(x_i^*, y_i) | x_i = x_i^* \text{ and } y_i \geq 0\}$. Since indifference curves of the same preference ordering do not intersect, we can find a sequence $(x_i^*, y_i^{'k}) \in \mathbb{E}'$ such that $IC(\bar{R}_i, (x_i^*, y_i^{'k})) = IC(\bar{R}_i, (x_i^k, y_i^k))$, where $x_i^* > x_i^k$ and $y_i^{'k} < y_i^k$. Since preferences are strictly monotonic $IC(\bar{R}_i, (x_i^*, y_i^{'k}))$ is a strictly increasing sequence of real numbers.

Let $IC(\bar{R}_i, (x_i^*, y_i^{'k})) \rightarrow \infty$ as $k \rightarrow \infty$. Then note that there exists an indifference curve $IC(\bar{R}_i, (x_i^*, y_i^{'k}))$ that passes through (x_i^{**}, y_i^*). This holds because $IC(\bar{R}_i, (x_i^{**}, y_i^*))$ is a finite number and preferences are continuous. Note that this indifference curve cannot be $IC(\bar{R}_i, (x_i^{**}, y_i^*))$ because any bundle in $SEQ_i((x_i^*, y_i^*))$ is in the interior of $LC(\bar{R}_i, (x_i^{**}, y_i^*))$. If $IC(\bar{R}_i, (x_i^{**}, y_i^*))$ is the indifference curve under consideration then it must cut $IC(\tilde{R}_i, (x_i^{**}, y_i^*))$ twice. But this means that two indifference curves of $\bar{R}_i$ intersect, which is a contradiction.

Therefore we assume that $\sup\{IC(\bar{R}_i, (x_i, y_i)) | (x_i, y_i) = (x_i^k, y_i^k), \text{ for some } k\} < \infty$. Since the sequence is increasing and bounded above it must converge and converge to its supremum. Let $IC(\bar{R}_i, (x_i^{'}, y_i^*))$ be the supremum where $x_i^* < x_i^{'} < x_i^{**}$. We claim that $IC(\bar{R}_i, (x_i^{'}, y_i^*))$ cuts $IC(\tilde{R}_i, (x_i^{'}, y_i^*))$ from below. Let for the sake of contradiction our claim is false. Since indifference curves of the same preference do not intersect, $IC(\bar{R}_i, (x_i^*, y_i^{'k})) \cap \mathbb{E} \neq \emptyset$ for all k. Hence we can choose a monotonically increasing sequence $(x_i^{''k}, y_i^{''k})$ ($x_i^{''k}$ increasing and $y_i^{''k} = y_i^*$) from $\mathbb{E}$ such that $IC(\bar{R}_i, (x_i^*, y_i^{'k})) = IC(\bar{R}_i, (x_i^{''k}, y_i^*))$ and $\{(x_i^{''k}, y_i^*)\}$ converges to $(x_i^{'}, y_i^*)$. Note that by Step 1, $IC(\tilde{R}_i, (x_i^{''k}, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^{''k}, y_i^*))$ from above, for all k.

Since $IC(\tilde{R}_i, (x_i^{'}, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^{'}, y_i^*))$ from below by continuity and strict monotonicity of preferences there exists an $\epsilon > 0$ such that, by Step 1 for all $x_i \in (x_i^{'} - \epsilon, x_i^{'})$ $IC(\tilde{R}_i, (x_i, y_i^*))$ cuts $IC(\bar{R}_i, (x_i, y_i^*))$ from below. Since the sequence $(x_i^{''k}, y_i^*)$ converge to $(x_i^{'}, y_i^*)$ this ϵ can be chosen in way that for some k large enough the bundle $(x_i^{''k}, y_i^*)$ lies in $(x_i^{'} - \epsilon, x_i^{'})$. Since indifference curves of the same preference do not intersect we will contradict the single-crossing property for such a k.

Now taking $(x_i^{'}, y_i^*)$ to be our initial bundle we can again argue that there exists another bundle $(x_i^{''}, y_i^*)$ where $x_i^{**} > x_i^{''} > x_i^{'}$ such that $IC(\bar{R}_i, (x_i^{''}, y_i^*))$ cuts $IC(\tilde{R}_i, (x_i^{''}, y_i^*))$ from below. [*]

Now we will show that this process of constructing a sequence will give us a sequence that converges to (x_i^{**}, y_i^*). To see this define the following set,

$$\widetilde{\mathbb{E}} = \{(x_i, y_i^*) \in \mathbb{E} | IC(\tilde{R}_i, (x_i, y_i^*)) \text{ cuts } IC(\bar{R}_i, (x_i, y_i^*)) \text{ from above}\}.$$

First we will show that there does not exist a bundle $(x_i^{***}, y_i^*) \in \mathbb{E}$ such that at any other bundle (x_i, y_i^*) with $x_i \geq x_i^{***}$, $IC(\tilde{R}_i, (x_i^{***}, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^{***}, y_i^*))$ from below. Let for the sake of contradiction this claim is false. Choose a monotonically increasing sequence (x_i^k, y_i^k) (x_i^k increasing and $y_i^k = y_i^*$) from $\widetilde{\mathbb{E}}$ which converges to (x_i^{***}, y_i^*). If $IC(\tilde{R}_i, (x_i^{***}, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^{***}, y_i^*))$ from below then by using a similar argument as above we can show a violation of the single-crossing property.

Now we go back to our claim that we can find a sequence $\{(x_i^n, y_i^*)\}$ such that $IC(\tilde{R}_i, (x_i^n, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^n, y_i^*))$ from above and converges to (x_i^{**}, y_i^*). Choose a monotonically increasing sequence $S^1 = \{(x_i^{1k}, y_i^*)\}_{k=1}^{\infty}$ such that $IC(\tilde{R}_i, (x_i^{1k}, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^{1k}, y_i^*))$ from above and converges to (x_i^1, y_i^*) such that $x_i^1 < x_i^{**}$. Note that such a sequence exists by [*]. By our above argument $IC(\tilde{R}_i, (x_i^1, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^1, y_i^*))$ from above. Let the first term of the desired sequence be (x_i^1, y_i^*). By [*] we can construct another sequence $S^2 = \{(x_i^{2k}, y_i^*)\}_{k=1}^{\infty}$ such that $IC(\tilde{R}_i, (x_i^{2k}, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^{2k}, y_i^*))$ from above and converges to (x_i^2, y_i^*) such that $x_i^1 < x_i^2 < x_i^{**}$. Let the second term of the desired sequence be (x_i^2, y_i^*). This process gives us the desired sequence.

Call this sequence $S^* = \{(x_i^n, y_i^*)\}_{n=1}^{\infty}$. Note that by Step 1 and the fact that $(x_i^n, y_i^*) \rightarrow (x_i^{**}, y_i^*)$, we can conclude that for some large enough n, $IC(\bar{R}_i, (x_i^n, y_i^*))$ will cut $IC(\tilde{R}_i, (x_i^{**}, y_i^*))$ from above, according to our assumption. But the sequence $\{(x_i^n, y_i^*)\}_{n=1}^{\infty}$ has been constructed in a way such that $IC(\tilde{R}_i, (x_i^n, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^n, y_i^*))$ from above. Hence two indifference curves of $\tilde{R}_i$ intersect. This is a contradiction.

Diagram 3.4 below is helpful in understanding this argument. In the diagram, the darker indifference curves represent $\tilde{R}_i$ and the lighter ones $\bar{R}_i$.

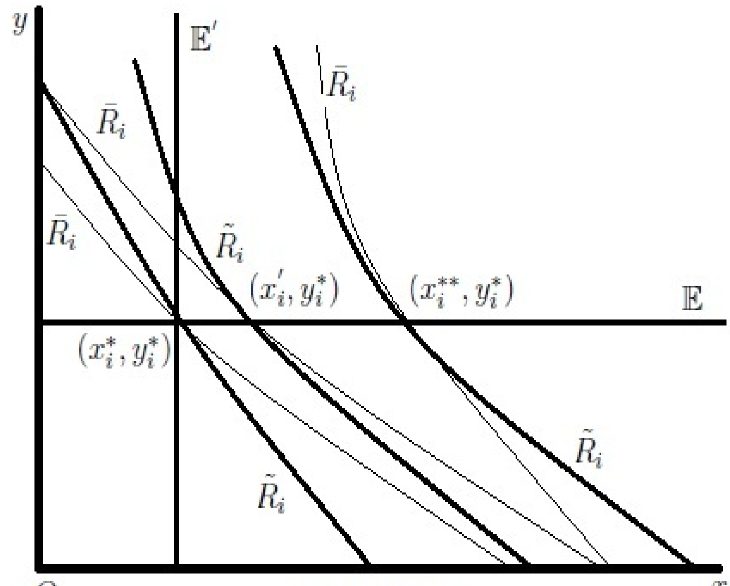

Diagram 3.4

Step 3: Consider the subset $\mathbb{E}' = \{(x_i, y_i) | x_i = x_i^*, y_i \geq 0\}$. Using the same argument as above it follows that indifference curves of $\tilde{R}_i$ must cut the indifference curves of $\bar{R}_i$ at all the bundles of $\mathbb{E}'$ from above.

Step 4: From Step 2 and Step 3 it follows that along every horizontal and vertical line, the indifference curves of $\tilde{R}_i$ must cut the indifference curves of $\bar{R}_i$ at all the bundles from from above. This establishes the Proposition. ■

Let $R_i', R_i'' \in \mathbb{D}^s$. We say $R_i' \succ R_i''$ if the indifference curves of R_i' cut the indifference curves of R_i'' from above at all bundles. Proposition 3.2.3 ensures that the order $\succ$ is well-defined.

The following Lemma describes an important set theoretic feature of a single-crossing domain which is rich.

Definition 3.2.5 *Let $\mathbb{D}^s$ be an arbitrary single-crossing domain. We say that $\mathbb{D}^s$ is a* **Linear Continuum** *if the following conditions hold.*

1. *If $R_i^{''} \succ R_i^{'}$ then there exists $R_i^{'''}$ such that $R_i^{''} \succ R_i^{'''} \succ R_i^{'}$.*

2. *The domain $\mathbb{D}^s$ has the least upper bound property.* [9]

Lemma 3.2.2 *If $\mathbb{D}^s$ is rich, then it is a linear continuum.*

Proof: Let $R_i^{''} \succ R_i^{'}$ and let (x_i, y_i) be arbitrary. Choose $(x_i^{'}, y_i^{'}) \in int\ UC(R_i^{'}, (x_i, y_i)) \cap int\ LC(R_i^{''}, (x_i, y_i))$. Note that (x_i, y_i) and $(x_i^{'}, y_i^{'})$ are diagonal. Since $\mathbb{D}^s$ is rich, there exists $R_i^{'''}$ such that it has an indifference curve that passes through these two bundles. Note that since $\mathbb{D}^s$ is a single-crossing domain, it will cut the indifference curve of $R_i^{'}$ from above and cut the indifference curve of $R_i^{''}$ from below at (x_i, y_i). Hence, $R_i^{''} \succ R_i^{'''} \succ R_i^{'}$.

Now we show that $\mathbb{D}^s$ has the least upper bound property. Choose an allocation $(x_i, y_i) \gg (0, 0)$. Consider a closed ball $N_\delta((x_i, y_i)) \subset \Re^2_{++}$ of (x_i, y_i) with radius δ. Observe that for all $R_i \in \mathbb{D}^s$, there exists an indifference curve passing thorough (x_i, y_i). Now consider the arc of the ball in $SEQ_i((x_i, y_i))$. Let the arc intersect the vertical axis through (x_i, y_i) at (x_i, y_i^*) and let it intersect the horizontal axis at (x_i^*, y_i). Note that by the single crossing property, indifference curves connecting every two diagonal bundles is unique. Hence we obtain a bijection $G : \mathbb{D}^s \rightarrow (x_i^*, x_i)$ where $G(R_i^{'}) = x_i^{'}$ if $IC(R_i^{'}, (x_i, y_i)) = IC(R_i^{'}, (x_i^{'}, y_i^{'}))$ with $x_i^{'} \in (x_i^*, x_i)$ and $y_i^{'} \in (y_i, y_i^*)$. Since (x_i^*, x_i) has the least upper bound property so also does $\mathbb{D}^s$.

■

From now on, we will assume that the topology on $\mathbb{D}^s$ is the order topology generated by the collection of open intervals of the form $(R_i^{'}, R_i^{''}) = \{R_i | R_i^{''} \succ R_i \succ R_i^{'}\}$. Since $\mathbb{D}^s$ is a linear continuum, $\mathbb{D}^s$ is connected in this order topology. [10] It follows that if F is a continuous SCF defined on $\mathbb{D}^s$, then $\Re_F$ is connected.

[9] A ordered set S is said to have the least upper bound property if every bounded subset of S has the supremum in S.

[10] See Munkres (2005).

According the the next Proposition, any interior allocation in Δ can be sustained as a Pareto-efficient allocation for some preference profile.

Proposition 3.2.4: Let $\mathbb{D}^s$ be rich. Consider an arbitrary allocation $((x_i, y_i), (x_j, y_j))$ in the interior of Δ. Then there exists $(R_i, R_j) \in [\mathbb{D}^s]^2$ such that $((x_i, y_i), (x_j, y_j)) \in PE(R_i, R_j)$.

Proof: Since $(x, y) = ((x_i, y_i), (x_j, y_j))$ is in the interior of Δ we can choose an $R_j \in \mathbb{D}^s$ such that $(x^1, y^1) \in IC(R_j, (x, y))$ in the interior of Δ such that (x^1, y^1) is in the $SEQ_i((x, y))$. Choose R_i^1 such that $IC(R_i^1, (x^1, y^1)) = IC(R_i^1, (x, y))$. Let $(x^2, y^2) \in PE(R_i^1, R_j)$ on $IC(R_j, (x, y))$. Then choose R_i^2 such that $IC(R_i^2, (x_i^2, y_i^2)) = IC(R_i^2, (x_i, y_i))$. Note that $R_i^2 \succ R_i^1$. In this way we construct an increasing sequence of preferences $\{R_i^k\}_{k=1}^{\infty}$ and an associated sequence of Pareto-efficient allocations for the profiles $\{(R_i^k, R_j)\}_{k=1}^{\infty}$ on $IC(R_j, (x, y))$ such that Euclidean distance between (x, y) and the Pareto-efficient allocations monotonically converge to zero. Now note that $\{R_i^k\}_{k=1}^{\infty}$ is bounded above by any $R_i \in \mathbb{D}^s$ which cuts $IC(R_j, (x, y))$ from above at (x, y). Since $\mathbb{D}^s$ has the least upper bounded property $\{R_i^k\}_{k=1}^{\infty}$ converges to its supremum. Since the Euclidean distance between (x, y) and $\{(x^k, y^k)\}_{k=1}^{\infty}$ monotonically converges to zero the supremum must be the R_i such that $(x, y) \in PE(R_i, R_j)$. ■

Remark 3.2.1: From the previous chapter we know that allocating zero amount of good x and positive of good y is not a Pareto-efficient allocation for the single-crossing domain in Example 3.2.2. Therefore, Proposition 3.2.4 is not true for non-interior allocations.

3.2.2 Discussion

In this subsection we compare our model with that of Saporiti (2009). He assumes a finite set of alternatives with cardinality at least three. He assumes strict orderings and that the domain is an ordered set. His definition of single-crossing property is as follows.

Definition 3.2.6 *(Saporiti (2009)) A set of preferences (strict) $\mathbb{L}$ exhibits the single-crossing property on the set of alternatives X if there is a linear order $>$ on X and a liner order $\succ$*

on $\mathbb{L}$ *such that for all* $a, b \in X$ *and for all* $P_i, P_i^{'} \in \mathbb{L}$

$$(SC1)\ [b > a, P_i^{'} \succ P_i \ and\ bP_ia] \Rightarrow bP_i^{'}a$$

and

$$(SC2)\ [b > a, P_i^{'} \succ P_i \ and\ aP_i^{'}b] \Rightarrow aP_ib.$$

We remark below on the differences between our model and his.

- Since we are concerned with allocations in an Edgeworth box, we do not assume finiteness of alternatives. The assumption of strict preferences is also inappropriate in our model because it rules out classical preferences.

- We do not directly impose an order on the domain. Instead we provide a more primitive definition of the single-crossing property in economic environments and derive an order on the domain.

- We also do not assume an order on $\Re_F$ (in his case, the set X). However, we show (by assuming the SCFs to be continuous in the order topology) that strategy-proofness and individual-rationality imply that $\Re_F$ is diagonal. Since there is a natural order on diagonal sets, we, in effect derive an ordering on $\Re_F$.

- Over a diagonal set, our notion of the single-crossing property implies the version of the property in Definition 3.2.6. Consider $SC(1)$. Suppose $IC(R_i^{'}, b)$ cuts $IC(R_i, b)$ from above in the sense of our definition. We then have $bP_i^{'}a$. Similarly for $SC(2)$, if $IC(R_i, a)$ cuts $IC(R_i^{'}, a)$ from below we have aP_ib.

3.3 Non Fixed-Price Trading Rules

In this section we give examples of SCFs that are strategy-proof and individually rational but are non FPT rules. These examples show that Barberà and Jackson (1995) characterization does not hold over single-crossing domains and highlights the role of concavification in their arguments. In addition, various features of these examples will illustrate the role of rich domains and continuity of the SCFs.

Example 3.3.1: In this example allocations a, b and c are as shown in Diagram 3.3. Preferences over this triple for agents 1 and 2 belong to the pair $(\mathbb{D}^1(\{a, b, c\}), \mathbb{D}^2(\{a, b, c\})$ (Proposition 3.2.1). Let the endowment ω be the allocation a. The SCF is given in Table 3.1 below.

	cP_2bP_2a	aP_2bP_2c	cI_2bI_2a
cP_1bP_1a	c	a	c
bP_1cP_1a	b	a	b
bP_1aP_1c	b	a	b
aP_1bP_1c	a	a	a
bI_1aP_1c	b	a	b
bP_1cI_1a	b	a	b
cI_1bP_1a	b	a	b

Table 3.1: A Non FPT Rule

It is easily verified that the SCF in Table 3.1 is strategy-proof, individually rational and has three elements in the range. It is not a FPT rule because all the allocations in the range lie on one side of the endowment but are not collinear. The range contains the endowment.

Consider a preference domain where aP_ibP_ic is not admissible for both $i = 1, 2$. The domain remains single-crossing although it is no longer rich. The SCF over the restricted domain is still strategy-proof and individually rational. However, the endowment is no longer in its range. This example in conjunction with our characterization result makes it clear that richness of the domain is critical in ensuring that the endowment lies in the range of the SCF.

The next example shows that agents' preferences need not be single-peaked on both sides of the endowment.

Example 3.3.2: Consider a rich domain of single-crossing preferences. The SCF is shown in Diagram 3.5. In the diagram ω and d are diagonal with $d \in FOQ_i(\omega)$. Consider a preference R_j^* such that $IC(R_j^*, \omega) = IC(R_j^*, d)$. Choose another allocation c in the interior of $LC(R_j^*, \omega)$ such that $c \in FOQ_i(\omega) \cap SEQ_i(d)$. Consider the two line segments joining ω and c and c and d. We will define a strategy-proof and individually rational SCF F such that $\Re_F = \overline{\omega c} \cup \overline{cd}$.

By the single-crossing property $Top(R_j, \overline{\omega c} \cup \overline{cd}) \in \{\omega, d\}$ for all R_j. Define the SCF F as follows:

$$F(R_i, R_j) = \begin{cases} \omega, & \text{if } \omega_j \in Top(R_j, \overline{\omega c} \cup \overline{cd}); \\ x, & \text{if } x_i = Top(R_i, \overline{\omega c} \cup \overline{cd}) \text{ and } b_j = Top(R_j, \overline{\omega c} \cup \overline{cd}) \end{cases}$$

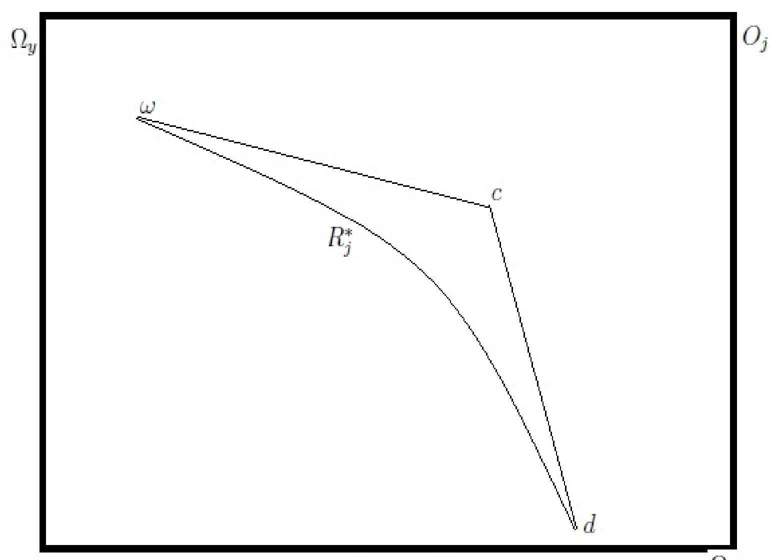

Diagram 3.5

Since the domain of F is rich there exists R_i such that $x = Top(R_i, \bar{\omega c} \cup \bar{cd})$ for all $x \in \bar{\omega c} \cup \bar{cd}$. We claim that F is strategy-proof. Note that agent i does not have any incentive to deviate when the outcome is ω because he cannot change the outcome by changing his announcement and when the outcome is other than ω he gets his best.

Agent j is not going to change her announcement when the outcome is ω because she is getting her best allocation. When the outcome is not ω she can change it to ω only. But if $d = Top(R_j, \overline{\omega c} \cup \overline{cd})$, then ω is the worst allocation for agent j in $\overline{\omega c} \cup \overline{cd}$ because of the single-crossing property. This SCF is also individually rational because an allocation other than ω is chosen only when both the agents are better-off relative to ω.

Observe that F is not continuous. To see this choose a profile (R_i^1, R_j^1) such that $d = Top(R_i, \overline{\omega c} \cup \overline{cd})$ for all i. Such a profile exists by richness and since $d \in int\ \Delta$.[11] Also consider R_j^2 such that $\omega = Top(R_j, \overline{\omega c} \cup \overline{cd})$. According to the construction of the order $\succ$ on the domains $R_j^2 \succ R_j^1$. By definition $F(R_i^1, R_j^1) = d$ and $F(R_i^1, R_j^2) = \omega$. Also note that $F(R_i^1, R_j) \in \{\omega, d\}$ for all R_j. Hence, continuity for agent j is violated.

We will show in our characterization result that if continuity is additionally imposed on the SCF, then agent preferences over the range on both sides of the endowment, are single-peaked. The next example shows that the range of a strategy-proof and individually rational SCF need not be piece-wise linear even if it is continuous.

Example 3.3.3: Let agent i's preferences be given by utility functions of the form $u_i(x_i, y_i; \theta_i) = \theta_i \sqrt{x_i} + y_i$ with $\theta_i > 0$ and $i = 1, 2$. In Example 3.2.2 we have shown that this is a rich domain of single-crossing preferences.

Let, $\Omega_x = 8$ and $\Omega_y = 4$. Let $\omega^1 = (4, 4 - 4^{\frac{2}{3}})$ and $\omega^2 = (4, 4^{\frac{2}{3}})$.

Let $B = \{((x_1, y_1)(x_2, y_2)) | y_1 + x_1^{\frac{2}{3}} = 4, ((x_1, y_1)(x_2, y_2)) \in FOQ_1(\omega)\}$. Pictorially the set B is depicted in Diagram 3.6.

[11] If d is not an interior allocations in Δ then this may not be true. We will discuss about this later.

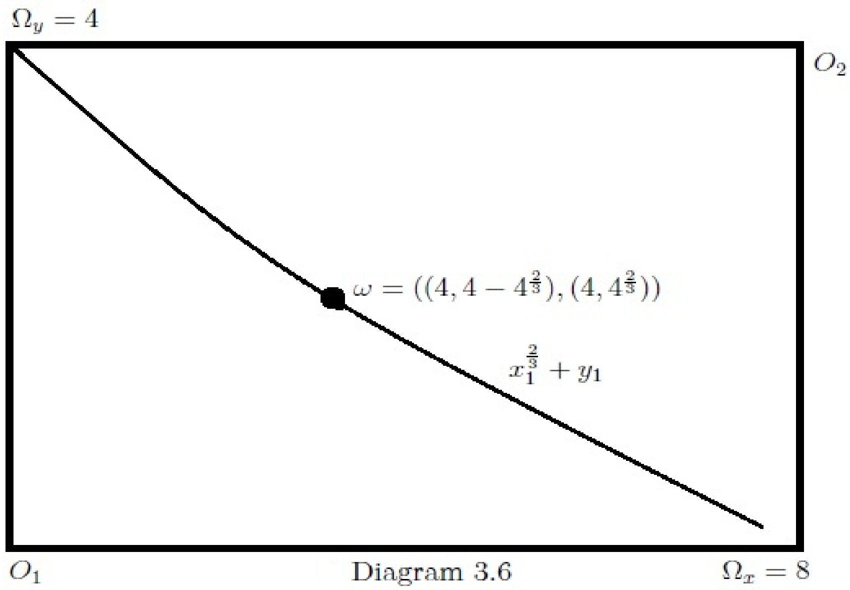

Diagram 3.6

Let $B' = B \setminus \{((8,0),(0,4)),((0,4),(8,0))\}$. Observe that $Top(\theta_i, B')$ is well defined for $i = 1,2$. By our choice of the domain and B', $\left(\left(\frac{3}{4}\theta_1\right)^6, 4 - \left(\frac{3}{4}\theta_1\right)^4\right)$ solves the problem $\text{Max}_{(0<x_1<8,0<y_1<4)}\theta_1\sqrt{x_1}+y_1$ s.t. $x_1^{\frac{2}{3}}+y_1 = 4$. In other words, $Top(\theta_1, B')$ is unique for every θ_1.

For agent 2 the optimization problem is $\text{Max}_{(0<x_2<8,0<y_2<4)}\theta_2\sqrt{x_2}+y_2$ s.t. $(8-x_2)^{\frac{2}{3}}+4-y_2 = 4$. The first order condition is given by the equation, $\theta_2 = \frac{4}{3}\frac{\sqrt{x_2}}{(8-x_2)^{\frac{1}{3}}}$. Note that derivative in the right hand side of the equation is $\frac{2}{3(8-x_2)^{\frac{1}{3}}\sqrt{x_2}} + \frac{4\sqrt{x_2}}{9(8-x_2)^{\frac{4}{3}}}$. It is continuous and strictly positive at all $x_2 \in (0,8)$ and hence by the Inverse Function Theorem, the solution of x_2 is a continuous and one to one function of θ_2. Hence, from the constraint the optimal solution for y_2 is also a continuous and one to one function of θ_2. In other words, $Top(\theta_2, B')$ is unique for every θ_2 and a continuous function of θ_2.

Define the SCF F as,

$$F(\theta_1,\theta_2) = \text{median}\{Top(\theta_1, B'), Top(\theta_2, B'), \omega\}.$$

Clearly, $\Re_F = B'$. It is easy to see that the SCF F is strategy-proof and individually rational. Since $Top(\theta_i, B')$ is continuous for $i = 1,2$, F is also continuous. Observe that in this example we do not include the allocations $((8,0),(0,4))$ and $((0,4),(8,0))$ in the range. This is because for any preference of agent 2 the allocation $(0,4)$ and $(8,0)$ cannot be attained as maximum on B. We discuss this issue in more detail in Example 3.3.5.

Saporiti (2009) shows that if a SCF is strategy-proof, (along with anonymous and unanimous) then the SCF must be a *Peak* rule. For two agent economies a Peak rule ensures that $F(R_i, R_j) \in \text{median}\{Top(R_i, \Re_F), Top(R_j, \Re_F), \tau)\}$, where $\tau = Top(R_*, \Re_F)$ for some $R_* \in \mathbb{D}^s$. The next example shows, in single-crossing domains as we have defined, that a Peak rule need not be continuous.

Example 3.3.4: Consider the same domain as in Example 3.3.3. Choose θ_2^*, $\theta_2^{'}$ and $B = \overline{d^{'}\omega} \cup \overline{\omega d}$ such that $Top(\theta_2^*, B) = \{d^{'}, d\}$. This is depicted in Diagram 3.7. Note that by single-crossing property, if $\theta_2 \neq \theta_2^*$ then $Top(\theta_2, B)$ is unique and $Top(\theta_2, B) \in \{d^{'}, d\}$.

Define the following SCF F,

$$F(\theta_1, \theta_2) = \begin{cases} d^{'}, & \text{if } d^{'} = Top(\theta_i, B) \text{ and } \theta_j = \theta_j^*; \\ d, & \text{if } d = Top(\theta_i, B) \text{ and } \theta_j = \theta_j^*; \\ \text{median}\{Top(\theta_1, B), Top(\theta_2, B), \omega\}, & \text{otherwise.} \end{cases}$$

Clearly $B = \Re_F$ and $\omega \in \Re_F$. Note that there exists R_* such that $\omega = Top(R_*, \Re_F)$. The SCF is a Peak rule because $F(\theta_1, \theta_2) \in \text{median}\{Top(\theta_1, \Re_F), Top(\theta_2, \Re_F), \omega)\}$ or according to the tie breaking rule that selects either $d^{'}$ or d. It is also strategy-proof and individually rational. But it is not continuous due to the reasons similar to in Example 3.3.2.

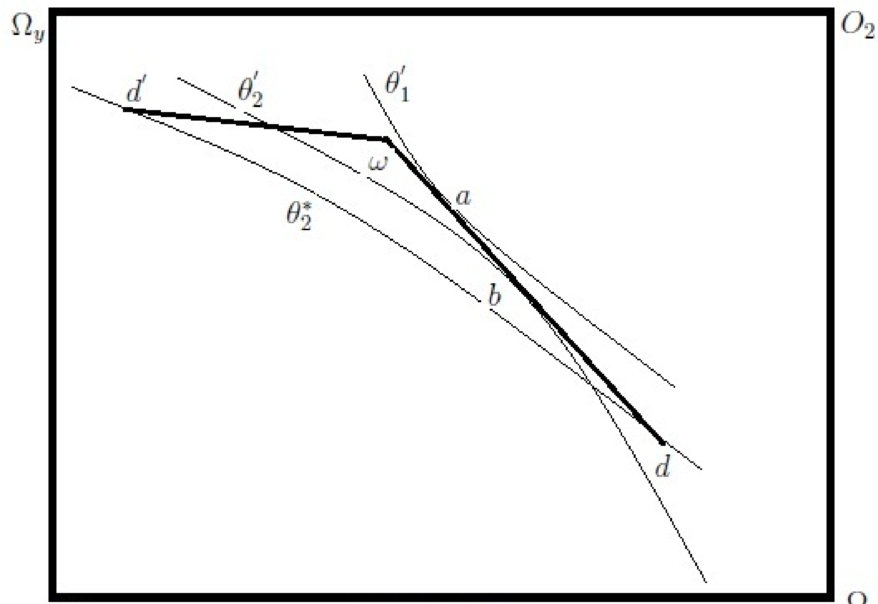

Diagram 3.7

In our characterization result we will assume that the range of a SCF is in the interior of Δ. Our next example shows that in the absence of this assumption characterization is hard even with the assumption of continuity of SCFs.

Example 3.3.5: Consider the domain in Example 3.3.3. The slope of an indifference curve of an utility function of this domain is $-\frac{\theta_i}{2\sqrt{x_i}}$. Note that $\lim_{x_i \to 0} -\frac{\theta_i}{2\sqrt{x_i}} = -\infty$. Consider any straight line $y_i = -mx_i + c$, where $m > 0, c > 0$. The consumption bundle $(0, c)$ cannot be sustained as a top on this straight line for any $\theta_i > 0$. This is because for any $\theta_i > 0$ there exists $\epsilon > 0$ such that $-\frac{\theta_i}{2\sqrt{x_i}} < -m$ for all $x_i \in (0, \epsilon)$.

Consider the set $\overline{d'\omega} \cup \overline{\omega d}$ in Diagram 3.7. Suppose instead of assuming d' to be an interior allocation, let d' lie on the x axis i.e. $d_1'^x = 0$ and $d_1'^y > 0$. According to the definition of GT rule (Definition 3.4.1) $d' \in \Re_F$ only if $d' \in Top(\theta_1, \overline{d'\omega} \cup \overline{\omega d})$. But $d' \notin Top(\theta_1, \overline{d'\omega} \cup \overline{\omega d})$ due to the reasons that were discussed in the previous paragraph. Hence, a GT rule defined on this domain cannot have a range that is $\overline{d'\omega} \cup \overline{\omega d}$, where $d_1'^x = 0$ and $d_1'^y > 0$.

These examples show that if the domain of a SCF is rich and single-crossing then whether the boundary allocations of Δ are contained in the range depends on the domain. Such complications do not arise in Barberà and Jackson (1995) because the domain in their analysis is rich enough to permit all kinds of corner allocations to be tops.

Remark 3.3.1: In the light of the discussion in Example 3.3.5, we will assume $\Re_F \subset int\ \Delta$.

3.4 A Characterization Result

In this section we will provide a characterization of SCFs, defined on rich single-crossing domains, that are strategy-proof, individually rational and continuous. We provide a characterization for a class of trading rules that we *Generalized Trading* rules. We define such rules as follows.

Definition 3.4.1 *Let $F : [\mathbb{D}]^2 \rightarrow \Delta$ be a SCF, where $\mathbb{D} \subset \mathbb{D}^c$. We say F is a* **Generalized Trading (GT)** *rule if the following takes place.*

1. $\omega \in \Re_F$.
2. $\Re_F$ *is diagonal.*
3. *Agent preferences are single-peaked on both sides of the endowment.*
4. *There exists an agent i such that $F(R_i, R_j) = median\{Top(R_i, \Re_F), Top(R_j, SEQ_i(\omega) \cap \Re_F), \omega)\}$ or $F(R_i, R_j) = median\{Top(R_i, \Re_F), Top(R_j, FOQ_i(\omega) \cap \Re_F), \omega\}$.*

From Definition 3.4.1 it follows that F is a GT rule then $\omega \in \Re_F$. The range of a GT rule is diagonal and agent preferences are single-peaked on both sides of the endowment. However, the range of a GT rule need not be piece-wise linear. Example 3.3.3 is an example of a GT rule. In this example the range of the SCF is not piece-wise linear. Following Remark 3.3.1, we provide a characterization of GT rules with an additional assumption that $\Re_F \subset int\ \Delta$. We will also assume that the SCF F is continuous. Hence $\Re_F$ is connected because $\mathbb{D}^s$ is connected. Because of the assumption $\Re_F \subset int\ \Delta$ and that the range is connected, our characterization is true only if $\omega_i \gg (0, 0)$. Therefore we will assume that $\omega_i \gg (0, 0)$ for

$i = 1, 2$. We show that if a SCF defined on a rich single-crossing domain is strategy-proof, individually rational and continuous then it must be a GT rule.

Theorem 3.4.1 *Let $\omega_i \gg (0,0)$ for $i = 1, 2$ and let $F : [\mathbb{D}^s]^2 \to \Delta$ be a continuous SCF such that $\Re_F \subset int\ \Delta$ and $\Re_F$ is a closed set. Then F is strategy-proof and individually rational if and only if it is a GT rule.*

A GT rule by definition is strategy-proof and individually rational. The other direction of the proof of this result is contained in Appendix 3.6.[12]

Observation 3.4.1 Nicolò (2004) examines a two-good, two-agent exchange economy with the domain of Leontief preferences.[13] He shows that the range of a strategy-proof and individually rational SCF must consist of a set of diagonal allocations containing the endowment. Our result is intermediate between the Nicolò (2004) and Barberà and Jackson (1995) result in the sense that the range of a SCF on a rich single-crossing domain (satisfying continuity) must satisfy some restrictions but can be more general than being piece-wise linear. It must however contain the endowment like in the other cases. Furthermore, agent preferences on either side of the endowment must be single-peaked.

3.5 Conclusion

In this chapter we have formulated the notion of rich single-crossing domains for a two-good exchange economies. We have characterized the class of strategy-proof, individually rational and continuous SCFs and identified them to be the class of Generalized Trading rules. This class is wider than the class of Fixed-Price Trading rules identified in Barberà and Jackson (1995). This result illustrates the importance of domains that admit concavification for the Barberà and Jackson (1995) result.

[12] Although we have assumed the range to be closed, with some small changes in the proof the result can be proved without assuming the range to be closed.

[13] Observe that these preferences are not classical.

An important question for further research is whether the GT rule result can be extended to the case of an arbitrary number of agents with additional assumptions such as anonymity and non-bossiness as in Barberà and Jackson (1995) or in the previous chapter of the thesis.

3.6 Appendix

We prove Theorem 3.4.1. Good x will be measured along the horizontal axis and that Good y is measured along the y axis.

Proof: We first show that the endowment is in the range. This result does not require either strategy-proofness or continuity.

Lemma 3.6.1 *Let $\mathbb{D}^s$ be a rich domain. Let $\omega_i \gg (0,0)$ for all i. Let the SCF $F : [\mathbb{D}^s]^2 \rightarrow \Delta$ be individually rational. Then $\omega \in \Re_F$.*

Proof: Since $\mathbb{D}^s$ is rich, we know from Proposition 3.2.4 that there exists a profile (R_i, R_j) such that $\omega \in PE(R_i, R_j)$. Note that indifference curves are strictly convex. Therefore, for any allocation other than ω at the profile (R_i, R_j), F will violate individual-rationality for at least one agent. Hence, $F(R_i, R_j) = \omega$ i.e. $\omega \in \Re_F$. ■

We now establish a monotonicity result with reference to the order defined earlier on $\mathbb{D}^s$.

Lemma 3.6.2 *Let $\mathbb{D}^s$ be a rich domain. Let $F : [\mathbb{D}^s]^2 \rightarrow \Delta$ be a strategy-proof SCF. If $R_i' \succ R_i''$ then $x_i(R_i', R_j) \geq x_i(R_i'', R_j)$.*

Proof: By strategy-proofness $F_i(R_i', R_j) \in LC(R_i'', F_i(R_i'', R_j)) \cap UC(R_i', F_i(R_i'', R_j))$. Hence, $x_i(R_i', R_j) \geq x_i(R_i'', R_j)$. ■

Next we show that if a SCF is continuous, strategy-proof and individually rational, then its range is diagonal. We will denote the set $\{(R_i, R_j^*) | R_i \in [\underline{R}_j, \bar{R}_j]\}$ by $[\underline{R}_j, \bar{R}_j; R_j^*]$.

Lemma 3.6.3 *Let $\mathbb{D}^s$ be rich. If the SCF $F : [\mathbb{D}^s]^2 \rightarrow \Delta$ is strategy-proof, individually rational and continuous, then $\Re_F$ is diagonal.*

Proof:

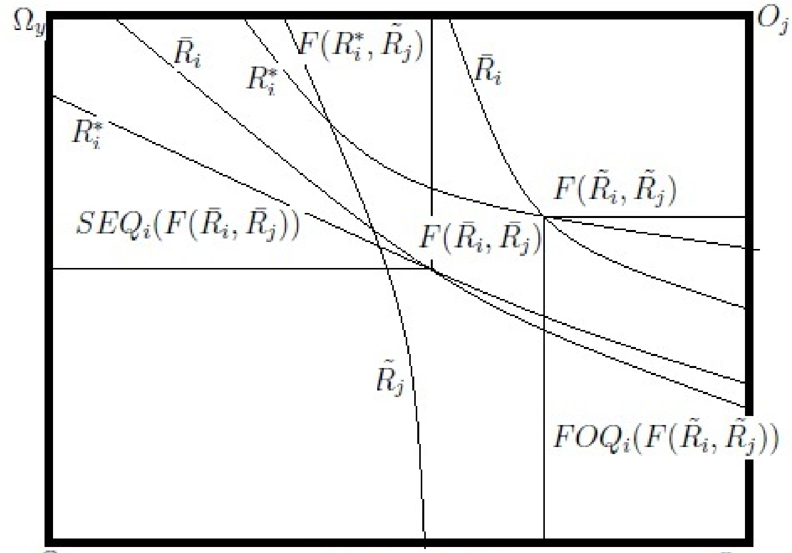

Diagram 3.8

We will prove this Lemma by contradiction. Without the loss of generality we can assume that $F_i(\tilde{R}_i, \tilde{R}_j) \gg F_i(\bar{R}_i, \bar{R}_j)$ as shown in Diagram 3.8. Note that by strategy-proofness $\bar{R}_k \neq \tilde{R}_k$ for all $k \in I$. From Lemma 3.6.2 it follows that if $\tilde{R}_i \succ \bar{R}_i$ then $\tilde{R}_j \succ \bar{R}_j$. Similarly, if $\bar{R}_i \succ \tilde{R}_i$ then $\bar{R}_j \succ \tilde{R}_j$. Therefore, without loss of generality we assume that $\tilde{R}_i \succ \bar{R}_i$. Also, note by individual-rationality, either $\omega \in int\ FOQ_i(F(\tilde{R}_i, \tilde{R}_j))$ or $\omega \in int\ SEQ_i(F(\bar{R}_i, \bar{R}_j))$. We will prove the Lemma with the help of 8 claims.

Claim 1: $\omega \in int\ FOQ_i(F(\tilde{R}_i, \tilde{R}_j))$.

Proof of Claim 1: We explain the argument with the help of Diagram 3.8. Choose R_i^* such that $\bar{R}_i \succ R_i^*$ and $UC(R_i^*, F(\tilde{R}_i, \tilde{R}_j)) \cap SEQ_i(F(\tilde{R}_i, \tilde{R}_j)) \cap UC(R_i^*, F(\bar{R}_i, \bar{R}_j)) \cap LC(\bar{R}_i, F(\bar{R}_i, \bar{R}_j)) \cap \Delta = \emptyset$. We can find such an R_i^* since $\mathbb{D}^s$ is rich and $F(\bar{R}_i, \bar{R}_j)$ and $F(\tilde{R}_i, \tilde{R}_j)$ are not diagonal.

Note that $\tilde{R}_i \succ R_i^*$ because $\tilde{R}_i \succ \bar{R}_i$ and transitivity of $\succ$. Since $\tilde{R}_i \succ R_i^*$, by Lemma 3.6.2, $F(R_i^*, \tilde{R}_j) \in SEQ_i(F(\tilde{R}_i, \tilde{R}_j))$. Since $\tilde{R}_j \succ \bar{R}_j$, by Lemma 3.6.2 $F(R_i^*, \bar{R}_j) \in FOQ_i(F(R_i^*, \tilde{R}_j))$. By strategy-proofness $F(R_i^*, \bar{R}_j) \in UC(R_i^*, F(\bar{R}_i, \bar{R}_j)) \cap LC(\bar{R}_i, F(\bar{R}_i, \bar{R}_j))$ and $F(R_i^*, \bar{R}_j) \in LC(\tilde{R}_j, F(R_i^*, \tilde{R}_j))$. Since indifference curves are downward sloping the indifference curve of $\tilde{R}_j$ passing through $F(R_i^*, \tilde{R}_j)$ must also intersect $UC(R_i^*, F(\bar{R}_i, \bar{R}_j)) \cap$

$LC(\bar{R}_i, F(\bar{R}_i, \bar{R}_j))$, as shown in Diagram 3.8. Therefore, for individual-rationality to be satisfied for agent j at $F(\tilde{R}_i, \tilde{R}_j)$, $\omega \in int\ FOQ_i(F(\tilde{R}_i, \tilde{R}_j))$. This establishes Claim 1.

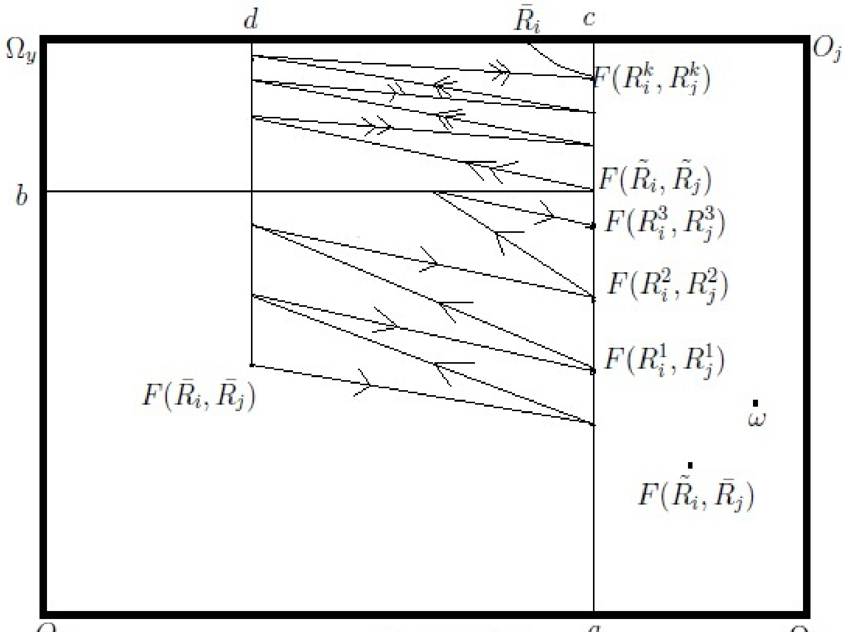

Diagram 3.9

To prove the Lemma we construct a sequence of profiles $\{(R_i^k, R_j^k)\}_{k=1}^{\infty}$ such that $(R_i^k, R_j^k) \succ\succ (\tilde{R}_i, \tilde{R}_j)$ [14] and $F(R_i^k, R_j^k)$ converges to the boundary of Δ along the line segment joining $F(\tilde{R}_i, \tilde{R}_j)$ and c, (Diagram 3.9). Then for large enough k we will find $F(R_i^k, R_j^k)$ such that the indifference curve of $\bar{R}_i$ passing through $F(R_i^k, R_j^k)$ lies in $int\ SEQ_i(F(\tilde{R}_i, \tilde{R}_j)) \cap int\ FIQ_i(F(\bar{R}_i, \bar{R}_j))$ as shown in Diagram 3.9. Then by Lemma 3.6.2, $F(\bar{R}_i, R_j^k)$ lies in the triangular region as shown in Diagram 3.9 which is the intersection of $UC(\bar{R}_i, F(R_i^k, R_j^k))$ and $SEQ_i(F(R_i^k, R_j^k))$. Then agent j will manipulate F, at $(\bar{R}_i, R_j^k)$ via $\bar{R}_j$.

However, before this we need to show existence of another sequence of profiles such that, $(\tilde{R}_i, \tilde{R}_i) \succ\succ (R_i^{k+1}, R_j^{k+1}) \succ\succ (R_i^k, R_j^k) \succ\succ (\bar{R}_i, \bar{R}_j)$, $(R_i^k, R_j^k) \to (\tilde{R}_i, \tilde{R}_j)$ and $F(R_i^k, R_j^k)$ lie on the line segment that connects a and $F(\tilde{R}_i, \tilde{R}_j)$ for all k (Diagram 3.9). Claims 2, 3 and

[14] Here, $(R_i^{'}, R_j^{'}) \succ\succ (R_i^{''}, R_j^{''})$ signifies $R_i^{'} \succ R_i^{''}$ and $R_j^{'} \succ R_j^{''}$ and $(R_i^{'}, R_j^{'}) \succeq (R_i^{''}, R_j^{''})$ signifies $R_i^{'} \succeq R_i^{''}$ and $R_j^{'} \succeq R_j^{''}$.

4 are important intermediate steps in order to construct this sequence. Define the following set,

$\mathbb{A} = \{(R_i, R_j) | x_i(R_i, R_j) = a_i^x, a_i^y \leq y_i(R_i, R_j) \leq y_i(\tilde{R}_i, \tilde{R}_j),$ for all i and $(\tilde{R}_i, \tilde{R}_j) \succeq (R_i, R_j) \succeq (\bar{R}_i, \bar{R}_j)\}$.

Claim 2 shows that there exists profiles other than $(\tilde{R}_i, \tilde{R}_j)$ that lie in $\mathbb{A}$.

Claim 2: There exists $(R_i, R_j) \in \mathbb{A}$ such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i, R_j) \succ\succ (\bar{R}_i, \bar{R}_j)$.

Proof of the Claim: Since F is strategy-proof $F(\tilde{R}_i, \bar{R}_j) \in int\ FOQ(F(\tilde{R}_i, \tilde{R}_j))$. Therefore, by continuity of F there exists R_i^* such that $\tilde{R}_i \succ R_i^* \succ \bar{R}_i$ and $F(R_i^*, \bar{R}_j)$ lies on the line segment joining a and $F(\tilde{R}_i, \tilde{R}_j)$.[**]

Since F is continuous, $F(R_i^*, R_j) \neq F(R_i^*, \bar{R}_j)$ for all $R_j \in (\bar{R}_j, \tilde{R}_j)$. If $F(R_i^*, R_j) = F(R_i^*, \bar{R}_j)$ for all $R_j \in (\bar{R}_j, \tilde{R}_j)$ then by continuity of F, $F(R_i^*, \tilde{R}_j) = F(R_i^*, \bar{R}_j)$. Then agent i will manipulate F at $(R_i^*, \tilde{R}_j)$ via $\tilde{R}_i$. Choose R_j^{**} such that $F(R_i^*, R_j^{**}) \in int\, THQ_i(F(\tilde{R}_i, \tilde{R}_j))$. By [**] there exists R_i^{**} such that $\tilde{R}_i \succ R_i^{**} \succ R_i^*$ and $F(R_i^{**}, R_j^{**})$ lies on the line segment joining a and $F(\tilde{R}_i, \tilde{R}_j)$. Hence the Claim is established.

The following Claim is another important step in proving the existence of the first sequence.

Claim 3: Let $(R_i^{'}, R_j^{'})$ be such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i^{'}, R_j^{'})$ and $(R_i^{'}, R_j^{'}) \in \mathbb{A}$. Then there exists $(R_i^{''}, R_j^{''})$ such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i^{''}, R_j^{''}) \succ\succ (R_i^{'}, R_j^{'})$ and $(R_i^{''}, R_j^{''}) \in \mathbb{A}$.

Proof of the Claim: The proof follows immediately by repeating the arguments that has been used to prove Claim 1. In the first step increase agent j's preference and then in the second step increase agent i's.

The following Claim in conjunction with Claim 2 and 3 will give us the desired sequence.

Claim 4: There does not exist a profile (R_i^*, R_j^*) such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i^*, R_j^*) \succ\succ (\bar{R}_i, \bar{R}_j)$ and $(R_i, R_j) \notin \mathbb{A}$ if $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i, R_j) \succ\succ (R_i^*, R_j^*)$.

Proof of the Claim: Since F is continuous $\mathbb{A}$ is a closed set in the order topology. Therefore set $\mathbb{A}$ contains all its limit points. Therefore by Claim 3 th proof follows.

Note that we do not consider any profile (R_i, R_j) such as $R_i = \tilde{R}_i$ and $\tilde{R}_j \succ R_i$ in the above Claims because they violate strategy-proofness.

Consider a sequence $S^1 = \{(R_i^{1n}, R_j^{1n})\}_{n=1}^{\infty} \subset \mathbb{A}$ such that $(R_i^{1(n+1)}, R_j^{1(n+1)}) \succ\succ (R_i^{1n}, R_j^{1n})$. Since each component of the sequence is strictly monotonic and bounded above they must converge to their respective supremum. Let $S^1 \to (R_i^1, R_j^1)$. If $(R_i^1, R_j^1) = (\tilde{R}_i, \tilde{R}_j)$ then we are done. Let $(R_i^1, R_j^1) \neq (\tilde{R}_i, \tilde{R}_j)$. By Claim 4 $(R_i^1, R_j^1) \in \mathbb{A}$. Define the first term of the desired sequence to be (R_i^1, R_j^1). By Claim 3 we will obtain another sequence S^2 that converges to $(R_i^2, R_j^2) \succ\succ (R_i^1, R_j^1)$. Let the second term of the desired sequence to be (R_i^2, R_j^2). By continuing this process we obtain the desired sequence.

Now we show how a sequence of profiles can be constructed such that its outcome belong to the vertical line that joins $F(\tilde{R}_i, \tilde{R}_j)$ and c (Diagram 3.9) and converges to c. Along with the existence of the sequence in the previous paragraph claims 1, 5, 6, 7 and 8 are the important intermediate steps to construct this sequence.

Claim 5: For some $R_j \succ \tilde{R}_j$, $F(\tilde{R}_i, R_j) \in int\ SEQ_i(F(\tilde{R}_i, \tilde{R}_j))$.

Proof of the Claim: Assume for the sake of contradiction that this does not happen. Then for all $R_j \succ \tilde{R}_j$, $F(\tilde{R}_i, R_j) = F(\tilde{R}_i, \tilde{R}_j)$. Consider $\tilde{\tilde{R}}_j \succ \tilde{R}_j$. By the argument above, we can construct an increasing sequence of profiles whose outcomes lie in $THQ_i(F(\tilde{R}_i, \tilde{R}_j))$ and which converges to $(\tilde{R}_i, \tilde{\tilde{R}}_j)$. Observe that for k large enough we can choose R_j^k such that $\tilde{\tilde{R}}_j \succ R_j^k \succ \tilde{R}_j$ and $F(R_i^k, R_j^k) \in THQ_i(F(\tilde{R}_i, \tilde{\tilde{R}}_j))$. But $F(\tilde{R}_i, R_j^k) = F(\tilde{R}_i, \tilde{R}_j) P_i^k F(R_i^k, R_j^k)$. Hence agent i will manipulate F at (R_i^k, R_j^k) via $\tilde{R}_i$. Hence, the Claim is established.

Since $F(\bar{R}_i, R_j) \in SEQ_i(F(\bar{R}_i, \bar{R}_j))$ for all $R_j \succ \bar{R}_j$ and F is continuous by Claim 5 there exists $R_j^* \succ \tilde{R}_j$ such that $F(\tilde{R}_i, R_j^*)$ picks an allocation from the line segment joining $F(\bar{R}_i, \bar{R}_j)$ and d (Diagram 3.9). Now we prove the following important claims.

Claim 6: There exists $(R_i^*, R_j^*) \succ\succ (\tilde{R}_i, \tilde{R}_j)$ such that $F(R_i^*, R_j^*)$ picks from the line segment joining c and $F(\tilde{R}_i, \tilde{R}_j)$.

Proof of the Claim: By Claim 5, there exists $R_j^* \succ \tilde{R}_j$, $F(\tilde{R}_i, R_j^*) \in int\ SEQ_i(F(\tilde{R}_i, \tilde{R}_j))$. Note that $\omega \in int\ FOQ(F(\tilde{R}_i, \tilde{R}_j))$. If $F(R_i, R_j^*) = F(\tilde{R}_i, R_j^*)$ for all $R_i \succ \tilde{R}_i$ then for

R_i large enough (according the order $\succ$) individual-rationality of agent i will be violated. Therefore, $F(R_i, R_j^*) \neq F(\tilde{R}_i, R_j^*)$ for all $R_i \succ \tilde{R}_i$. By continuity of F and applying the argument of individual-rationality of agent i the proof follows.

The following Claim is important.

Claim 7: There does not exist $(R_i^{***}, R_j^{***}) \succ\succ (\tilde{R}_i, \tilde{R}_j)$ such that for all profiles greater than (R_i^{***}, R_j^{***}), the SCF F chooses allocation off the line segment connecting c and $F(\tilde{R}_i, \tilde{R}_j)$.

Proof of the Claim: Define the set,

$$\tilde{\mathbb{A}} = \{(R_i, R_j) | (R_i, R_j) \succeq (\tilde{R}_i, \tilde{R}_j), x_i(R_i, R_j) = c_i^x \text{ and } c_i^y \geq y_i(R_i, R_j) \geq y_i(\tilde{R}_i, \tilde{R}_j), \text{ for all } i\}.$$

Consider an sequence from $\tilde{\mathbb{A}}$ that converges to (R_i^{***}, R_j^{***}). Since F is continuous $(R_i^{***}, R_j^{***}) \in \tilde{\mathbb{A}}$. Repeating the arguments in Claim 6 it follows that for every profile $(R_i^{'}, R_j^{'}) \succ\succ (\tilde{R}_i, \tilde{R}_j)$ for which the SCF F chooses an allocation from the line segment joining c and $F(\tilde{R}_i, \tilde{R}_j)$ we can find $(R_i^{''}, R_j^{''}) \succ\succ (R_i^{'}, R_j^{'})$ such that $F(R_i^{''}, R_j^{''})$ lies on the line segment joining c and $F(\tilde{R}_i, \tilde{R}_j)$. Hence, the proof follows.

Note that the outcome paths in $SEQ_i(F(\tilde{R}_i, \tilde{R}_j))$ are such that when we fix agent i's preference and change j's and then when we change i's, the latter one lies above the former. Consider the following arguments to see this. Observe that since F is continuous $F([\tilde{R}_j, R_j^1; \tilde{R}_i])$[15] and $F([\tilde{R}_i, R_i^1; R_j^1])$ are connected paths in Δ.

Claim 8: The path $F([\tilde{R}_i, R_i^1; R_j^1])$ does not lie below $F([\tilde{R}_j, R_j^1; \tilde{R}_i])$ (from agent i's origin).

Proof of the Claim: Let for the sake of contradiction $F([\tilde{R}_i, R_i^1; R_j^1])$ lies below $F([\tilde{R}_j, R_j^1; \tilde{R}_i])$. Then we will obtain two profiles (R_i^*, R_j^1) and $(\tilde{R}_i, R_j^*)$ such that: (i) $R_i^* \succ \tilde{R}_i$ and $R_j^1 \succ R_j^*$ (ii) $F(R_i^*, R_j^1)$ lies on $F([\tilde{R}_i, R_i^1; R_j^1])$ and $F(\tilde{R}_i, R_j^*)$ lies on $F([\tilde{R}_j, R_j^1; \tilde{R}_i])$ and (iii) $F(R_i^*, R_j^1)$ and $F(\tilde{R}_i, R_j^*)$ are non-diagonal. Then by Claim 1, $\omega \in SEQ_i(F(R_i^*, R_j^1))$ and hence by individual-rationality of F, $\omega \in SEQ_i(F(\bar{R}_i, \bar{R}_j))$. This cannot happen because $(\bar{R}_i, \bar{R}_j)$ and $(\tilde{R}_i, \tilde{R}_j)$ are non-diagonal allocations. Hence, we reach a contradiction.

[15] $[\mathrm{R}_i^{'}, R_i^{''}; R_j^{'}] = \{(R_i, R_j) | R_i^{''} \succeq R_i \succeq R_i^{'}, R_j = R_j^{'}\}$.

Moving back and forth along a single path and/or existence of an allocation on the line segment such that beyond that allocation the SCF F does not choose an allocation from the line segment are not possible. If any of these happens then for R_i large enough (according to the order $\succ$) F will violate individual-rationality for agent i because $\omega \in int\ FOQ_i(F(\tilde{R}_i, \tilde{R}_j))$.

■

Remark 3.6.1: Note that we have not used closedness of $\Re_F$ in the above Lemma.

Remark 3.6.2: Since $\mathbb{D}^s$ is connected in the order topology and F is continuous, $\Re_F$ is connected. Hence, the diagonal property of $\Re_F$ implies that $\Re_F$ can be written as $\widetilde{d'\omega} \cup \widetilde{\omega d}$ where $d' \in SEQ_i(\omega)$ and $d \in FOQ_i(\omega)$, for some i.

The following Lemma demonstrates a relationship between the end elements on a diagonal set and single-crossing preferences.

Lemma 3.6.4 *Let $\mathbb{D}^s$ be rich. Let $\Re_F \subset int\ \Delta$ be closed and is given by $\widetilde{d'\omega} \cup \widetilde{\omega d}$. Then there exists profiles (R_i^*, R_j^*) and (R_i^{**}, R_j^{**}) such that,*

1. *dP_i^*b and dP_j^*b for all $b \in \Re_F$, $d'P_i^{**}b$ and $d'P_j^{**}b$ for all $b \in \Re_F$.*

2. *We can choose (R_i^*, R_j^*) and (R_i^{**}, R_j^{**}) in a way that for all i, $\omega P_i^* b$ for all $b \in \widetilde{d'\omega} \setminus \{\omega\}$ and $\omega P_i^{**} b$ for all $b \in \widetilde{\omega d} \setminus \{\omega\}$.*

Proof: We follow the proof in Diagram 3.10.

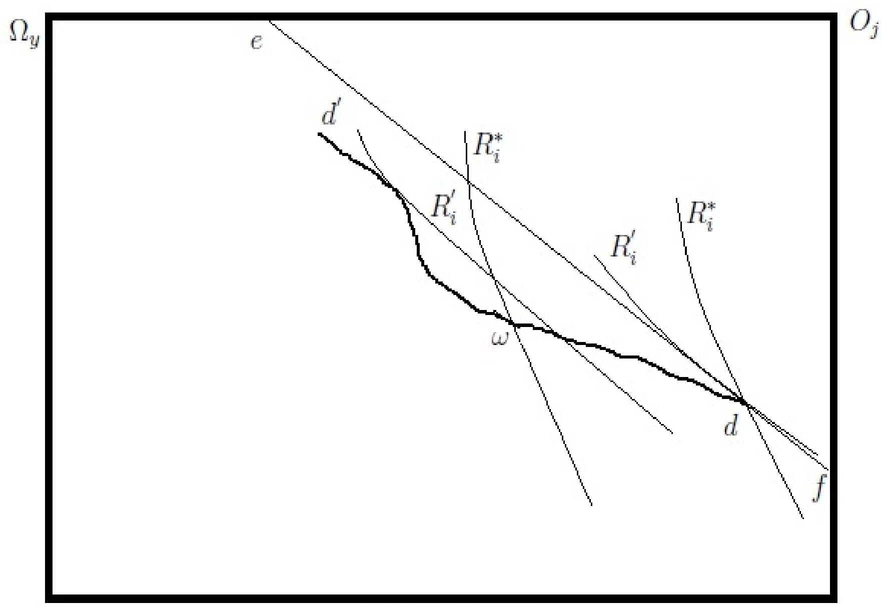

Diagram 3.10

In this diagram the darker diagonal set is the range of the SCF F. Since, $\Re_F \subset int\ \Delta$ and is closed we can find $\overline{ef}$ such that the range lies entirely below this line (from agent i's origin). Since d is an interior allocation and $\mathbb{D}^s$ is rich we can replicate the argument in Proposition 3.2.4 to show that there exists $R_i^{'}$ such that $d = Top(R_i^{'}, \overline{ef})$. Hence (1) follows.

However note that existence of $R_i^{'}$ such that $d = Top(R_i^{'}, \Re_F)$ does not mean that $\omega P_i^{'} b$ for all $b \in \widetilde{d'\omega} \setminus \{\omega\}$. But by richness of $\mathbb{D}^s$ we can choose R_i^* such that $IC(R_i^*, \omega)$ cuts $IC(R_i^{'}, \omega)$ and $\overline{ef}$ from above. Hence by Proposition 3.2.3, (2) follows.

■

A feature of FPT rules is that their range is a closed set. The following property of closed range is useful.

Lemma 3.6.5 *Let $\mathbb{D}^s$ be rich. Let the SCF $F : [\mathbb{D}^s]^2 \rightarrow \Delta$ be strategy-proof, individually rational and continuous. Let $\Re_F \subset int\ \Delta$ be closed and is given by $\widetilde{d'\omega} \cup \widetilde{\omega d}$. If the profiles (R_i^*, R_j^*) and (R_i^{**}, R_j^{**}) are such that they satisfy (1) and (2) in Lemma 3.6.4 then $F(R_i^*, R_j^*) = d$ and $F(R_i^{**}, R_j^{**}) = d^{'}$.*

Proof: We prove the Lemma by contradiction. Without the loss of generality consider d.

Assume for the sake of contradiction $F(R_i^*, R_j^*) = b^* \neq d$. Note that by individual-rationality of F, $b^* \in FOQ_i(\omega)$.

We will show that $d \notin \Re_F$. Consider the following four cases:

Case (a): $R_i \succ R_i^*$ and $R_j \succ R_j^*$.

Case (b): $R_i^* \succ R_i$ and $R_j \succ R_j^*$.

Case (c): $R_i \succ R_i^*$ and $R_j^* \succ R_j$.

Case (d): $R_i^* \succ R_i$ and $R_j^* \succ R_j$.

Case (a): Since $dP_i^* b^*$, by strategy-proofness $F(R_i, R_j^*) \neq d$. If $F(R_i, R_j^*) = d$ then agent i will manipulate at (R_i^*, R_j^*) via R_i. Since $R_i \succ R_i^*$, by Lemma 3.6.2 $F(R_i, R_j^*) \in \widetilde{\omega d} \setminus \{d\}$. Since $R_j \succ R_j^*$, by Lemma 3.6.2 $F(R_i, R_j) \in SEQ_i(F(R_i, R_j^*))$. Hence, $F(R_i, R_j) \neq d$.

Case (b): Since $R_i^* \succ R_i$ by Lemma 3.6.2 $F(R_i, R_j^*)$ is in $SEQ_i(b^*)$. By Lemma 3.6.2 again $F(R_i, R_j) \in SEQ_i(F(R_i, R_j^*))$ since $R_j \succ R_j^*$. Therefore, $F(R_i, R_j) \in SEQ_i(b^*)$. Hence, $F(R_i, R_j) \neq d$.

Case (c): Since $dP_i^* b^*$, by strategy-proofness $F(R_i, R_j^*) \neq d$. If $F(R_i, R_j^*) = d$ then agent i will manipulate at (R_i^*, R_j^*) via R_i. Since $R_i \succ R_i^*$, by Lemma 3.6.2 $F(R_i, R_j^*) \in \widetilde{\omega d} \setminus \{d\}$. Again by strategy-proofness $F(R_i, R_j) \neq d$ since $dP_j^* F(R_i, R_j^*)$. Hence $F(R_i, R_j) \neq d$.

Case (d): Since $dP_j^* b^*$, by strategy-proofness $F(R_i^*, R_j) \neq d$. Since $R_i^* \succ R_i$, by Lemma 3.6.2 $F(R_i, R_j) \in SEQ_i(F(R_i^*, R_j))$. Hence, $F(R_i, R_j) \neq d$.

Also note that by Lemma 3.6.2, if $R_i^* \succ R_i$ then $F(R_i, R_j^*) \in SEQ_i(b^*)$ and if $R_j \succ R_j^*$ then $F(R_i^*, R_j) \in SEQ_i(b^*)$. The cases above shows that $d \notin \Re_F$, which is a contradiction. Hence $F(R_i^*, R_j^*) = d$. ■

The following Lemma proves that agent preferences are single-peaked on both sides of the endowment.

Lemma 3.6.6 *Let $\mathbb{D}^s$ be rich. Let the SCF $F : [\mathbb{D}^s]^2 \to \Delta$ be strategy-proof, individually rational and continuous. Let $\Re_F \subset int\ \Delta$ be closed and is given by $\widetilde{d'\omega} \cup \widetilde{\omega d}$. Then agent preferences are single-peaked on $SEQ_i(\omega)$ and $FOQ_i(\omega)$.*

Proof: We prove the Lemma in four steps. In the first step we show that any allocation such that $c \in \widetilde{d'\omega}$ or $c \in \widetilde{\omega d}$ can be sustained as a top in $\widetilde{d'\omega}$ or $\widetilde{\omega d}$ respectively. In the second step we show that no agent has isolated tops on any side of the endowment. In the third step we show that both agents' preferences have unique top on both sides of the endowment. In the fourth step we show that preferences are in fact, single-peaked on both sides of the endowment.

Note that both $\widetilde{d'\omega}$ and $\widetilde{\omega d}$ are compact. Since, preferences are continuous on each of these segments a maximum exists under all R_i. In Step 1 we show that all the allocations in each side of ω can be sustained as a top under some R_i.

Step 1: Let $c \in \widetilde{d'\omega}$ or $c \in \widetilde{\omega d}$. Then there exists R_i such that $c \in Top(R_i, \widetilde{d'\omega})$ or $c \in Top(R_i, \widetilde{\omega d})$.

Proof of Step 1: Without the loss of generality let $c \in \widetilde{\omega d}$. Choose R_i' and R_j' such that it satisfies (1) and (2) in Lemma 3.6.4. By Lemma 3.6.5 $F(R_i', R_j') = d$. By continuity and individual-rationality of F and since $\mathbb{D}^s$ is rich, there exists an interval $[R_j', R_j''; R_i']$ such that $F([R_j', R_j''; R_i']) = \widetilde{\omega d}$, where $F(R_i', R_j'') = \omega$. By the Intermediate Value Theorem, there exists $R_j''' \in (R_j', R_j'')$ such that $F(R_i', R_j''') = c$. By strategy-proofness $F(R_i', R_j''') \in Top(R_j''', \widetilde{\omega d})$.

Step 2: We follow Diagram 3.11.

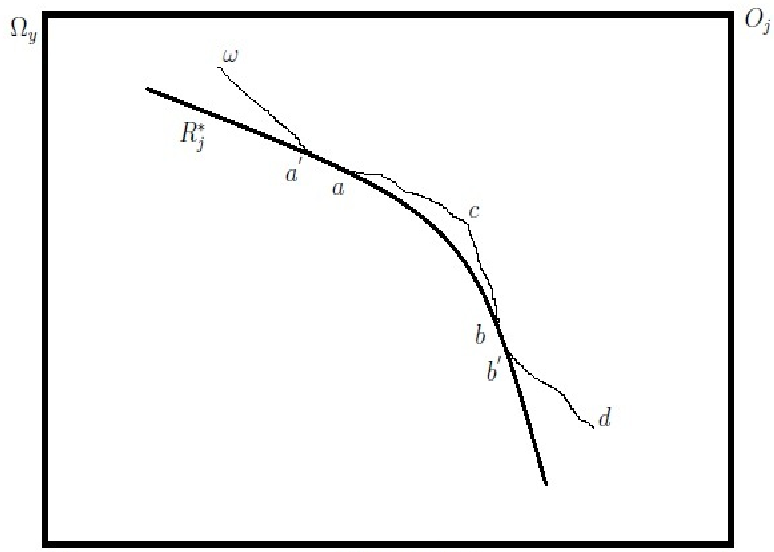

Diagram 3.11

We will prove Step 2 by contradiction. Suppose F has isolated plateaus on $\widetilde{\omega d}$. Let $\widetilde{a'a}$ and $\widetilde{bb'}$ be two plateaus. Consider R_j^* such that $int\ \widetilde{ab} \subset LC(R_j^*, a) = LC(R_j^*, b)$. Choose R_i' and R_j' such that it satisfies (1) and (2) in Lemma 3.6.4. By Lemma 3.6.5 $F(R_i', R_j') = d$.

Let $c \in int\ \widetilde{ab}$. By Lemma 3.6.4 there exists $R_j''' \succ R_j'$ such that $\omega P_j''' z$ for all $z \in \widetilde{\omega d} \backslash \{\omega\}$. By individual-rationality of F, $F(R_i', R_j''') = \omega$. Hence, by Lemma 3.6.2 and continuity of F there exists $R_j'' \succ R_j'$ such that $F(R_i', R_j'') = c$.

From the single-crossing property it follows that either $aP_j''c$ or $bP_j''c$. Consider $aP_j''c$. By strategy-proofness $F(R_i', R_j) \notin \widetilde{ac}$ for all $R_j \succ R_j''$. Hence, $F_j([R_j'', R_j'''; R_i'])$ is not connected but $[R_j'', R_j'''; R_i']$ is a connected set in $\mathbb{D}^s$. Hence, the range of F_j restricted to $[R_j'', R_j'''; R_i']$ is not connected. Therefore F_j restricted to $[R_j'', R_j'''; R_i']$ is not continuous, which is a contradiction to the assumption that F is continuous. Now consider $bP_j''c$. Again note that $[R_j', R_j''; R_i']$ is a connected set in $\mathbb{D}^s$. Since $F(R_i', R_j') = d$ continuity of F will be violated in this case also.

Step 3: It follows from Step 2 that agent preferences have unique plateaus on both sides of ω. We show that preferences admit unique maximal element on both sides of ω. Let $\widetilde{a'a}$ be a plateau for some preference R_j^*. Note that $\widetilde{a'a} \subset IC(R_j^*, a') = IC(R_j^*, a)$. By the single-crossing property either (i) $a' = Top(R_j, \widetilde{a'a})$ or (ii) $a = Top(R_j, \widetilde{ab})$ for $R_j \neq R_j^*$. The proof follows by repeating the arguments similar to the ones in Step 2.

We have proved that on both sides of the endowment each agent i has a unique top. However this is not enough to conclude that agent preferences are single-peaked. We now show this.

Step 4: Suppose for the sake of contradiction preferences are not single-peaked. We follow Diagram 3.12.

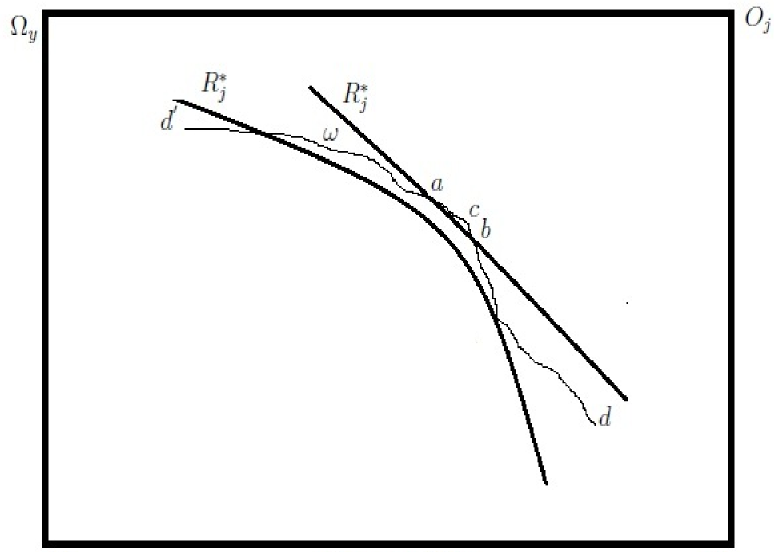

Diagram 3.12

Suppose R_j^* is not single-peaked but has a top on $\widetilde{\omega d}$. Therefore there exists an indifference curve of the preference R_j^* such that the indifference curve cuts $\widetilde{\omega d}$ at least twice and $Top(R_j^*, \widetilde{\omega d})$ is in the interior of the upper contour set of this indifference curve. Let a and b be two points of intersection. Hence, under R_j^* there exists allocations in $\widetilde{\omega d}$ between a and b that are in the lower contour set of the indifference curve that passes through these two allocations. Choose such an allocation c. From the Step 1 and Step 2 there exists R_j such that $c = Top(R_j, \widetilde{\omega d})$. Hence, indifference curves of R_j and R_j^* cuts twice.

■

We now show that at most only one agent can have tops on both sides of the endowment.

Lemma 3.6.7 *Let $\mathbb{D}^s$ be rich. Let the SCF $F : [\mathbb{D}^s]^2 \to \Delta$ be strategy-proof, individually rational and continuous. Let $\Re_F \subset int\ \Delta$ be closed and is given by $\widetilde{d'\omega} \cup \widetilde{\omega d}$. Then there exists an agent i such that his preferences are single-peaked on $\Re_F$.*

Proof: By Lemma 3.6.6 agent preferences are single-peaked on both sides of the endowment. Suppose agent j has a preference R_j^* such that $a \in int\ \widetilde{\omega d}$ and $b \in int\ \widetilde{d'\omega}$ are tops on $\widetilde{\omega d}$ and $\widetilde{d'\omega}$ respectively under R_j^*. Consider the profiles (R_i^1, R_j^1) and (R_i^2, R_j^2) such that they satisfy (1) and (2) in Lemma 3.6.4. By Lemma 3.6.5 $F(R_i^1, R_j^1) = d$ and $F(R_i^2, R_j^2) = d'$.

By individual-rationality of agent i, $F(R_i^1, R_j^*) \in \widetilde{\omega d}$ and $F(R_i^2, R_j^*) \in \widetilde{d'\omega}$. By individual-rationality of F, $F(R_i^1, R_j^2) = \omega$ and $F(R_i^2, R_j^1) = \omega$. Since agent j's preferences are single-peaked on each side of the endowment, continuity and strategy proofness imply $F(R_i^1, R_j^*) = a$ and $F(R_i^2, R_j^*) = b$. Note that $R_i^1 \succ R_i^2$. By continuity there exists R_i^* such that $F(R_i^*, R_j^*) = \omega$. By, strategy proofness $\omega = Top(R_i^*, \widetilde{\omega d})$ and $\omega = Top(R_i^*, \widetilde{d'\omega})$. Therefore, $\omega = Top(R_i^*, \Re_F)$. By Lemma 3.6.6 agent i's preferences are also single-peaked on both sides of the endowment. Since $\omega = Top(R_i^*, \Re_F)$ by the single-crossing property, agent i's preferences must be single-peaked on $\Re_F$.

■

From the preceding Lemma it follows that there exists an agent i such that for some R_i^*, $Top(R_i^*, \Re_F) = \omega$. Hence, $Top(R_i, \Re_F) \in \widetilde{\omega d}$ if $R_i \succ R_i^*$ and $Top(R_i, \Re_F) \in \widetilde{d'\omega}$ if $R_i^* \succ R_i$. Therefore by individual-rationality $F(R_i, R_j) \in \widetilde{\omega d}$ if $R_i \succ R_i^*$ and $F(R_i, R_j) \in \widetilde{d'\omega}$ if $R_i^* \succ R_i$. By individual-rationality $F(R_i^*, R_j) = \omega$ for all R_j. Now consider the following partition of $[\mathbb{D}^s]^2$: $K(1) = \{(R_i, R_j)|R_i = R_i^*\}$, $K(2) = \{(R_i, R_j)|R_i \succ R_i^*\}$ and $K(3) = \{(R_i, R_j)|R_i^* \succ R_i\}$. Hence it follows that if $(R_i, R_j) \in K(2)$ then $F(R_i, R_j) \in \widetilde{\omega d}$ and if $(R_i, R_j) \in K(3)$ then $F(R_i, R_j) \in \widetilde{d'\omega}$. Now note that in both $\widetilde{\omega d}$ and $\widetilde{d'\omega}$ preferences of agent j are also single-peaked.

Now consider $k(1) \cup K(2)$. We know by individual-rationality of F that if $(R_i, R_j) \in K(2)$ then $F(R_i, R_j) \in \widetilde{\omega d}$. Now we know that,

$F(R_i, R_j) =$

$$\min\Big\{a_\emptyset, \max\{Top(R_i, \Re_F), a_{\{i\}}\}, \max\{Top(R_j, \widetilde{\omega d}), a_{\{j\}}\}, \max\{Top(R_i, \Re_F), Top(R_j, \widetilde{\omega d}), a_{\{i,j\}}\}\Big\}.$$

From Theorem 2 in Barberà and Jackson (1994) if $s \subset s^{'} \subset \{i, j\}$ then $a_s \geq a_{s'}$, where a_s is an extended real number for all $s \subset \{i, j\}$. [***]

Claim 1: $a_\emptyset \geq d$ and $a_{\{i,j\}} \leq \omega$.

Proof of Claim 1: Let for the sake of contradiction $a_\emptyset < d$. From Lemma 3.6.4 and Lemma 3.6.5 we can choose a profile such that under F,

$$\min\Big\{a_\emptyset, \max\{d, a_{\{i\}}\}, \max\{d, a_{\{j\}}\}, \max\{d, d, a_{\{i,j\}}\}\Big\} = d.$$

By [***], $\min\Big\{a_\emptyset, d, d, d\Big\} < d$, which is a contradiction to the above equality.

Now, let for the sake of contradiction $a_{\{i,j\}} > \omega$. By individual-rationality of F from our earlier discussion we can choose a profile such that,

$$\min\Big\{a_\emptyset, \max\{\omega, a_i\}, \max\{\omega, a_j\}, \max\{\omega, \omega, a_{i,j}\}\Big\} = \omega.$$

By [***] $\min\Big\{a_\emptyset, a_{\{i\}}, a_{\{j\}}, a_{\{i,j\}}\Big\} > \omega$, which is a contradiction to the above equality. This establishes Claim 1.

Since $F(R_i, R_j) \in \widetilde{\omega d}$ for all $(R_i, R_j) \in K(1) \cup K(2)$, the definition of min-max function in conjunction with Claim 1 imply, $a_\emptyset = d$ and $a_{\{i,j\}} = \omega$.

Claim 2: $a_{\{i\}} = \omega$ and $a_{\{j\}} = \omega$.

Proof of Claim 2: Let for the sake of contradiction $a_{\{i\}} > \omega$. We know that by individual-rationality of F, $\min\Big\{d, \max\{d, a_{\{i\}}\}, \max\{\omega, a_{\{j\}}\}, \max\{d, \omega, \omega\}\Big\} = \omega$. Hence, $a_{\{j\}} = \omega$. Again by, individual-rationality of F, $\min\Big\{d, \max\{\omega, a_{\{i\}}\}, \max\{d, \omega\}, \max\{\omega, d, \omega\}\Big\} = \omega$, i.e. $\min\Big\{d, \max\{\omega, a_{\{i\}}\}, d, d\Big\} = \omega$, which is a contradiction.

Let for the sake of contradiction $a_{\{j\}} > \omega$. By individual-rationality of F,

$$\min\Big\{d, \max\{d,\omega\}, \max\{\omega, a_{\{j\}}\}, \max\{d,\omega,\omega\}\Big\} = \omega.$$

This is a contradiction since, $\max\{\omega, a_{\{j\}}\} > \omega$. This establishes the Claim 2.

Hence, from the claims above it follows that $a_{\emptyset} = d$ and $a_s = \omega$ for $s \subset \{i,j\}$ and $s \neq \emptyset$. Therefore,

$$F(R_i, R_j) = \min\Big\{d, \max\{Top(R_i, \Re_F), \omega\}, \max\{Top(R_j, \widetilde{\omega d}), \omega\}, \max\{Top(R_i, \Re_F), Top(R_j, \widetilde{\omega d}), \omega\}\Big\}.$$

Since none of the tops can be higher than d so we can write,

$$F(R_i, R_j) = \min\Big\{\max\{Top(R_i, \Re_F), \omega\}, \max\{Top(R_j, \widetilde{\omega d}), \omega\}, \max\{Top(R_i, \Re_F), Top(R_j, \widetilde{\omega d}), \omega\}\Big\}.$$

$$\text{Equivalently, } F(R_i, R_j) = \text{median}\Big\{Top(R_i, \Re_F), Top(R_j, \widetilde{\omega d}), \omega\Big\}.$$

Analogously, if $(R_i, R_j) \in K(1) \cup K(3)$ then

$$F(R_i, R_j) = \text{median}\Big\{Top(R_i, \Re_F), Top(R_j, \widetilde{d'\omega}), \omega\Big\}.$$

This completes the proof of Theorem 3.4.1. ■

Chapter 4

Procurement Auctions: Technical Bids, Subjective Evaluation and Corruption

Introduction

In this chapter, we study a class of procurement auctions for a single good where the quality of the good is variable. The buyer/procurer is therefore concerned both with the quality of the good supplied as well as the price charged by the supplier. We introduce the issues we wish to study in the chapter by referring to the specific case of the Delhi and Mumbai Airport Privatization (DMAP) auctions carried out by the Government of India between 2003-06 (for details see Jain et al. (2007)).

The objective of this exercise was to out-source the task of modernizing the Delhi and Mumbai airports to private firms which were selected by an auction mechanism. The selected firms were responsible for modernizing the airports. They were also required to manage the airport for thirty years with an agreement to share a percentage of the annual revenue with the Government. Two issues were of primary importance in this process, viz. (a) the quality of the new airports and (b) the Government's share of annual revenue after the airports were built.

The notion of the quality of an airport is of course, very difficult to express objectively. Moreover, the quality can be judged only after the completion of construction whereas a developer has to be chosen before the start of the construction. The selection procedure was based on a list of desirable characteristics that the developer was expected to possess. This included, for instance, their size and experience in airport development based on the hypothesis that larger firms with better experience would be more likely to build a better quality airport. Each firm was asked to submit a document called a *technical bid* that included the firm's design of the airport as well as the firms' characteristics. It also had to simultaneously submit a *financial bid* which represented the percentage of revenue that the supplier was willing to share with the Government.

Bids were evaluated in two rounds. In the first round technical bids were evaluated which was done was follows. Each item in the technical bid was graded from a maximum possible score for that item and then aggregated into an overall *quality-score*. (The maximum score of each item or its weight was announced beforehand.) If a technical bid obtained the minimum quality-score then the bidder qualified for the subsequent round. Here the financial bids of the qualified bidders were opened. The overall winner was the one whose financial bid was the highest amongst the bidders who qualified for the second round.

The idea behind the two rounds of bidding is obviously to increase the level of competition in both the quality and price dimensions. It is a pervasive practice in procurement auctions conducted by the Government of India. Its implications are straightforward in the standard model (Che (1993)) where the firm has a single-dimensional cost parameter that is private information. Here setting a minimum quality standard has obvious consequences. All firms has a dominant strategy to bid the minimum quality level. The overall winner is then decided by a first price auction in the price dimension.

In this chapter we begin by examining a variant of this model where a certain predetermined proportion of bidders qualifies for the second round. Our main result is that there does not exist a continuous, symmetric and increasing pure strategy Bayes-Nash Equilibrium

(BNE) in this model. We therefore focus attention on a model where the minimum quality standard is postulated, but the evaluation of technical bids is *subjective*.

We wish to emphasize the point that in the DMAP auction, even though the weighting scheme was common knowledge, the exact evaluation procedure i.e. the "what kind of technical bid would obtain what score" information, could not/cannot be conveyed. For instance, it seems reasonable to assume the Government cannot specify *all* design aspects completely. Indeed, it may not be sure of the "exact" design it was looking for. Once all the designs were received then it may have found one among them to be the best, although ex-ante this design was neither conceived and nor conveyed to the the bidders. Note that if the details of the design selected could be conveyed before the bidding then all the bidders would have submitted the same design. The fact that the evaluation of the technical bid was subjective in the case of the DMAP auctions, has been recognized (see Jain et al. (2007)) widely.

A subjective evaluation procedure has two important consequences. The first is that it can lead to the choice of an inefficient (or high cost) supplier because the efficient supplier/s may fail to qualify. This may happen because the efficient bidder/s misreads the technical requirements. The second consequence is that it facilitates corruption of a particular kind. The buyer may collude with a particular bidder by choosing an evaluation procedure that favors the qualification over other bidders of that bidder for the second round. Importantly such manipulations cannot be challenged legally if subjectivity is accepted. In fact the collusion between a buyer and a particular bidder can be a common knowledge.

The subjectivity story is well illustrated in the DMAP auctions. Many bidders failed to qualify in the technical round. In fact there were many re-assessments of the technical bids. In the initial evaluation, Reliance Airport Developers Private Limited (PADPL) qualified for Mumbai airport. But due to objections raised by the Gajendra Haldea committee, technical bids were re-evaluated. In the second evaluation PADPL also qualified. But again complaints were raised by some political parties that consultants (ones appointed to evaluate technical bids) had close business ties with bidders. Therefore they demanded that the entire evaluation process be scrapped. A new committee was formed and technical bids were again evaluated. This time PADPL did not qualify for either Mumbai and Delhi airports. PADPL then filed a writ petition in Delhi High Court against the Airport Authority of India (AAI) alleging that "An arbitrary decision making process which was discriminatory in practice was adopted for awarding the contract. An open transparent procedure for the evaluation and consideration of tenders was not followed and the terms of conditions of evaluation were changed on an ad hoc basis, only to exclude PADPL and favor the GVK Industries Limited." The High Court dismissed the writ petition saying that Empowered Group of Minsters (EGoM) had absolute discretion in the matter of choosing the modalities. PADPL appealed to the Supreme Court of India which also dismissed their petition. For more details see Jain et al. (2007).

In order to understand the interplay of subjectivity, inefficiency and corruption we study a model where minimum quality is specified exogenously. Each bidder receives a signal about the *subjectivity parameter* to be used by the buyer. The parameter is a real number and it can be interpreted as being either "strict" or "lenient". We believe that an appropriate solution concept in this model is *Subjective Nash Equilibrium* (SNE) developed in Kalai and Lehrer (1993) and Kalai and Lehrer (1995). We show that in equilibrium all bidders qualify for the second stage and they anticipate a subjectivity parameter that is no "stricter" than what they receive. Notice that in equilibrium the bidders need not correctly anticipate the subjectivity parameter. The equilibrium is efficient in the sense that the lowest cost bidder wins the auction.

Inefficiency can however arise in equilibrium if corruption is introduced here. We consider a two bidder version of the model where it is common knowledge that the buyer colludes with one of the bidders. In equilibrium the honest bidder bids according to the strict subjectivity parameter while the dishonest bidder bids according to the lenient parameter. The buyer announces the lenient parameter and both bidders qualify but the dishonest bidder has a cost advantage over the honest bidder. Standard first price auction theoretic techniques can be used to characterize the equilibrium price bid functions. Ex-post inefficiency can arise if the honest bidder has lower cost than the dishonest bidder. We provide an example illustrating this phenomenon.

A widely studied class of procurement auctions related to ours is the class of quasi-linear scoring auctions. A scoring rule is a real valued function whose domain is the set of all two-tuples of technical and financial bids. This model was introduced by Che (1993) and extended in different ways by Asker and Cantillon (2010), Branco (1997) and Naegelen (2002).

A related model is Ganuza and Pechlivanos (2000). In this paper the buyer announces a design "a priori" and the firms then directly reveal their cost parameters. The paper characterizes optimal Bayesian incentive-compatible mechanisms. This differs from ours in the sense that the design is not specified a priori nor is the bidding in terms of the cost parameters.

Another related stream of literature relates to the subjective evaluation of agents' performance in a principal-agent settings - for instance, Baker et al. (1994), MacLeod (2003) and Levin (2003). In these models, agents may perform many tasks that are not amenable to objective evaluation (for example, an employee's contribution to the reputation of a company). These papers address questions such as the optimal mix of subjective and objective performance measures (Baker et al. (1994)) and the nature of contracts that emerge when subjective measures are used (MacLeod (2003) and Levin (2003)). In our model, the buyer subjectively evaluates the quality proposal of a bidder and not that of a completed task or project and our objective is to understand issues created by this fact. We also note that our notion of equilibrium SNE differs from theirs.

In Section 4.1, we propose a basic procurement model without subjectivity. In Section 4.2 we study a variant of the model with subjective evaluation of technical bids. In Section 4.3 we study the the impact of subjectivity in corruption. In Section 4.4 we conclude.

4.1 The Basic Model

In this section we investigate a standard model of procurement without subjectivity. There is a single buyer who wishes to purchase a single good of variable quality. There is a set $B = \{1, 2, ..., n\}$ of potential suppliers or bidders. The auction specifies a positive integer λ where $2 \leq \lambda \leq n-1$. Each bidder simultaneously submits a quality or "technical" bid t which is a real number and a price bid p. The technical bids are ordered from the highest to the lowest. The top λ technical bids are deemed to have qualified for the second (or "financial") stage of the auction. In a λ-*lowest price auction* (or λ-LP) auction, the winner is the second stage bidder whose price bid is the lowest amongst bidders who have qualified for the second stage. The winning bidder delivers the product at the quality level of her bid and charges the price that she bid. In a λ-*second lowest price auction* (or λ-SLP) auction, the winner is again the second stage bidder whose price bid is the lowest amongst bidders who have qualified for the second stage. The winning bidder also delivers the product at the quality level of her bid but charges a price equal to the price bid of lowest rejected bidder in the second stage.

We assume cost function of all the bidders is given by $c(t, \theta)$ where $\theta \in [\underline{\alpha}, \bar{\alpha}] \subset \Re_{++}$. Here θ represents the technology that a bidder uses for producing the product and is private information of the bidder. We shall refer to θ realized by the bidder as her *cost-type*. We assume that all first and second-order partial derivatives of the cost function are non-negative; in particular $c_t > 0$, $c_\theta > 0$, $c_{tt} \geq 0$, $c_{t\theta} \geq 0$ and $c_{\theta\theta} \geq 0$. In addition $c(0, \theta) = 0$ for all θ. We assume that the θ's are drawn independently and identically across bidders with a strictly increasing differentiable distribution function F with an associated density function f which is continuously differentiable.

Consider bidder i of cost-type θ who bids (q_i, p_i) while bidders other than i bid (q_{-i}, p_{-i}). Then i's payoff function can be written as follows:

The payoff function of the i^{th} bidder is:

$$\pi((q_i, p_i), (q_{-i}, p_{-i}); \theta) = \begin{cases} p_r - c(q_i, \theta), & \text{if the bidder wins;} \\ 0, & \text{otherwise} \end{cases}$$

where p_r is the price that the winning bidder pays and depends on the nature of the auction. Clearly p_r depends on the bids of all bidders.[1]

A strategy for bidder i consists of a pair of functions $q_i : [\underline{\alpha}, \bar{\alpha}] \rightarrow \Re_{++}$ and $p_i : [\underline{\alpha}, \bar{\alpha}] \rightarrow \Re_{++}$. In line with the literature, we shall focus on *symmetric*, *strictly monotonic* and *continuous* pure strategy BNE in the model. Note that strict monotonicity in this case requires the q_i functions to be strictly decreasing and the p_i functions to be strictly increasing. Symmetry and strict monotonicity ensure that the equilibrium is ex-post efficient in the sense that the the lowest cost bidder wins the auction. However the next two results demonstrate that such equilibria do not exist in either λ-LP or λ-SLP auctions.

Proposition 4.1.1: There does not exist a symmetric, strictly monotonic and continuous BNE in a λ-LP auction.

Proof: We prove the result by contradiction. Suppose not, i.e. let (q^*, p^*) be a symmetric, continuous and strictly monotonic equilibrium. Let $q^*([\underline{\alpha}, \bar{\alpha}])$ be the range of q^*. Note that by continuity of q^*, $q^*([\underline{\alpha}, \bar{\alpha}])$ is a closed interval. Let $q^*([\underline{\alpha}, \bar{\alpha}]) = [a, b]$.

Since q^* is strictly monotonic, $q^{*-1} : [a, b] \rightarrow [\underline{\alpha}, \bar{\alpha}]$ is defined and strictly monotonic. Similarly p^{*-1} is defined and strictly monotonic.

Consider bidder i's of cost-type θ with $\theta \in (\underline{\alpha}, \bar{\alpha})$. We will show that bidder i's expected utility is not maximized at $(q^*(\theta), p^*(\theta))$ given that other bidders are playing (p^*, q^*). Suppose bidder i bids price $p^*(\theta)$ and quality m where $m \in (a, b)$. Bidder i's payoff as a function of m can be written as under:[2]

[1] All ties are broken with equal probability.

[2] This expression is cumbersome. However Example 4.5.1 in Appendix 4.5 for the case $n = 3$ and $\lambda = 2$ makes the calculations clear.

$$\pi\Big(m;\theta,(q^*,p^*)\Big) = \begin{cases} [p^*(\theta)-c(m,\theta)][1-F(\theta)]^{(n-1)} \\ \qquad\qquad \text{if } m \geq q^*(\theta) \\ \\ \times\Big[\sum_{j=0}^{\lambda-1}\binom{n-1}{j}[F(q^{*-1}(m))-F(\theta)]^j[1-F(q^{*-1}(m))]^{(n-1-j)}\Big] \\ \qquad\qquad \text{if } m < q^*(\theta) \end{cases}$$

Observe that $\pi : [a,b] \to \Re$. Since $\theta \in (\underline{\alpha}, \bar{\alpha})$ so $q^*(\theta) \in (a,b)$ i.e. $q^*(\theta)$ is an interior point of the domain of π. We will show that π is not maximized at $m = q^*(\theta)$.

Since the cost function is differentiable, the right hand derivative of π exists for all $m \in (a,b)$; in particular it exists at $m = q^*(\theta)$. This derivative is $-\frac{d}{dm}c(q^*(\theta),\theta)[1-F(\theta)]^{(n-1)}$ which is negative since $\frac{d}{dm}c(q^*(\theta),\theta) > 0$. Since q^{*-1} is strictly monotonic, q^{*-1} is differentiable almost everywhere in $[a,b]$. Without the loss of generality, suppose q^{*-1} is differentiable at $m = q^*(\theta)$. Since F is differentiable, $\sum_{j=0}^{\lambda-1}\binom{n-1}{j}[F(q^{*-1}(m))-F(\theta)]^j[1-F(q^{*-1}(m))]^{(n-1-j)}$ is differentiable at $m = q^*(\theta)$. We now show that π is differentiable at $m = q^*(\theta)$.

We have

$$\frac{d}{dm}\sum_{j=0}^{\lambda-1}\binom{n-1}{j}[F(q^{*-1}(m))-F(\theta)]^j[1-F(q^{*-1}(m))]^{(n-1-j)}$$

$$= \frac{d}{dm}[1-F(q^{*-1}(m))]^{(n-1)} + \frac{(n-1)!}{(n-2)!}\frac{d}{dm}[F(q^{*-1}(m))-F(\theta)]^1[1-F(q^{*-1}(m))]^{(n-2)} +$$

$$\frac{d}{dm}\sum_{j=2}^{\lambda-1}\binom{n-1}{j}[F(q^{*-1}(m))-F(\theta)]^j[1-F(q^{*-1}(m))]^{(n-1-j)}$$

$$= -(n-1)[1-F(q^{*-1}(m))]^{(n-2)}f(q^{*-1}(m))\tfrac{d(q^{*-1}(m))}{dm}$$

$$+(n-1)[1-F(q^{*-1}(m))]^{(n-2)}f(q^{*-1}(m))\frac{d(q^{*-1}(m))}{dm} +$$

$$[F(q^{*-1}(m)) - F(\theta)]\frac{d}{dm}[1 - F(q^{*-1}(m))]^{(n-2)} +$$

$$\frac{d}{dm}\sum_{j=2}^{\lambda-1}\binom{n-1}{j}[F(q^{*-1}(m)) - F(\theta)]^j[1 - F(q^{*-1}(m))]^{(n-1-j)}.$$

The first two terms in the expression above get canceled. Note that from the fourth term onwards, $j \geq 2$. Therefore, after taking the derivative, the expression involves $[F(q^{*-1}(m)) - F(\theta)]$. This implies that the expression vanishes at $m = q^*(\theta)$. Consequently, the derivative of the truncated π function for $m < q^*(\theta)$, is also $-\frac{d}{dm}c(q^*(\theta), \theta)[1 - F(\theta)]^{(n-1)}$. Applying Lemma 4.5.1 in Appendix 4.5, it follows that π is differentiable at $m = q^*(\theta)$. Moreover, the derivative of π at $m = q^*(\theta)$ is negative. Since $q^*(\theta)$ is an interior point of π, it follows that π is not maximized at $q^*(\theta)$. This follows from the Interior Extremum Theorem, Bartle and Sherbert (2005). This implies that whenever q^{*-1} is differentiable, the expected payoff of bidder i is not maximized at $m = q^*(\theta)$ contradicting the assumption that (p^*, q^*) is an equilibrium. ■

We now show that a similar result holds for an λ-SLP auction.

Proposition 4.1.2: There does not exist a symmetric, strictly monotonic and continuous BNE in a λ-SLP auction.

Proof: We first establish an intermediate step.

Lemma 4.1.1 *Let (q^*, p^*) be an equilibrium in a λ-SLP auction. Let $p^{**}(\theta) = c(q^*(\theta), \theta)$ for all $\theta \in [\underline{\alpha}, \bar{\alpha}]$. Then (q^*, p^{**}) is another equilibrium in the same auction.*

Proof: Let b_j denote an arbitrary price bid of bidder j. Suppose bidder i of cost-type θ makes the quality bid $q^*(\theta)$. Observe that since the quality bid functions for bidder i in (q^*, p^*) and (q^*, p^{**}) are same, i qualifies for the second round at every realization of cost-types of all bidders. Therefore, the payoff to bidder i is the same under the two strategies for cost-type realizations of all players where i does not qualify. We therefore only need to consider realization of cost-types where i qualifies for the second stage. Now consider the following mutually exhaustive cases.

Case (i): $c(q^*(\theta), \theta) < \min_{j \neq i} b_j$. If bidder i bids a price higher than $\min_{j \neq i} b_j$, his payoff will be zero because he will lose in the second stage. So i cannot do strictly better by bidding other than $c(q^*(\theta), \theta)$.

Case (ii): $c(q^*(\theta), \theta) \geq \min_{j \neq i} b_j$. If bidder i bids low enough and if wins the second stage his payoff will be negative. Therefore i cannot do strictly better by bidding other than $c(q^*(\theta), \theta)$.

It is clear from the two cases above that (q^*, p^{**}) is an equilibrium. ■

If bidders bid according to p^{**} then bidder i's expected payment is

$$E[c(q^*(Y_L^{(n-1)}), Y_L^{(n-1)})|Y_L^{(n-1)} > \theta]$$

where $Y_L^{(n-1)}$ is the lowest order statistic in $(n-1)$ draws of cost-types.

Now plugging this price in place of p_r in the function π, the analysis in Proposition 4.1.1 can be replicated to conclude that (q^*, p^{**}) is not an equilibrium. ■

Observe that there is no equilibrium in any auction where $\lambda = 1$. In this case, every losing bidder can make a quality bid high enough be the only qualified bidder in the second stage and bid a price high enough to cover costs. This deviation would make the bidder strictly better-off.

An implication of our results is that it is hard to first screen bidders in the quality dimension and then choose a winner among the selected bidders by evaluating their bids on price. A part of the difficulty arises because the "winning" quality level is endogenous. In the next section, we study a model where a minimum quality level is specified exogenously but the evaluation of technical bids is subjective.

4.2 A Model with Subjective Evaluation of Technical Bids

In this section we study a procurement auction model where the evaluation of technical bids involves subjective judgment.

In the model, a minimum quality cut-off level is specified exogenously. The winner of the auction is the bidder who bids the lowest price while satisfying the quality requirements. The winner's payment is the price that it bids. However, the evaluation of technical bids involves subjectivity. Therefore bidders are unsure of what "quality-score" their technical bids will fetch. Thus two bidders can have different interpretations of the technical requirements in a Tender Notice. The subjective assessments are independent of the cost-types.

For modeling purposes we will assume that there are a finite number of possible "interpretations" of bids. These "interpretations" correspond to various positive real numbers $\eta_1, \eta_2, \ldots$ from the set Ω. We shall refer to elements of the set Ω as *subjectivity parameters*. Each bidder receives a particular subjectivity parameter as a signal. On receiving $\eta_k \in \Omega$, she believes that a technical-bid t will be evaluated as having a quality-score $\eta_k t$. The buyer also realizes a subjectivity parameter which he uses to evaluate. This parameter is unobserved by the bidders. For instance, if he realizes η^*, then a bid t is given a score $\eta^* t$. The set Ω and the class of functions $q(t, \eta) = \eta t$ is of course, commonly known. The winner is committed to its technical bids.[3]

For simplicity we assume further that $\Omega = \{\eta_1, \eta_2\}$ where $\eta_1 < \eta_2$. If bidder i receives η_1 while the buyer receives η_2 then a technical bid is evaluated "more leniently" by the buyer than anticipated by the bidder. The reverse is true if the buyer receives η_1 and the bidder η_2.

We consider the model where the buyer specifies a minimum cut-off of quality-score $\underline{q} > 0$. In the absence of subjectivity it is a strictly dominant strategy to bid the minimum quality-score. Thus every bidder bids the same technical bid and competition takes place only in

[3]For instance consider the example of privatization of Delhi and Mumbai airports. Although the quality-score that the winner anticipates for the submitted design and the quality-score that the buyer assigns may differ, the winner is obliged to carry out the design that she bids.

the price dimension. Standard auction theory techniques can be used to show the existence of a symmetric and strictly increasing (i.e. strictly increasing in cost-type) equilibrium bidding strategy in financial bids. Such an equilibrium is ex-post efficient. In the presence of subjectivity, efficiency may not be achieved because the most efficient bidder may not qualify on the technical score. However, we will show that under an appropriate equilibrium notion in this setting, inefficiency cannot be an equilibrium phenomenon. We now introduce our equilibrium notion which is SNE which we believe is appropriate in our context. This notion is developed in Kalai and Lehrer (1993) and Kalai and Lehrer (1995).

We consider a reformulation of the problem where there exists a dummy player (called Nature or N) who reveals one of the elements of Ω to the buyer after the bids are submitted. The set of actions of Nature is Ω. Nature does not have any payoff. A *joint strategy* of bidder i and N is an $n+1$ tuple of strategies that bidder i *believes* that all the players will play. This belief need not be correct. For instance, a bidder need not be correct regarding Nature's choice of subjectivity parameter. Formally, a joint strategy of bidder i is $s^i = ((t_1^i, p_1^i), \ldots, (t_i^i, p_i^i), \ldots, (t_n^i, p_n^i), \eta_k^i)$. Here, (t_j^i, p_j^i) denotes the bidding strategy that bidder i believes that bidder j will play and η_k^i corresponds to the subjectivity parameter that i believes the buyer is going to use. Similarly, a joint strategy for Nature is $s^N = ((t_1^N, p_1^N), \ldots, (t_n^N, p_n^N), \eta_k)$. Here we interpret s_i^N as the strategy actually played by bidder i. Also s_{n+1}^N refers to the play of Nature in the joint strategy s^N. We will let s_{-i}^i to denote the strategies of other players listed in the joint strategy s^i.

All bidders face three possible consequences - "not qualify", "qualify but lose", "qualify and win" which we denote by $0, 1$ and 2 respectively. We let $C = \{0, 1, 2\}$.

We assume that at the end of the game each bidder will be told about his own consequences and not those about others.[4] For every cost-type θ_i, his own bid and a joint strategy a bidder can calculate the probability distribution over consequences.This will be denoted by $prob(l|t_i, p_i, s_{-i}^i, \eta_k^i, \theta_i)$ for all $l \in C$. The payoff of bidder i is given by

$$\pi_i((t_i, p_i, s_{-i}^i, \eta_k^i)|\theta_i) = (p_i - c(t_i, \theta_i))prob(2|t_i, p_i, s_{-i}^i, \eta_k^i, \theta_i). \tag{4.1}$$

[4]This assumption helps us to focus our analysis on the individual consequences only.

An important constituent of SNE is a *subjectively-rational* strategy, which we define below in the context of our model.

Definition 4.2.1 *A strategy* $(t_i^* : [\underline{\alpha}, \bar{\alpha}] \rightarrow \Re_{++}, p_i^* : [\underline{\alpha}, \bar{\alpha}] \rightarrow \Re_{++})$ *of bidder* i *is* ***Subjectively-Rational*** *with respect to* (s_{-i}^i, η_k^i) *if* $(t_i^*(\theta_i), p_i^*(\theta_i))$ *maximizes* $\pi_i((t_i, p_i, s_{-i}^i, \eta_{ik})|\theta_i))$ *in Equation 4.1 for all* θ_i.

Another important constituents of SNE are *probability distribution induced on* C *induced by true play of the game and by bidder* i*'s joint strategy.*

Since, bidder i's belief about other players' strategies need not be correct, these distributions also need not be same. Let $s = ((t_1^1, p_1^1), \ldots, (t_n^n, p_n^n), \eta^{\mathrm{N}})$ be the strategies played by bidders and Nature. Note that s refers to the true play of the game. Let μ_s denote the probability distribution induced by s on C. Also let μ_{s^i} denote the probability distribution induced by joint strategy s^i on C. A SNE requires these two distributions to be the same i.e. $\mu_s = \mu_{s^i}$. Formally,

Definition 4.2.2 *An* $n+1$ *tuple of strategies* $s^* = ((t_1^*, p_1^*), \ldots, (t_n^*, p_n^*), \eta_k)$ *is a SNE if there exists an* $n+1$ *tuple of joint strategy vectors* $(s^1 \ldots s^n, s^N)$ *such that*

1. $s_i^i = (t_i^*, p_i^*)$, $s_i^N = (t_i, p_i)$ *for all* $i = 1, 2, \ldots, n$ *and* $s_{n+1}^N = \eta_k$.
2. (t_i^*, p_i^*) *is subjectively-rational for all* i.
3. $\mu_s = \mu_{s^i}$ *for all* i.

In a SNE the i^{th} component of the joint strategy s^i is the actual bid of bidder i and it is subjectively optimal. Moreover, the probability distributions over consequences generated by the bids is the same as probability distribution generated by the joint strategies s^i for all $i = 1, 2 \ldots n$. An important feature of a SNE is that bidders belief about the subjectivity parameter chosen by Nature need not be the be parameter chosen by Nature i.e. $\eta_k^i = \eta_k$ need not hold.

A special case of a SNE is a BNE. In this case $s_j^i = (t_j, p_j)$ for all i, j and $\eta_k^i = \eta_k$ for all i. In other words, a player's beliefs about the strategies of other players must coincide with their actual play. Later we will demonstrate SNE that are not BNE. Below we identify some salient properties of SNE.

Proposition 4.2.1: Let $s^* = ((t_1^*, p_1^*), \ldots, (t_n^*, p_n^*), \eta_k)$ be a SNE. Then,

(a) $t_i^* = \frac{q}{\eta_k^i}$ for all $i = 1, 2, \ldots, n$.

(b) All bidders qualify for the second stage.

(c) $\eta_k^i \leq \eta_{n+1}^N$ for all $i = 1, 2, \ldots, n$.

Proof: We first prove (a). Fix bidder i and let s^i be a joint strategy of bidder i. Fix θ_i and let bidder i bid (t_i, p_i). If i bids below $\frac{q}{\eta_k^i}$ then she does not qualify for sure. On the other hand bidding strictly greater than $\frac{q}{\eta_k^i}$ drives up cost. Therefore, bidding $\frac{q}{\eta_k^i}$ is uniquely subjectively-rational.

Now we prove (b). Suppose i does not qualify. Fix her cost-type at θ_i. Since she does not qualify, the true play of the game puts probability 1 on the consequence 0 i.e. μ_s puts probability 1 on i not qualifying. However, μ_{s^i} puts probability zero on this consequence, violating requirement 3 in the definition of SNE.

Observe that part (c) is a direct consequence of (b).

■

According to part (c) of Proposition 4.2.1, a feature of a SNE is that all bidders' signals must be at least as "harsh" as the one that Nature chooses for the buyer. We now show that we can construct a SNE where bidders' beliefs are different from that of the buyer.

Example 4.2.1: Let $s^* = ((t^*, p^*), \ldots, (t^*, p^*), \eta_2)$, $s^i = ((t^*, p^*), \ldots, (t^*, p^*), \eta_1)$ for all $i = 1, 2, \ldots, n$ and $s^N = ((t^*, p^*), \ldots, (t^*, p^*), \eta_2)$ where

$$
\begin{aligned}
t^*(\theta) &= \frac{q}{\eta_1} \\
p^*(\theta) &= \frac{1}{[1-F(\theta)]^{n-1}} \int_{\theta}^{\bar{\alpha}} c(\frac{q}{\eta_1}, r)(n-1)[1-F(r)]^{(n-2)} f(r) dr.
\end{aligned}
$$

It follows from standard arguments that p^* is a best response of bidder i if other bidders play p^* (see Che (1993)). This implies that (t^*, p^*) is subjectively-rational. Note that according to the true play as well as according to the bidders' beliefs, the probability of consequence 0 is zero. Since all the bidders qualify and they follow the same strategies and beliefs about what others bidders' play is correct, the probabilities calculated by a bidder on his win is the same as what the true play generates. Hence s^* is a SNE. It is not a BNE because bidders' beliefs about Nature's play is not correct.

Remark 4.2.1: Observe that in our analysis we have not interpreted the subjectivity parameter as a private information of the bidders. Instead we have interpreted it as a strategy of the buyer. This interpretation is appropriate in our model because a subjectivity parameter is important for a bidder only to the extent that the buyer's use of one affects the payoff of the bidder.

We have shown that in equilibrium there is no inefficiency in the sense that lower cost-type bidders fail to qualify. However another kind of inefficiency may arise. Suppose that equilibrium of the kind arises where all the bidders realize η_1 and Nature chooses η_2. Now the winning bidder has to supply $\frac{q}{\eta_1}$ which is greater than $\frac{q}{\eta_2}$ which is what is actually required.

In the next section, we analyze the interplay of subjectivity and corruption in this model.

4.3 Corruption and Subjectivity

In this section we will analyze the effect of subjectivity on corruption. An important aspect of subjectivity is that it cannot be legally challenged. The buyer can formulate a project proposal in suitably vague terms and disqualify various "unwanted" bidders on technical grounds. These bidders typically have limited legal remedies available.[5]

We consider a two-bidder version of the model developed in the previous section. However, the buyer colludes with one of the bidders, say bidder 2. In fact, it may be convenient to think of bidder 2 and the buyer as the same player. This player makes a quality bid as a bidder and also announces a subjectivity parameter as a buyer after the bids have been received. This collusion is common knowledge. The game proceeds as follows. Each bidder receives a subjectivity parameter signal from the set $\Omega = \{\eta_1, \eta_2\}$ $(\eta_1 < \eta_2)$ and bids accordingly. The buyer then chooses a subjectivity parameter and selects the bidder with the lowest price-bid amongst the bidders who qualify on the quality-score. A BNE of this game with strictly increasing and differentiable equilibrium bidding functions in price, is characterized below.

Proposition 4.3.1: Let $((t_1^*, p_1^*), (t_2^*, p_2^*))$ be a BNE. Then equilibrium quality bids are given by $t_1^*(\theta) = \frac{q}{\eta_1}$ and $t_2^*(\theta) = \frac{q}{\eta_2}$ respectively. The equilibrium price bid functions p_1^* and p_2^* are solutions to the following differential equations:

$$p_j^{*-1'}(p_i) = \frac{1 - F(p_j^{*-1}(p_i))}{f(p_j^{*-1}(p_i))(p_i - c(\frac{q}{\eta_i}, p_j^{*-1}(p_i))}, i = 1, 2 \tag{4.2}$$

with boundary conditions $p_1(\underline{\alpha}) = p_2(\underline{\alpha})$ and $p_1(\bar{\alpha}) = p_2(\bar{\alpha})$.

The buyer chooses subjectivity parameter η_2.

Proof: Note that in equilibrium $p_1^*(\underline{\alpha}) = p_2^*(\underline{\alpha})$ and $p_1^*(\bar{\alpha}) = p_2^*(\bar{\alpha})$. If bidder i bids below $p_j^*(\underline{\alpha})$ then he wins for sure. But by increasing a small amount he increases pay-off without reducing his probability of winning. Similarly, bidder i will not bid above $p_j^*(\bar{\alpha})$. If he bids above $p_j^*(\bar{\alpha})$ then he does not win for sure.

Now, suppose bidder 1 bids $\frac{q}{\eta_2}$ in equilibrium. The best response of the other player is to bid $\frac{q}{\eta_1}$ as bidder and announce η_1 as a buyer. Observe that bidder 1 does not qualify and gets a payoff of zero. However by deviating and bidding $\frac{q}{\eta_1}$ and the same price function

[5]For instance, refer to the case of Reliance Airport Developers Private Limited in the case of the Delhi Airport Project discussed in the Introduction.

as bidder 2 he can get a strictly positive payoff. Therefore 1 must bid $\frac{q}{\eta_1}$ in equilibrium. Therefore bidder 2 must bid $\frac{q}{\eta_2}$ and the buyer must announce η_2.

In view of this argument it follows that the auction reduces to a lowest price auction where bidder 1's valuation lies in the set $V_1 = \{c(\frac{q}{\eta_1}, \theta) | \theta \in [\underline{\alpha}, \bar{\alpha}]\}$ and bidder 2's valuation lies in the set $V_2 = \{c(\frac{q}{\eta_2}, \theta) | \theta \in [\underline{\alpha}, \bar{\alpha}]\}$. Since cost functions are continuous, V_1 and V_2 are intervals. A typical element in V_i is denoted by v_i. The transformed distribution functions are F_1 and F_2. The payoff function of of bidder i is $p_i - v_i$ if he wins and zero otherwise. By Lizzeri and Persico (2000), strictly increasing and differentiable BNE exists in this game. The equilibrium strategies p_1^{**} and p_2^{**} are solutions to the following differentiable equations,[6]

$$p_j^{**-1'}(p_i) = \frac{1 - F_j(p_j^{**-1}(p_i))}{f_j(p_j^{**-1}(p_i))(p_i - p_j^{**-1}(p_i))}, i = 1, 2.$$

Writing these equations as functions of cost-type gives us the system of differentiable equations 4.2.

■

The equilibrium price bid functions in Proposition 4.3.1 are not same. This follows from the observation that derivative of the bid functions are not same for $p_1^*(\theta) = p_1^*(\theta)$, because $t_1^* \neq t_2^*$. Since these two bidding functions are not same the inefficient bidder may be chosen. The next Proposition shows that in equilibrium the honest bidder bids a higher price for every realization of cost-type. Thus, it says that even if the honest bidder is more efficient than the inefficient one he may not be chosen.

Proposition 4.3.2: The equilibrium bidding functions in price satisfy $p_1^*(\theta) > p_2^*(\theta)$ for all $\theta \in (\underline{\alpha}, \bar{\alpha})$.

[6]Note that an implicit assumption for this existence result is that for every t the inverse of $c(t, \theta)$ is twice continuously differentiable. Together with the assumption of continuous differentiablity of f then F_i has density f_i and f_i is continuously differentiable, for $i = 1, 2$. This differentiablity assumption is required in order to use the results in Lizzeri and Persico (2000) in our model. An example of a cost function with these properties is $c(t, \theta) = t\theta$.

Proof: First we show that if p_1^* and p_2^* intersect in the interior of $[\underline{\alpha}, \bar{\alpha}]$ then they will intersect at most once.

Suppose $p_1^{*-1}(p) = p_2^{*-1}(p) = \theta$. Since $t_1^* > t_2^*$,

$$p_2^{*-1'}(p) = \frac{1}{(p - c(t_1^*, \theta))} > p_1^{*-1'}(p) = \frac{1}{(p - c(t_2^*, \theta))}.$$

In other words, $\frac{dP_2^*(\theta)}{d\theta} < \frac{dP_1^*(\theta)}{d\theta}$. This means that if the bidding functions intersect, then the bidding function of bidder 1 cuts the bidding function of bidder 2 from below. Hence the intersection takes place at most once in $(\underline{\alpha}, \overline{\alpha})$.

Now we prove our claim. Assume for the sake of contradiction that there exists θ such that $p_1(\theta) \leq p_2(\theta)$. This means that for all $x \in (\underline{\alpha}, \theta)$ we must have $p_1(x) < p_2(x)$. Let $p_1(\underline{\alpha}) = p_2(\underline{\alpha}) = \underline{p}$.

This implies that for all p close enough to $\underline{p}, p_1^{*-1}(p) > p_2^{*-1}(p)$. [*]

Since F is a strictly increasing distribution function, $1 - F(p_1^{*-1}(p)) < 1 - F(p_2^{*-1}(p))$ for all p close enough to $\underline{p}$. Note that $F(p_1^{*-1}(p))$ and $F(p_2^{*-1}(p))$ are different functions because they are different compositions. For the sake of convenience let $H(p) = [1 - F(p_1^{*-1}(p))] - [1 - F(p_2^{*-1}(p))]$. Clearly, $H(\underline{p}) = 0$. Note that for all p close enough to $\underline{p}$, $H(p) < 0$. We prove the following claim, which helps us to reach a contradiction.

Claim: For every $\epsilon > 0$ there exists $p(\epsilon) \in (\underline{p}, \underline{p} + \epsilon)$ such that $\frac{dHp(\epsilon)}{dp} < 0$.

Proof of the Claim: Fix an $\epsilon > 0$. By the Mean Value Theorem, there exists $p(\epsilon) \in (\underline{p}, \underline{p} + \epsilon)$ such that $\epsilon \frac{dHp(\epsilon)}{dp} = H(\underline{p} + \epsilon) - H(\underline{p}) = H(\underline{p} + \epsilon) < 0$. Since $\epsilon > 0$, $\frac{dHp(\epsilon)}{dp} < 0$.

From the above Claim it follows that there exists a p close enough to $\underline{p}$ such that $p_1^{*-1}(p) > p_2^{*-1}(p)$ and $\frac{dH(p)}{dp} < 0$. This means that, $-f(p_1^{*-1}(p))p_1^{*-1'}(p) + f(p_2^{*-1}(p))p_2^{*-1'}(p) < 0$. Hence,

$$f(p_1^{*-1}(p))p_1^{*-1'}(p) > f(p_2^{*-1}(p))p_2^{*-1'}(p) > 0. \text{ [**]}$$

Observe that the first order condition for bider i can be written as,

$$f(p_j^{*-1}(p_i))p_j^{*-1'}(p_i)(p_i - c(t_i^*, p_i^{*-1}(p_i))) = [1 - F(p_j^{*-1}(p_i))],$$

$$\Rightarrow f(p_j^{*-1}(p_i))p_j^{*-1'}(p_i)p_i - [1 - F(p_j^{*-1}(p_i))] = f(p_j^{*-1}(p_i))p_j^{*-1'}(p_i)c(t_i^*, p_i^{*-1}(p_i))).$$

Hence,

$$c(t_i^*, p_i^{*-1}(p_i)) = p_i - \frac{[1 - F(p_j^{*-1}(p_i))]}{f(p_j^{*-1}(p_i))p_j^{*-1'}(p_i)}. \; [***]$$

Since $[1 - F(p_2^{*-1}(p))] > [1 - F(p_1^{*-1}(p))]$ for all p close enough to $\underline{p}$, by [**] we can choose p such that $\frac{[1-F(p_2^{*-1}(p))]}{f(p_2^{*-1}(p))p_2^{*-1'}(p)} > \frac{[1-F(p_1^{*-1}(p))]}{f(p_1^{*-1}(p))p_1^{*-1'}(p)} > 0$. This observation together with [***] gives,

$$c(t_1^*, p_1^{*-1}(p)) = p - \frac{[1 - F(p_2^{*-1}(p))]}{f(p_2^{*-1}(p))p_2^{*-1'}(p)} < c(t_2^*, p_2^{*-1}(p)) = p - \frac{[1 - F(p_1^{*-1}(p))]}{f(p_1^{*-1}(p))p_1^{*-1'}(p)}.$$

Therefore $c(t_1^*, p_1^{*-1}(p)) < c(t_2^*, p_2^{*-1}(p))$. Since $t_1^* > t_2^*$ we must have $p_1^{*-1}(p) < p_2^{*-1}(p)$. This is a contradiction to [*].[7]

■

The Proposition above implies that the honest bidder may not win even if he is more efficient than the dishonest one with positive probability. We give a numerical example illustrating this phenomenon. A numerical solution of the equilibrium price bid functions are presented in Diagram 4.1.

Example 4.3.1: In this example we assume $\underline{\alpha} = 1$ and $\bar{\alpha} = 2$, F to be uniform, $c(t, \theta) = t\theta$. Also, $\underline{q} = 1$, $\eta_1 = 1$, $\eta_2 = \frac{4}{3}$. Therefore, $t_1^* = 1$ and $t_2^* = \frac{3}{4}$. Therefore, $V_1 = [1, 2]$ and $V_2 = [\frac{3}{4}, \frac{2}{3}]$. The initial condition is $p_1^*(1) = p_2^*(1) = 1.5$. At cost-type 2, both the strategies asymptote.

[7] Note that in the case of a model with more than 2 bidders the nature of equilibria will not change, i.e. in any equilibrium there exists at least one bidder whose technical bid is according to η_1. Therefore, the inefficiency that is observed in a two-bidder model will also be found in a n-bidder model.

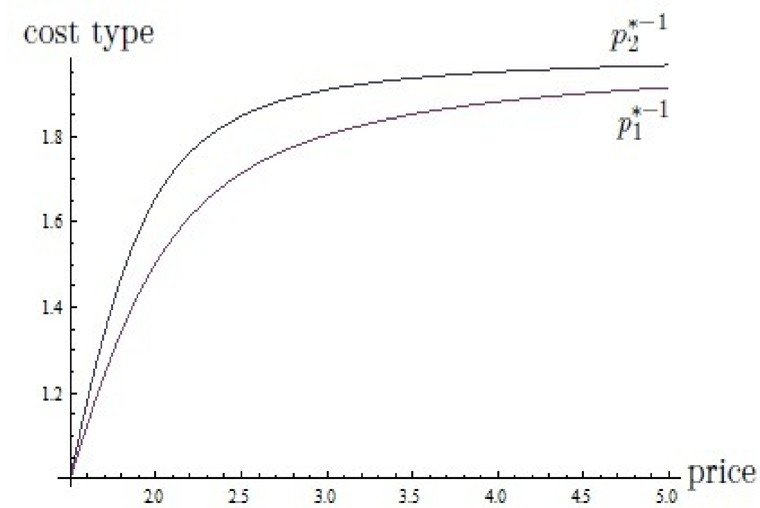

Diagram 4.1

In Diagram 4.1 inverse equilibrium price bid functions are plotted for Example 4.3.1. The system of differential equations for equilibrium are solved by using Mathematica. These bidding functions demonstrate that for lower cost-types of bidder 1 relative to bidder 2, bidder 1 bids higher price. Therefore, even if he is more efficient than bidder 2, bidder 1 will not win.

4.3.1 Discussion

There is a large literature on corruption/collusion in auctions. In Laffont and Tirole (1991) an auctioneer is asked to choose a firm to carry out a public project. Before the bids are submitted, the agency receives a signal about the quality of the firms but can supply wrong information to the buyer about their quality. In contrast, we describe an endogenous process that determines the quality of a firm. The buyer does not have to supply wrong information about the quality that is determined by the process. In fact, the quality of a bidder perceived by the buyer is made public. But even so, the collusion cannot be legally challenged.

Arozamena and Weinschelbaum (2009) consider corruption in first price auction when it is known among the other bidders that the auctioneer favors one of the bidders. The dishonest bidder is allowed to revise his bid upward or downward by the auctioneer. Burguet and Che

(2004) consider a scoring auction where the relevant bids for the buyer are two dimensional, quality and price. They assume that both the bidders are dishonest - along with quality and price they bid a bribe. The auctioneer can manipulate the quality bid by inflating it by some number m i.e. if bidder i bids quality q_i then the auctioneer can manipulate it to $q_i + m$. If the bribe of bidder i is more than that of bidder j, then the auctioneer manipulates i's quality and make him the winner. However the winner supplies the quality that he bids; so it is not clear how this corruption can be sustained. In our model, the auctioneer does not manipulate technical bid directly. Instead he manipulates the evaluation procedure to favor his preferred bidder. Since the evaluation process is subjective this collusion cannot be verified in the court of law. Further issues relating to corruption in procurement are discussed in Wolfstetter and Lengwiler (2006).

4.4 Conclusion

In this chapter we study some theoretical properties of a class of procurement auctions used widely by the Government in India. These auctions first screen the bidders in terms of technical qualifications. We have shown that no symmetric, continuous and strictly monotonic pure strategy Bayes-Nash equilibrium exists in these auctions. We have also analyzed auctions with a pre-specified minimum quality-score where technical bids are evaluated subjectively. Subjective evaluation of technical bids also leads to a natural model of bidder-buyer collusion which we have analyzed.

4.5 Appendix

Lemma 4.5.1 *Let g and h be two real valued functions defined on $[a, b]$. Fix $c \in (a, b)$. Define,*

$$f(x) = \begin{cases} g(x), & \text{if } x \geq c; \\ h(x), & \text{if } x < c \end{cases}$$

Let $g(c) = h(c)$ and $g'(c) = h'(c)$. Then f is differentiable at c and $g'(c) = h'(c) = f'(c)$.

Proof:

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{g(x) - g(c)}{x - c} = g^{'}(c)$$

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{h(x) - g(c)}{x - c} = \lim_{x \to c^-} \frac{h(x) - h(c)}{x - c} = h^{'}(c)$$

Hence the result follows.

■

Example 4.5.1: We follow this example in Diagram 4.2.

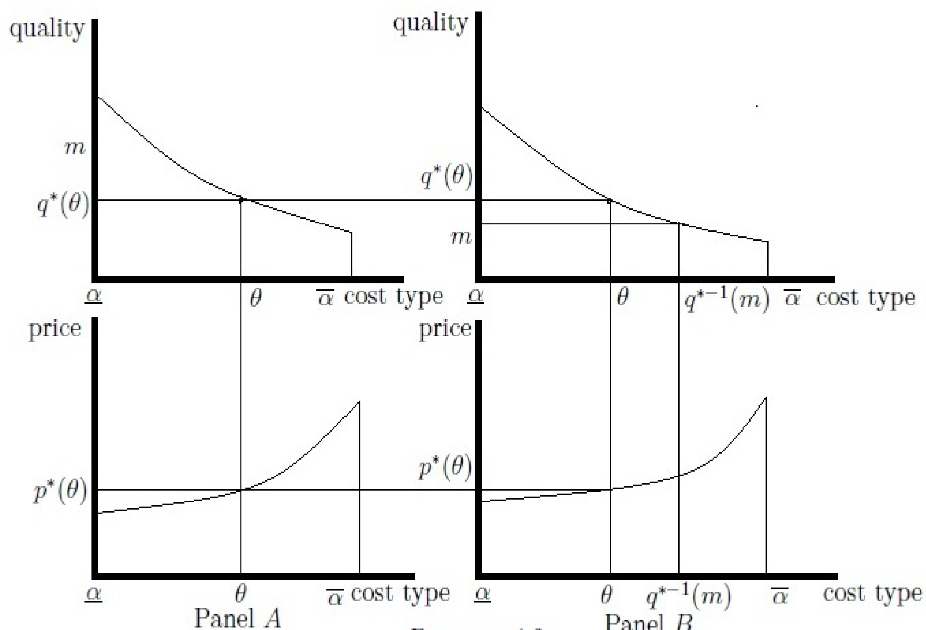

Diagram 4.2

Let the equilibrium be denoted by (q^*, p^*). Consider bidder 1 whose type is θ. Suppose he bids quality-score and price $(m, p^*(\theta))$. If $m > q^*(\theta)$ (Panel A) then bidder 1 wins if and only if no other bidders' cost-type is lower than θ. Since all other bidders follow the suggested equilibrium strategy so if any of the two bidders type is lower than θ then he bids a price lower than $p^*(\theta)$. Since the buyer considers exactly two technical bids so bidder 1 does not win and hence his expected payoff is zero in that case. Therefore the objective function takes the form,

$$[p^*(\theta) - c(m,\theta)][1 - F(\theta)]^2, \text{ if } m \geq q^*(\theta).$$

On the other hand if $m < q^*(\theta)$ (Panel B) then bidder 1's probability of win is,

probability that both the bidders cost-type is greater than $q^{*-1}(m)$ + probability that one bidder's cost-type is greater than $q^{*-1}(m)$ and one bidder's cost-type is between θ and $q^{*-1}(m)$ and this occurs in 2 ways.

Therefore, the objective function takes the form

$$[p^*(\theta) - c(m,\theta)]\Big\{[1 - F(q^{*-1}(m))]^2 + 2[F(q^{*-1}(m)) - F(\theta)][1 - F(q^{*-1}(m))]\Big\}, \text{ if } m < q^*(\theta).$$

The expected payoff function written in Section 4.1 is a generalization of the two expressions above.

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