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SOME INVARIANT SEQUENTIAL AND NONSEQUENTIAL RULES FOR IDENTIFYING A MULTIVARIATE NORMAL POPULATION

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SUMMARY. The problem considered here is an identification problem, where the populations are multivariate normal differing in their unknown means and the common variance-covariance matrix Σ may be known or unknown. A sample of fixed size is given from the population π_0 which is to be identified with one of the two other populations π_1 and π_2, from which sampling can be done sequentially or non-sequentially. This is an extension of the univariate version of the problem taken up by Ghosh and Ray Chaudhuri (1984) where a truncated invariant SPRT was proposed as a solution. The one sided univariate version of the problem was also considered by Ghosh and Mukhopadhyay (1980). Here the invariant SPRT for the multivariate case is studied and it is seen that the error probabilities can be bounded as in the univariate case. A general theorem regarding asymptotic distribution of a class of stopping time is given, from which the asymptotic distribution of the stopping time of the invariant SPRT for the known Σ case, follows. Termination properties are also studied for the proposed invariant SPRT.

1. Introduction

Let X, Y, Z with suffixes denote random variables associated with π_0, π_1 and π_2 respectively. Each of the populations π_0, π_1 and π_2 are p-variate ($p \geqslant 2$) normal with mean μ, μ_1 and μ_2 respectively and the common variance-covariance matrix Σ which may be known or unknown.

The problem can be formulated in the following way :

$$\text{Test } H_0 : \mu = \mu_1 \text{ versus } H_1 : \mu = \mu_2$$

$$\left.\begin{aligned} \text{s.t. } P_{H_0}(\text{Rejection of } H_0) &= \alpha \\ P_{H_1}(\text{Rejection of } H_1) &= \beta \end{aligned}\right\} \quad \ldots \quad (1.1)$$

A parameter Δ_0 is introduced (as in the univariate case) to specify the indifference zone and the following hypotheses are tested,

$$\left.\begin{aligned} H_0 : \mu = \mu_1, \mu \neq \mu_2, \|\mu_1 - \mu_2\|_\Sigma = \Delta_0 \\ H_1 : \mu \neq \mu_1, \mu = \mu_2, \|\mu_1 - \mu_2\|_\Sigma = \Delta_0 \end{aligned}\right\} \quad \ldots \quad (1.2)$$

where $$\|\mu_1 - \mu_2\|_\Sigma = ((\mu_1 - \mu_2)' \Sigma^{-1} (\mu_1 - \mu_2))^{1/2} \quad \ldots \quad (1.3)$$

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It is natural to expect that any reasonable procedure for testing the hypotheses described in (1.2) will also work (in fact in a better way perhaps) when the true $\|\mu_1-\mu_2\|_\Sigma > \Delta_0$. Some practical motivation is provided in Ghosh and Mukhopadhyay (1980) and Schaafsma and Van Vark (1977).

The following three schemes are considered here :

S1 : Three fixed samples of size k, (pre-determined, $k \geqslant k_0$) n_0 and n_0 are taken from π_0, π_1 and π_2 respectively. Here k_0 denotes the minimum sample size from π_0, needed for the indentification problem subject to condition (1.1) (vide Section 2.1 of Ghosh and Mukhopadhyay, 1980). Clearly k_0 depends on α, β and Δ_0.

S2 : A sample of fixed size $k(k \geqslant k_0$ as given in S1) is taken from π_0 where π_1 and π_2 are sampled sequentially.

S3 : All the three populations are sampled sequentially.

Under sampling scheme S1, the best invariant fixed sample procedure is considered. This procedure has error probabilities monotonically decreasing as $\|\mu_1-\mu_2\|_\Sigma$ increases (vide Dasgupta, 1974) when the cutoff constant c is one i.e. when $\alpha = \beta$.

Under sampling scheme S2 and S3, (for the known Σ case) the invariant SPRT's based on the maximal invariant are considered once with $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n)$ and once with $(\bar{X}_n, \bar{Y}_n, \bar{Z}_n)$ as sufficient statistics for (μ, μ_1, μ_2). The error probabilities of both these SPRT's can be bounded as in the univariate case vide Ghosh and Ray Chaudhuri (1984).* The termination properties are studied in Section 4. In Section 5, a general theorem is given to study the asymptotic behavior of a class of stopping times. The asymptotic distribution of the stopping time of the invariant SPRT for scheme S2 for the known Σ case, follows from this general theorem. The effect of truncation on the stopping time is also discussed.

2. Procedures for known Σ case

If Σ is known, it may be assumed to be I_p without loss of generality.

Now the hypotheses described in (1.2) can be restated as

$$\left.\begin{array}{l} H_0 : \theta = (\Delta_0, \Delta_0, 1) \\ H_1 : \theta = (\Delta_0, \Delta_0, -1) \end{array}\right\} \quad \ldots (2.1)$$

*These results are discussed in Section 2, and Section 3 deals with similar results with the unknown Σ.

where $$\theta = \left(\|2\mu-\mu_1-\mu_2\|, \|\mu_1-\mu_2\|, \frac{(2\mu-\mu_1-\mu_2)'(\mu_1-\mu_2)}{\|2\mu-\mu_1-\mu_2\|\cdot\|\mu_1-\mu_2\|}\right)$$

... (2.2)

and $$\|X\| = \|X\|_I.$$

Here $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n)$ is sufficient for (μ, μ_1, μ_2) with k and n defined as follows for different schemes. For scheme S1, $k \geqslant k_0$, $n = n_0$ where both k and n_0 are fixed, k is predetermined and n_0 is determined subject to (1.1). Here such a choice of n_0 is possible as $k \geqslant k_0$, k_0 as described in S1. For scheme S2, k is predetermined and $k \geqslant k_0$ as in S2 and $n = 1, 2, 3, \ldots$ and for scheme S3, $k = n = 1, 2, \ldots$.

The group of transformation applied here is $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n) \to (B\bar{X}_k+C, B\bar{Y}_n+C, B\bar{Z}_n+C)$ where B is $p\times p$ orthogonal and C is $p\times 1$ scalar vector. The maximal invariant under this transformation is $A_n = (\|2\bar{X}_k-\bar{Y}_n-\bar{Z}_n\|^2, \|\bar{Y}_n-\bar{Z}_n\|^2, (2\bar{X}_k-\bar{Y}_n-\bar{Z}_n)'(\bar{Y}_n-\bar{Z}_n))$. By the basic theorem of Hall et al (1965) A_n is invariantly sufficient for θ. Let $\psi_{p\times p}$ be orthogonal such that $\psi(\mu_1-\mu_2) = (\Delta_0, 0, 0, \ldots, 0)'$. Then $S = \psi(2\bar{X}_k-\bar{Y}_n-\bar{Z}_n)$ is normally distributed with mean $(\Delta_0, 0, 0, \ldots, 0)'$ under H_0 and $(-\Delta_0, 0, 0, \ldots, 0)'$ under H_1 and variance-convariance matrix $(4k^{-1}+2n^{-1})I_p$. And $T = \psi(\bar{Y}_n-\bar{Z}_n)$ is independent of S and is normally distributed with mean $(\Delta_0, 0, 0 \ldots, 0)'$ and variance-covariance matrix $2n^{-1}I_p$.

The distribution of $A_n = (\|S\|^2, \|T\|^2, S'T)$ is noncentral Wishart (vide section 3 of Anderson and Girshick (1944)) with the density

$$f_{H_m}(A_n) = \frac{e^{-\frac{1}{2}k_n^2-\frac{1}{2}\sum_{i=1}^{2} b_{ii}^m}}{2^{p-\frac{1}{2}(p-2)}\pi^{1/2}\Gamma\left(\frac{p-1}{2}\right)} |b_{ij}^m|^{\frac{p-3}{2}} (k_n^2 b_{11})^{\frac{p-2}{4}} I_{\frac{1}{2}(p-2)}(k_n\sqrt{b_{11}}), \quad \text{for } m = 0,1$$

where $k_n^2 = \Delta_0^2(\sigma_S^{-1}+\sigma_T^{-1}), \sigma_S = 4k^{-1}+2n^{-1}, \sigma_T = 2n^{-1}$

$$b_{11}^m = \left(\frac{\sigma_T}{\sigma_S}\; S'S+\frac{\sigma_S}{\sigma_T}\; T'T+(-1)^m\, 2S'T\right)(\sigma_S+\sigma_T)^{-1}$$

$$b_{12}^m = ((-1)^{m+1}\sigma_T S'S+(\sigma_T-\sigma_S)S'T+(-1)^m\sigma_S T'T)(\sigma_S+\sigma_T)^{-1}(\sigma_S\sigma_T)^{-1/2}$$

$$b_{22}^m = (S'S+(-1)^{m+1}2S'T+T'T)\,(\sigma_S+\sigma_T)^{-1} \text{ under } H_m \text{ for } m = 0,\ 1$$

$$|b_{ij}^m| = b_{11}^m\, b_{22}^m-(b_{12}^m)^2.$$

$I_{\frac{1}{2}(p-2)}$ is the Bessel function of imaginary argument. The trace $\sum_1^2 b_{ii}^m$ and $|b_{ij}^m|$ both remain unchanged under the two hypotheses.

Define $Z_m = k_n(b_{11}^m)^{1/2}$

$$= \Delta_0 \left(\frac{S'S}{J_S^2} + \frac{T'T}{J_T^2} + (-1)^m \frac{2S'T}{\sigma_S \sigma_T} \right)^{1/2} \text{ for } m = 0, 1, \ldots \quad \ldots \quad (2.3)$$

Then the test statistics reduces to

$$W_{n,k}(\Delta_0) = \frac{f_{H_1}(A_n)}{f_{H_0}(A_n)} \quad \ldots \quad (2.4)$$

$$= \frac{Z_1^{-\frac{p-2}{2}} I_{\frac{1}{2}(p-2)}(Z_1)}{Z_0^{-\frac{p-2}{2}} I_{\frac{1}{2}(p-2)}(Z_0)} \quad \ldots \quad (2.5)$$

$$= \frac{w_p(z_1)}{w_p(z_0)} \quad \ldots \quad (2.6)$$

where $$w_p(x) = \int_0^1 \cosh(x\,t)\,(1-t^2)^{\frac{p-3}{2}}\,dt \quad \ldots \quad (2.7)$$

The equality of (2.5) and (2.6) is an easy consequence of the series representation of cosh (x) and $I_{\frac{p-2}{2}}(x)$ (vide Whittaker and Watson, 1958, page 373).

Remark 1 : One may obtain this form of density ratio (as in (2.6)) of maximal invariant A_n, by integrating over the group of transformation (vide Wijsman (1967, 1979)). One can avoid the complicated series expansion by adopting this technique.

For Scheme S1, the procedure is as follows :

Reject H_0 if $W_{n_0,k}(\Delta_0) > 0$ where Δ_0 and n_0 are chosen to satisfy (1.1). When $\alpha = \beta$, we have $c = 1$ and

$$W_{n,k}(\Delta_0) > 1 \Longleftrightarrow \sigma_S^{-1}\sigma_T^{-1}S'T < 0 \quad \ldots \quad (2.8)$$

By Theorem 2.1 of Dasgupta (1974), both types of error probabilities can be shown to be monotonically decreasing functions of $\|\mu_1 - \mu_2\|$. However, the monotonicity could not be shown for $c \neq 1$.

Now to implement S1, one needs the value of k_0 or at least an upper bound of k_0. Derivation of an exact value of k_0 involves tedious numerical calculation as the distribution of $W_{n,k}(\Delta_0)$ is extremely complicated, whereas an upper bound of k_0 can be obtained by a much simpler method as given below.

If $\alpha \neq \beta$, consider the harder problem with $\alpha' = \beta' = \alpha \wedge \beta$ (if $\alpha = \beta$ then $\alpha' = \beta' = \alpha$). Probability of correct identification for this harder problem is $P_{H_0}\left(\frac{S'T}{\sigma_S \sigma_T} > 0\right) \geqslant \left(\Phi\left(\frac{\Delta_0}{(p(4k^{-1}+2n^{-1}))^{1/2}}\right)\right)^{2p}$

(using independence of S and T and $\Phi(\,.\,)$ denotes the normal c.d.f.).

Now for having a solution in n for

$$\left(\Phi\left(\frac{\Delta_0}{(p(4k^{-1}+2n^{-1}))^{1/2}}\right)\right)^{2p} = 1-\alpha' \qquad \ldots \quad (2.9)$$

one needs to have

$$k > 4\tau_{\alpha p}^2 p\, \Delta_0^{-2} \qquad \ldots \quad (2.10)$$

where $\tau_{\alpha p}$ is s.t. $\Phi(\tau_{\alpha p}) = (1-\alpha')^{1/2p}$.

Define $k_1 = [4\tau_{\alpha p}^2 p \Delta_0^{-2}]$ = the smallest integer greater than or equal to $4\tau_{\alpha p}^2 p \Delta_0^{-2}$. Now one can take $k \geqslant k_1$ to implement Scheme S1.

Let $$n_1 = \left[\left(\frac{\Delta_0^2}{2\tau_{\alpha p}^2\, p} - 2k^{-1}\right)^{-1}\right] \text{ for } k \geqslant k_1. \qquad \ldots \quad (2.11)$$

Then n_1 works as an upper bound of n_0.

For scheme S2, the truncated invariant SPRT with test statistic $W_{n,k}(\Delta_0)$ is considered with the usual boundaries. Here the untruncated SPRT does not terminate with probability one (see Theorem 1 in Section 4), which emphasises the need for a truncation point. One can choose the truncation point $m_0 = 2n_1$, n_1 as given in (2.11).

For scheme S3, the invariant SPRT with usual boundaries is studied. The test statistic in this case is $W_{n,n}(\Delta_0)$. This SPRT terminates with probability one which is ensured by Theorem 2 in Section 4.

Both kinds of error probabilities of the invariant SPRT's for schemes S2 and S3, can be bounded as given in Proposition 2 of Ghosh and Ray Chaudhuri (1984). For applying Proposition 2 the following Lemma 1 is needed.

Lemma 1 : *For $A < 1$, $B > 1$ and $\Delta^* > \Delta_0 > 0$,*

(i) $W_{n,k}(\Delta_0) \leqslant A \Longrightarrow W_{n,k}(\Delta^*) < A$ *and*

(ii) $W_{n,n}(\Delta_0) \geqslant B \Longrightarrow W_{n,k}(\Delta^*) \geqslant B$.

The proof of Lemma 1 follows in exactly similar lines as the proof of Lemma 2 of Ghosh and Ray Chaudhuri (1984). Thus Lemma 1 ensures the fulfilment of condition (2.21) of Proposition 2 of the above mentioned paper and the following bounds can be obtained.

For scheme S2, we have

$$\left.\begin{aligned}\alpha^* &\leqslant \frac{\alpha}{1-\beta}(1-\beta^*)-\frac{\alpha}{1-\beta}P_{H_1^*}(N_1 \geqslant m_0, W_{m_0 \cdot k} > 1)+P_{H_0^*}(N_1 \geqslant m_0, W_{m_0 \cdot k} > 1)\\ \beta^* &\leqslant \frac{\beta}{(1-\alpha)}(1-\alpha^*)-\frac{\beta}{1-\alpha}P_{H_0^*}(N_1 \geqslant m_0, W_{m_0 \cdot k} \leqslant 1)+P_{H_1^*}(N_1 \geqslant m_0, W_{m_0 \cdot k} \leqslant 1)\end{aligned}\right\} \quad \dots (2.12)$$

where N_1 is the stopping time of the untruncated SPRT

$H_0^* := (\Delta^*, \Delta^*, 1)$, $H_1^* := (\Delta^*, \Delta^*, -1)$ with $\Delta^* > \Delta_0$,

θ as in (2.2) and m_0 the truncation point,

$\alpha^* = P_{H_0^*}$ (Rejection of H_0),

$\beta^* = P_{H_1^*}$ (Rejection of H_1).

For scheme S3, the bounds are much simpler, (as in page 46 of Wald, 1947). Then

$$\alpha^* \leqslant \frac{\alpha}{1-\beta}(1-\beta^*), \ \beta^* \leqslant \frac{\beta}{1-\alpha}(1-\alpha^*)$$

and thus
$$\alpha^*+\beta^* \leqslant \alpha+\beta.$$

3. Procedures for unknown Σ case

Σ being not known, the situation here is more complicated. The hypotheses tested here are as follows:

$$\left.\begin{aligned}H_0 &: \theta = (\Delta_0, \Delta_0, 1)\\ H_1 &: \theta = (\Delta_0, \Delta_0, -1)\end{aligned}\right\}, \quad \dots (3.1)$$

where
$$\theta = \left(\|2\mu-\mu_1-\mu_2\|_\Sigma, \|\mu_1-\mu_2\|_\Sigma, \frac{(2\mu-\mu_1-\mu_2)'\Sigma^{-1}(\mu_1-\mu_2)}{\|2\mu-\mu_1-\mu_2\|_\Sigma\|\mu_1-\mu_2\|_\Sigma}\right). \quad \dots (3.2)$$

Note $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n, S_n)$ here is sufficient for $(\mu, \mu_1, \mu_2, \Sigma)$ where

$$S_n = \sum_1^k (X_i-\bar{X}_k)(X_i-\bar{X}_k)'+\sum_1^n (Y_i-\bar{Y}_n)(Y_i-\bar{Y}_n)'+\sum_1^n (Z_i-\bar{Z}_n)(Z_i-\bar{Z}_n)'.$$

n and k for different schemes are here as defined in Section 2. The group of transformation considered is

$$(\bar{X}_k, \bar{Y}_n, \bar{Z}_n, S_n) \to (B\bar{X}_k+C, B\bar{Y}_n+C, B\bar{Z}_n+C, BS_nB')$$

where B is $p \times p$ nonsingular and C is $p \times 1$ scalar vector.

Now $B_n = (Y'_{1n} S_n^{-1} Y_{1n},\ Y'_{2n} S_n^{-1} Y_{2n},\ Y'_{1n} S_n^{-1} Y_{2n})$ is maximal invariant and by the basic theorem of Hall *et al.* (1965) B_n is invariantly sufficient for θ where

$$Y_{1n} = \frac{1}{\sqrt{4k^{-1}+2n^{-1}}} \cdot (2\bar{X}_k - \bar{Y}_n - \bar{Z}_n),\ Y_{2n} = \frac{\bar{Y}_n - \bar{Z}_n}{\sqrt{2n^{-1}}}.$$

The density of B_n under both hypotheses are given as follows (vide Sitgreaves, (1952)

$$f_{H_m}(B_n) = \frac{\Gamma\left(\frac{n^*+1}{2}\right) e^{-\frac{1}{2}\Delta_0^2(k_1^2+k_2^2)} |B|^{\frac{p-3}{2}}}{\Gamma\left(\frac{n^*-p+2}{2}\right) \Gamma\left(\frac{n^*-p+1}{2}\right) \Gamma\left(\frac{p-1}{2}\right) \Gamma\left(\frac{1}{2}\right) |I+B|^{\left(\frac{n^*+2}{2}\right)}}$$

$$\sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{n^*+2}{2}+j\right)}{j\,!\,\Gamma\left(\frac{p}{2}+j\right)} \left(\frac{1}{2}\right)^j (U_m)^j \quad \ldots \quad (3.4)$$

$$U_m = \Delta_0^2(k_1^2 b_{11}^* + 2k_1 k_2 b_{12}^*(-1)^m + k_2^2 b_{22}^*) \text{ for } m = 0, 1, \quad \ldots \quad (3.5)$$

$$\left.\begin{aligned} b_{11}^* &= b^{-1}(b_{11}+b_{11}b_{22}-b_{12}^2) \\ b_{22}^* &= b^{-1}(b_{22}+b_{11}b_{22}-b_{12}^2) \\ b_{12}^* &= b^{-1}b_{12} \\ b &= 1+b_{11}+b_{12}+b_{11}b_{12}-b_{12} \end{aligned}\right\} \quad \ldots \quad (3.6)$$

$$b_{11} = Y'_{1n} S_n^{-1} Y_{1n},\ b_{22} = Y'_{2n} S_n^{-1} Y_{2n},\ b_{12} = Y'_{1n} S_n^{-1} Y_{2n},$$

$$\left.\begin{aligned} B &= \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix},\ n^* = 2n+k-3,\ \Delta_0 = \|\mu_1-\mu_2\|_\Sigma \\ k_1 &= \frac{1}{\sqrt{4k^{-1}+2n^{-1}}},\ k_2 = \frac{1}{\sqrt{2n^{-1}}} \end{aligned}\right\}. \quad \ldots \quad (3.7)$$

The test statistic reduces to

$$V_{n,k}(\Delta_0) = \frac{f_{H_1}(B_n)}{f_{H_0}(B_n)} = \frac{\sum\limits_{j=0}^{\infty} \left(\Gamma\left(\frac{n^*+2}{2}+j\right) / j\,!\,\Gamma\left(\frac{p}{2}+j\right) \right) \left(\frac{1}{2}\right)^j (u_1)^j}{\sum\limits_{j=0}^{\infty} \left(\Gamma\left(\frac{n^*+2}{2}+j\right) / j\,!\,\Gamma\left(\frac{p}{2}+j\right) \right) \left(\frac{1}{2}\right)^j (u_0)^j}$$

$$= \frac{\int_0^\infty \int_0^1 \cosh(tv\, 2^{1/2} U_1^{1/2})\, (1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1} e^{-t^2}\, dv dt}{\int_0^\infty \int_0^1 \cosh(tv\, 2^{1/2} U_0^{1/2})\, (1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1} e^{-t^2}\, dv dt} \quad \ldots \quad (3.8)$$

Here also one may obtain the density ratio $V_{n,k}(\Delta_0)$ by integrating over the group of transformation.

The procedures for Schemes S1, S2 and S3 for the unknown Σ case are similar to those for the known Σ case with $V_{n,k}(\Delta_0)$ in place of $W_{n,k}(\Delta_0)$.

For scheme S1, $V_{n_0,k}(\Delta_0) > 1 \Longleftrightarrow b^*_{12} < 0 \Longleftrightarrow Y'_{1n} S_n^{-1} Y_{2n} < 0$.

Now $Y'_{1n}S_n^{-1} Y_{2n} = Y'_{1n} P' (P'^{-1} S_n^{-1} P^{-1}) PY_{2n}$ where P is $p \times p$ nonsingular s.t. $P\Sigma P' = I_p$ and $P(\mu_1 - \mu_2) = (\Delta_0, 0, \ldots, 0)'$. Invoking part (ii) of Theorem 2.2 of Dasgupta (1974), the monotonicity of both types of error for $\alpha = \beta$ case, can be obtained.

For scheme S3, the usual invariant SPRT with test statistic $V_{n,n}(\Delta_0)$ terminates with probability one (vide Remark 2).

Error probabilities of both kinds of the truncated SPRT (for scheme S2) as well as the untruncated SPRT (for scheme S3) can be bounded as in the known Σ case. For that the fulfilment of condition 2.21 of proposition 2 of Ghosh and Ray Chaudhuri (1984) is necessary, which is assured by the following Lemma 2.

Lemma 2 : *For* $A < 1$, $B > 1$ *and* $\Delta^* > \Delta_0 > 0$.

(i) $V_{n,k}(\Delta_0) \leqslant A \Longrightarrow V_{n,k}(\Delta^*) \leqslant A$

(ii) $V_{n,k}(\Delta_0) \geqslant B \Longrightarrow V_{n,k}(\Delta^*) \geqslant B$.

Proof : The test statistic can be written as

$$V_{n,k}(\Delta_0) = \frac{\int_0^\infty \cosh(u\, 2^{1/2}\, U_1^{1/2}) f(u)\, du}{\int_0^\infty \cosh(u\, 2^{1/2}\, U_0^{1/2}) f(u) du}$$

where $f(u) \geqslant 0$ for $0 < u < \infty$. The proof now follows in the exactly similar lines as the proof of Lemma 2 of Ghosh and Ray Chaudhuri (1984).

4. Termination properties of the SPRT's for various schemes

This section supplies the proofs of two Theorems as mentioned in preceding sections. Let us first prove Theorem 1.

Theorem 1 : *Let* $N_1 = \inf\{n : W_{n,k}(\Delta_0) \geqslant B$ *or* $W_{n,k}(\Delta_0) \leqslant A\}$

Then $P_{(\mu, \mu_1, \mu_2)}(N_1 = \infty) > 0$ *for fixed* (μ, μ_1, μ_2).

Proof: Let $W_{n,k}(\Delta_0) < A$

$$\Longrightarrow W^{-1}_{n,k}(\Delta_0) \geqslant A^{-1} > 1$$

$$\Longrightarrow \frac{\int_0^1 \cosh(Z_0 t)(1-t^2)^{\frac{p-3}{2}}dt}{\int_0^1 \cosh(Z_1 t)(1-t^2)^{\frac{p-3}{2}}dt} \geqslant A^{-1} > 1$$

$\Longrightarrow Z_0 > Z_1$ where Z_0, Z_1 are as given in (2.3)

$$\Longrightarrow S'T > 0.$$

Let $u_1 = \frac{S'S}{\sigma_S^2}$, $u_2 = \frac{T'T}{\sigma_T^2}$, $u = \frac{S'T}{\sqrt{(S'S,)(T'T)}}$ and in this case $0 < u \leqslant 1$.

Now $\int_0^1 \frac{\cosh(Z_0 t)}{\cosh(Z_1 t)} f(t)dt > A^{-1}$ where $f(t) = \frac{\cosh(Z_1 t)(1-t^2)^{\frac{p-3}{2}}}{\int_0^1 \cosh(Z_1 t)(1-t^2)^{\frac{p-3}{2}}dt}$

$\Longrightarrow \frac{\cosh Z_0}{\cosh Z_1} > A^{-1}$ as $\frac{\cosh Z_0 t}{\cosh Z_1 t}$ is an increasing function of t for $Z_0 > Z_1$.

$$\Longrightarrow \frac{\cosh \Delta_0 (u_1+u_2+2\sqrt{u_1 u_2 . u})^{1/2}}{\cosh \Delta_0 (u_1+u_2-2\sqrt{u_1 u_2 . u})^{1/2}} > A^{-1}$$

$$\Longrightarrow \frac{\cosh \Delta_0 (\sqrt{u_1}+\sqrt{u_2})}{\cosh \Delta_0 (\sqrt{u_1}-\sqrt{u_2})} > A^{-1}.$$

Thus $N_1 = n \Rightarrow \frac{\cosh \Delta_0 (\sqrt{u_1}+\sqrt{u_2})}{\cosh \Delta_0 (\sqrt{u_1}-\sqrt{u_2})} > B \wedge A^{-1}$

$\Longrightarrow 2\Delta_0\sqrt{u_1} > \log(B \wedge A^{-1})$ (following along similar lines as in proof of Theorem 1 of Ghosh and Ray Chaudhuri, 1984)

$$\Longleftrightarrow \Delta_0^2 u_1 > \left(\frac{1}{2}\log(B \wedge A^{-1})\right)^2$$

$\Longrightarrow \frac{\Delta_0^2}{\sigma_s^2} S_j^2 > \frac{1}{p}\left(\frac{1}{2}\log(B \wedge A^{-1})\right)^2$ for at least one j, $1 \leqslant j \leqslant p$

as $S'S = \sum_1^p S_j^2$

Let $M = \inf\left\{n : \frac{\Delta_0^2 S_j^2}{\sigma_S^2} > \frac{1}{p}\left(\frac{1}{2}\log(B \wedge A^{-1})\right)^2 \text{ for at least one } j\right\}$

$$M_j = \inf\left\{n : \frac{\Delta_0^2 S_j^2}{\sigma_S^2} > \frac{1}{p}\left(\frac{1}{2}\log(B \wedge A^{-1})\right)^2\right\} \text{ for } 1 \leqslant j \leqslant p.$$

Then $P(M = \infty) = P(M_1 = \infty, M_2 = \infty, \ldots, M_p = \infty)$, with $P = P_{(\mu, \mu_1, \mu_2)}$. Now $P(M_j = \infty) > 0 \; \forall \; j = 1, 2, \ldots, p$, by Theorem 2 of Ghosh and Mukhopadhyay (1980). Also M_j's are independently distributed which implies $P(M = \infty) > 0$ and $N_1 \geqslant M$ gives the required result.

Theorem 2 : *Let* $N_2 = \inf\{n : W_{n,n}(\Delta_0) \geqslant B \text{ or } W_{n,n}(\Delta_0) \leqslant A\}$.

Then $P_{(\mu,\mu_1,\mu_2)}(N_2 < \infty) = 1$ *for fixed* (μ, μ_1, μ_2).

Proof : It is enough to show $P_{(\mu,\mu_1,\mu_2)}(A < W_{n,n}(\Delta_0) < B) \to 0$.

Theorem 3.7 of Ghosh (1970) says it is enough to have convergence of $\sqrt{n^{-1}} \log W_{n,n}(\Delta_0)$ to a continuous *r.v.* or to $+\infty$ or to $-\infty$ in probability. For then $\sqrt{n^{-1}} \log A$ and $\sqrt{n^{-1}} \log B$ both go to zero and the convergence of $P_{(\mu,\mu_1,\mu_2)}(A < W_{n,n}(\Delta_0) < B)$ to zero is immediate.

Now $W_{n,n}(\Delta_0) = \dfrac{w_p(nZ_{1n})}{w_p(nZ_{0n})}$ where $w_p(\,.\,)$ is as given in (2.7), and

$$\begin{aligned} Z_{in} &= n^{-1} Z_i \text{ for } i = 0, 1 \text{ (with } k = n \text{ in } Z_i \text{ defined in (2.3))} \\ &= \Delta_0(6^{-2} S'S + 2^{-2} T'T + (-1)^i 6^{-1} S'T)^{1/2} \qquad \ldots \quad (4.1) \end{aligned}$$

with $S = \psi(2\bar{X}_n - \bar{Y}_n - \bar{Z}_n) \sim N_p(\psi(2\mu - \mu_1 - \mu_2), 6n^{-1} I_p)$

$$T = \psi(\bar{Y}_n - \bar{Z}_n) \sim N_p(\psi(\mu_1 - \mu_2), 2n^{-1} I_p).$$

The approximation formula (3.3.4) of page 255 of Wijsman (1979) simplifies the situation as follows,

$$\log A < \log W_{n,n}(\Delta_0) < \log B$$

$$\Longrightarrow \log A - c < n Z_{1n} - n Z_{0n} + 2^{-1}(p-1) \log \frac{1 + nZ_{0n}}{1 + nZ_{1n}} < \log B + c \qquad \ldots \quad (4.3)$$

with c a positive real number.

Let $Z_{in} \to a_i$ a.s. as $n \to \infty$ for $i = 0, 1$. Then the possible cases are (1) $a_0 \neq a_1$ (2) $a_0 = a_1$, since $Z_{in} \geqslant 0 \; \forall \; n$, $a_i \geqslant 0$ for $i = 0, 1$.

If $a_0 = 0$, then $n^{1/2} Z_{0n}$ converges to a continuous distribution (precisely to a normal distribution) and thus

$$n^{-1/2} \log (1+nZ_{0n}) = n^{-1/2} \log n^{1/2}+n^{-1/2} \log (n^{-1/2}+n^{1/2}Z_{0n}) = o_p(1).$$

If $a_0 > 0$, then $n^{-1/2} \log (1+nZ_{0n}) \to 0$ a.s. as $n \to \infty$. Similarly for

$$a_1 \geqslant 0, \; n^{-1/2} \log (1+nZ_{1n}) = o_P(1)$$

Thus the large sample behaviour of $n^{1/2} (Z_{1n}-Z_{0n})$ is of main interest. We now take up two different cases.

Case 1 : $a_0 \neq a_1 \Rightarrow n^{1/2}|Z_{1n}-Z_{0n}| \to \infty$ a.s. as $n \to \infty$, implying the required result.

Case 2*a* : $a_0 = a_1 = 0$.

Here $n^{1/2}(Z_{1n}-Z_{0n})$ converges to a continuous distribution (note that in this case the distribution of $n^{1/2}(Z_{1n}-Z_{0n})$ is free of n for each fixed n) implying the required result.

Case 2*b* : $a_0 = a_1 > 0$ $a_0 = a_1 \Longrightarrow (ES)'(ET) = 0$.

Now $$Z_{1n}-Z_{0n} = (Z_{1n}+Z_{0n})^{-1} \Delta_0^2 3^{-1}(S'T)$$

$$n^{1/2} S'T = n^{1/2} (S-ES)'(T-ET)+n^{1/2} S'ET+n^{1/2} T'ES-n^{1/2}(ES)'(ET).$$

Thus $n^{1/2} S'T$ is asymptotically normal and $Z_{1n}+Z_{0n} \to 2a_0 (>0)$ a.s. as $n \to \infty$.

Thus $n^{1/2} (Z_{1n}-Z_{0n})$ converges to a continuous distribution, and thus the proof of Theorem 2 is complete. □

Remark 2 : Results similar to Theorem 1 and Theorem 2 also hold for the unknown Σ case, but we omit the proof, to keep the paper in a concise form.

5. Asymptotic distributions of stopping times

This section is devoted to the study of the asymptotic distribution of the stopping time N_1 (truncated and untruncated). A general theorem regarding asymptotic distribution of a given class of stopping time is given first ; from which the limiting distribution of N_1 follows.

Theorem 3. *Let* $\{W_n^r\}_{n \geqslant 1}$ *denote a sequence of random variables for* $r \in [0, \infty)$.

Let $\{b_r\}$ *be a real sequence s.t.* $b_r \to \infty$ *as* $r \to \infty$. *Let*

$$\left.\begin{aligned} \tau_r &= \inf\{n : W_n^r \geqslant b_r\} \\ &= \infty \quad \textit{otherwise.} \end{aligned}\right\} \quad \ldots \quad (5.1)$$

Suppose the following conditions hold :

(A1) $\exists\, \mu > 0$ *s.t.* $b_r^{-1}\tau_r \to \mu^{-1}$ *in* P *as* $r \to \infty$.

For any sequence of positive integers $\{m_r\}$ *for which* $b_r^{-1}m_r \to \mu^{-1}$ *as* $r \to \infty$.
(A2) $\exists$ *a distribution function* $F(\,.\,)$ *and a real sequence* $\{\mu_r\}$ *converging to* μ (μ *as given in* (A1)) *as* $r \to \infty$, *such that the following holds for all continuity points* t *of* F,

$$P\{b_r^{-1/2}(W_{m_r}^r - m_r\mu_r) \leqslant t\} \to F(t) \text{ as } r \to \infty. \quad \ldots \quad (5.2)$$

(A3) *For given any* ε *and* $\eta \,\exists\, r_0$ *(large) and* c_0 *(small) such that* $\forall\, r \geqslant r_0$

$$P\left\{ \left| \frac{W_{m_r}^r}{m_r} - \frac{W_{m'}^r}{m'} \right| < \varepsilon\, m_r^{-1/2} \;\forall\, m' : \left| m' - m_r \right| < c_0\, m_r \right\} > 1 - \eta \ldots \quad (5.3)$$

Then (a) $P(\{\tau_r > n_{r,x}\}\Delta\{W_{n_{r,x}}^r < b\}) \to 0$ as $r \to \infty$ *where* $n_{r,x} = [b_r\mu_r^{-1} - b_r^{1/2}\mu^{-1}x]$, *with* x *a continuity point of* F.

$[y]$ *denotes the smallest integer greater than or equal to* y *and* $A\Delta B$ *denotes the symmetric difference of the sets* A *and* B.

(b) *Moreover for all sequences* $\{n_r\}$ *s.t.* $b_r^{-1}n_r \to \mu^{-1}$ *as* $r \to \infty$

$$\mu b_r^{-1/2}(\tau_r - b_r\mu_r^{-1}) = -b_r^{-1/2}(W_{n_r}^r - n_r\mu_r) + o_P(1)$$

and hence the limiting distribution of $-\mu b_r^{1/2}(\tau_r - n_r\mu_r)$ *is* F.

Remark 3 : In applications of Theorem 3, μ_r cannot be replaced by μ in general.

Remark 4 : Observe that if (A2) and (A3) are satisfied for one sequence $\{m_r\}$ s.t. $b_r^{-1}m_r \to \mu^{-1}$ as $r \to \infty$, then (A2) and (A3) are satisfied for all sequences $\{n_r\}$ s.t. $b_r^{-1}n_r \to \mu^{-1}$ as $r \to \infty$, with the same μ and F.

Remark 5 : Let $\tau_r' = \inf\{n : W_n^r + c \geqslant b_r\}$ where W_n^r, b_r are as in Theorem 3 and c is a real constant. Suppose (A1) (with τ_r' in place of τ_r), (A2) and (A3) are satisfied. Then $P(\{\tau_r' > n_{r,x}\}\ \Delta\ \{W_{n_{r,x}}^r < b_r\}) \to 0$ as $r \to \infty$, The proof is along similar lines as the proof of Theorem 3.

We now proceed to the proof of Theorem 3. Let us first state a lemma.

Lemma 3. *Let* $\{U_r, r\,\epsilon[0,\infty)\}$ *and* $\{V_r, r\,\epsilon[0,\infty)\}$ *be two stochastic processes satisfying the following conditions,*

(1) $P\{U_r \leqslant t\} \to G(t)$ *as* $r \to \infty$,

for all continuity point t *of* G, *where* G *is a distribution function.*

(2) *For all continuity point* t *of* G *and for all* $\epsilon > 0$,

$$\lim_{r\to\infty} P\{V_r < t-\epsilon,\, U_r > t\} = 0$$

$$\lim_{r\to\infty} P\{V_r > t,\, U_r < t-\epsilon\} = 0$$

Then $V_r - U_r = o_p(1)$.

The proof of Lemma 3 follows from the proof of Lemma 1 of Ghosh (1971).

Proof of Theorem 3 : *Proof of Part* (a) : For simplicity in notation let us denote $n_{r,x}$ by n_r, in the proof of Part (a).

$$P(\{\tau_r > n_r\}\,\Delta\{W'_{n_r} < b_r\})$$

$$= P\{\tau_r < n_r,\, W'_{n_r} < b_r\} \quad \text{(By the definition of } \tau_r \text{ given in (5.1)).}$$

$$\leqslant P\{\tau_r < n_r,\, W'_{n_r} < b_r,\, |\tau_r b_r^{-1}\mu_r - 1| < \epsilon_1\}$$

$$+P\{|\tau_r b_r^{-1}\mu_r - 1| \geqslant \epsilon_1\} \qquad \ldots \quad (5.4)$$

where $0 < \epsilon_1 < 1$ is to be chosen suitably later.

For any fixed $\epsilon_1 > 0$, the second term of (5.4) goes to zero as $r \to \infty$ (by (A1) and the fact that $\mu_r \to \mu$ as $r \to \infty$).

Fix $\epsilon_2 > 0$. Let n_1 be the smallest integer less than or equal to $(1-\epsilon_1)$ $b_r \mu_r^{-1}$. Thus n_1 is less than n_r for large r.

Now the first term on (5.4)

$$\leqslant P\{n_1 < \tau_r < n_r,\, W'_{n_r} < b_r\}$$

$$\leqslant P\left\{\max_{n_1 < j \leqslant n_r} W'_j > b_r,\, W'_{n_r} < b_r - \epsilon_2 b_r^{1/2}\right\}$$

$$+P\{b_r - \epsilon_2 b_r^{1/2} < W'_{n_r} < b_r\} \qquad \ldots \quad (5.5)$$

The second term of (5.5), can be made as small as we please if $\epsilon_2 > 0$, is chosen sufficiently small and then $r \to \infty$ (by using (A2) and the fact that x is a continuity point of F).

▲ 2–10

The first term on (5.5) $\leqslant P\left\{\max_{n_1<j<n_r} W^r_j - W^r_{n_r} > \epsilon_2 b_r^{1/2}\right\}$

$$\leqslant P\left\{\max_{n_1<j<n_r} j\left(\frac{W^r_j - j\mu_r}{j} - \frac{W^r_{n_r} - n_r\mu_r}{j}\right) > \epsilon_2 b_r^{1/2}\right\} \quad \text{(As } n_r > j \text{ and for large } r,\ \mu_r > 0)$$

$$= P\left\{\max_{n_1<j<n_r} j\left(\frac{W^r_j}{j} - \frac{W^r_{n_r}}{n_r}\right) + (W^r_{n_r} - n_r\mu_r)\left(\frac{j}{n_r} - 1\right) > \epsilon_2 b_r^{1/2}\right\}$$

$$\leqslant P\left\{b_r^{-1/2} \max_{n_1<j<n_r} n_r\left|\frac{W^j_r}{j} - \frac{W^r_{n_r}}{n_r}\right| > \epsilon_2/2\right\}.$$

$$+ P\left\{b_r^{-1/2}\ |\ W^r_{n_r} - n_r\mu_r| \cdot |\max_{n_1<j<n_r}\left(\frac{j}{n_r} - 1\right)| > \epsilon_2/2\right\} \quad \ldots \quad (5.6)$$

The first term n (5.6) goes to zero by (A3) and the fact $b_r^{-1} n_r \to \mu^{-1}$ as $r \to \infty$. The secc 1 term on (5.6) goes to zero by (A2) and the fact that $\max_{n_1<j<n_r}\left(\frac{j}{n_r} - 1\right)$ can be made arbitrarily small by first making ϵ_1 sufficiently small and then making $r \to \infty$. Thus Part (a) is proved.

Proof of Part (*b*) : Observe

$$P\{\tau_r \geqslant n_{r,x}, W^r_{n_{r,x}} \geqslant b\} = 0 \text{ (by definition of } \tau_r)$$
$$\Longrightarrow P\{-\mu b_r^{-1/2}(\tau_r - b_r\,\mu_r^{-1}) < x - \mu b_r^{-1/2},$$
$$b_r^{-1/2}(W^r_{n_{r,x}} - n_{r,x}\mu_r) > x\mu_r\,\mu^{-1}\} = 0 \quad \ldots \quad (5.7)$$

Using Part (a),

$$P\{-\mu b_r^{-1/2}(\tau_r - b_r\mu_r^{-1}) > x,\ b_r^{-1/2}(W^r_{n_{r,x}} - n_{r,x}\,\mu_r) < x\mu_r\mu^{-1} + \mu b_r^{-1/2}\} \to 0$$
$$\text{as } r \to \infty. \quad \ldots \quad (5.8)$$

Now condition (2) of Lemma 3, with $U_r = b_r^{-1/2}(W^r_{n_r} - n_r\,\mu_r)$ and $V_r = -\mu b_r^{-1/2}(\tau_r - b_r\mu_r^{-1})$ can be seen to be satisfied using (5.7) and (5.8), (A2) and the fact $b_r^{-1/2}\,(W^r_{n_r} - n_r\mu_r) - b_r^{-1/2}\,(W^r_{n_{r,x}} - n_{r,x}\,\mu_r) = o_P(1)$ (which follows from (A2), (A3) and Remark 4). Condition (1) of Lemma 3 follows from (A2) and thus the proof of Part (b) follows from Lemma 3. □

To name a few works in the area of asymptotic behaviour of a class of stopping times we have Siegmund (1968), Bhattacharya and Mallik (1973), Ghosh and Mukhopadhyay (1975), Lai and Siegmund (1977, 1979) and Woodroofe (1982). The last two works deal with the study of the second order asymptotic behaviour of a class of stopping times using the method of non-linear renewal theory. In this paper Theorem 3 can be thought of as a version of Theorem 2 of Bhattacharya and Mallik (1973), based on the ideas of Anscombe (1952) suitable for the present context.

We now apply Theorem 3, for obtaining asymptotic distribution of N_k, as $k \to \infty$ where k is the size of the fixed sample available from π_0

and
$$\left.\begin{aligned} N_k &= \inf\{n : |\log W_{n,k}(\Delta_0)| \geqslant b_k\} \\ &= \infty \qquad\qquad \text{otherwise.} \end{aligned}\right\} \quad \ldots \quad (5.9)$$

$$b_k \to \infty \text{ as } k \to \infty \text{ s.t. } k^{-1}b_k \to a_1 > 0. \quad \ldots \quad (5.10)$$

Let
$$\nu = \mu_1 - \mu_2 \quad \ldots \quad (5.11)$$

$$\theta_k = \Delta_0\|\nu\| - 2b_k k^{-1} \quad \ldots \quad (5.12)$$

$$\theta = \Delta_0\|\nu\| - 2a_1 \quad \ldots \quad (5.13)$$

$$\sigma = (2a_1\, \Delta_0^3\|\nu\|)^{1/2}\, (\Delta_0\|\nu\| - 2a_1)^{-2} \text{ for } \Delta_0\|\nu\| - 2a_1 > 0. \quad \ldots \quad (5.14)$$

The distribution of N_k is obtained for the two cases, $\mu = \mu_1$ and $\mu = \mu_2$. As the original problem is an identification problem these two cases are most important.

Theorem 4 : *For $\mu = \mu_1$ or $\mu = \mu_2$, $k^{-1/2}(N_k - b_k\, \theta_k^{-1})$ is asymptotically (as $k \to \infty$) normal with mean zero and variance σ^2 if $\Delta_0\|\nu\| > 2a_1$.*

Proof of Theorem 4 : For $\mu = \mu_1$, it is enough to consider
$$\left.\begin{aligned} N &= \inf\{n : \log W_{n,k}(\Delta_0) \leqslant -b_k\} \\ &= \infty \qquad\qquad \text{otherwise,} \end{aligned}\right\} \quad \ldots \quad (5.15)$$
as $P_{\mu=\mu_1}(N_k = N_k') \to 1$ as $k \to \infty$.

We now approximate N_k' by two other stopping times M_k and L_k which are simpler to handle.

$$\left.\begin{aligned} M_k &= \inf\left\{n : Z_1 - Z_0 + 2^{-1}(p-1)\log\frac{1+Z_0}{1+Z_1} - c \leqslant -b_k\right\} \\ &= \infty \qquad\qquad\qquad\qquad \text{otherwise,} \end{aligned}\right\} \quad \ldots \quad (5.16)$$

$$\left.\begin{aligned} L_k &= \inf\left\{n : Z_1 - Z_0 + 2^{-1}(p-1)\log\frac{1+Z_0}{1+Z_1} + c \leqslant b_k\right\} \\ &= \infty \end{aligned}\right\} \quad \ldots \quad (5.17)$$

where Z_0, Z_1 are as in (2.3).

$M_k \leqslant N_k' \leqslant L_k$ by (5.15), (5.16), (5.17) and the approximation formula 3.3.4 of Wijsman (1979).

Let us first study M_k.

$$M_k = \inf\left\{n : (2k^{-1}n+1)\left(Z_0 - Z_1 + 2^{-1}(p-1)\log\frac{1+Z_1}{1+Z_0}\right) - 2k^{-1}n(b_k - c) \geqslant b_k - c\right\}$$

$$= \infty \qquad \text{otherwise.} \qquad \ldots \quad (5.18)$$

First we shall show (a) and (b) of Theorem 3 are satisfied with $r = k$, $b_r = b_k' = b_k - c$,

$$\left.\begin{aligned} W_n^r &= W_n^k = (2k^{-1}n+1)\left(Z_0 - Z_1 + 2^{-1}(p-1)\log\frac{1+Z_1}{1+Z_0}\right) - 2k^{-1}n\, b_k' \\ \tau_r &= \tau_k = M_k \\ \mu_r &= \mu_k = \theta_k' = \Delta_0\|\nu\| - 2k^{-1} b_k' \\ \mu &= \theta = \Delta_0\|\nu\| - 2a_1 > 0 \text{ by hypotheses.} \end{aligned}\right\} \ldots \quad (5.19)$$

$F(x) = \Phi(x/\Delta_0(\theta^{-1}(4a_1\theta^{-1}+2))^{1/2})$, $\Phi(\cdot)$ denotes the normal c.d.f.

Now (A1) (with $\tau_r = \tau_k = M_k$), (A2) and (A3) with terms defined in (5.19) are satisfied vide Lemma 4, Lemma 5 and Lemma 6, given below.

Thus (a) and (b) of Theorem 3 hold with terms as described in (5.19).

Now from part (b) one gets, as $k \to \infty$,

$$\theta(b_r')^{1/2}(M_k - \theta b_k'(\theta_k')^{-1}) \Longrightarrow N(0, \Delta_0^2\theta^{-1}(4a_1\theta^{-1}+2))$$

$$\Longrightarrow \theta k^{-1/2}(M_k - b_k'(\theta_k')^{-1}) \Longrightarrow N(0, \Delta_0^2 a_1\theta^{-1}(4a_1\theta^{-1}+2))$$

$$\Longrightarrow k^{-1/2}(M_k - b_k\theta_k^{-1}) \Longrightarrow N(0, \Delta_0^2 a_1\theta^{-3}(4a_1\theta^{-1}+2)) \ldots \quad (5.20)$$

as $k^{-1/2}(b_k'(\theta_k')^{-1} - b_k\theta_k^{-1}) \to 0$ as $k \to \infty$.

Similarly one can show

$$k^{-1/2}(L_k - b_k\theta_k^{-1}) \Longrightarrow N(0, \Delta_0^2 a_1\theta^{-3}(4a_1\theta^{-1}+2)) \qquad \ldots \quad (5.21)$$

Observe,
$$\Delta_0^2 a_1\theta^{-3}(4a_1\theta^{-1}+2) = \Delta_0^2 a_1(\Delta_0\|\nu\| - 2a_1)^{-4}(2\Delta_0\|\nu\|) = \sigma^2.$$

Thus the proof for the case $\mu = \mu_1$ follows from (5.20), (5.21) and the fact that $M_k \leqslant N_k' \leqslant L_k$.

The proof for the case $\mu = \mu_2$ follows along similar lines. □

Let us now provide a motivation for the lemmas (mentioned in the above proof) suggested by the referee

Let

$$S = 2\bar{X}_k - \bar{Y}_n - \bar{Z}_n\,;\; T = \bar{Y}_n = \bar{Z}_n \qquad \ldots \quad (5.22)$$

Then
$$Z_0 = \Delta_0\left\|\left(\frac{S-\nu}{\sigma_S} + \frac{T-\nu}{\sigma_T}\right) + \nu\left(\frac{1}{\sigma_S} + \frac{1}{\sigma_T}\right)\right\| \qquad \ldots \quad (5.23)$$

for σ_S, σ_T defined as in Section 2, ν as in (5.11) and Z_0 as in (2.3).

For $\mu = \mu_1$, the first term on the RHS of (5.23) is expected to be smaller compared to the second for large n and k. Thus making first order expansion about $\nu(\sigma_S^{-1}+\sigma_T^{-1})$ and doing the same with Z_1, we get

$$Z_0 - Z_1 = 2\Delta_0 \sigma_S^{-1} \|\nu\|^{-1} S'\nu + R_{n,k} \quad \ldots \quad (5.24)$$

where $2\Delta_0^{-1} R_{n,k} = U'_{n,k} A_{n,k}(\nu'\nu + a_{n,k})^{-1/2} + 2U'_{n,k}((\nu'\nu+a_{n,k})^{-1/2} - \|\nu\|^{-1})$

$$-V'_{n,k} B_{n,k}(\nu'\nu+b_{n,k})^{-1/2} + 2V'_{n,k}((\nu'\nu+b_{n,k})^{-1/2} - \|\nu\|^{-1}) \quad \ldots \quad (5.25)$$

with $U_{n,k} = \dfrac{S-\nu}{\sigma_S} + \dfrac{T-\nu}{\sigma_T}$; $V_{n,k} = \dfrac{S-\nu}{\sigma} - \dfrac{T-\nu}{\sigma_T}$

$$\left.\begin{array}{l} A_{n,k} = \left(\dfrac{1}{\sigma_S} + \dfrac{1}{\sigma_T}\right)^{-1} U_{n,k};\ a_{n,k} = O^{(1)}_{n,k} A'_{n,k}(A_{n,k}+2\nu) \\ B_{\,,} = \left(\dfrac{1}{\sigma_T} - \dfrac{1}{\sigma_S}\right)^{-1} V_{n,k};\ b_{n,k} = O^{(2)}_{n,k}\, B'_{n,k}(B_{n,k} - 2\nu) \\ 0 < O^{(j)}_{n,k} < 1 \text{ for } j = 1,2 \text{ (appears from the first order expansion)} \end{array}\right\} \ldots \quad (5.26)$$

If n_k is a sequence of positive integer s.t. $k^{-1} n_k \to a$ ('a' positive real number) as $k \to \infty$, then it is easy to see from (5.25) and (5.26) that

$$R_{n_k, k} = o_p(n_k^{1/2}) \text{ for } \mu = \mu_1 \quad \ldots \quad (5.27)$$

Also for

$$\mu = \mu_1 \text{ and } \|\nu\| > 0, \log\left(\frac{1+Z_1}{1+Z_0}\right) \to \log\left(\frac{a}{1+a}\right) \text{a.s. } k \to \infty \quad \ldots \quad (5.28)$$

where Z_0, Z_1 as in (2.3), having n_k in place of n.

Similar results as in (5.24–5.28) can also be obtained for $\mu = \mu_2$. These facts will be used in the proof of Lemma 5 and Lemma 6. They also motivate Lemma 4 but the proof of Lemma 4 runs along a different line.

Lemma 4 : $(b'_k)^{-1} M_k \to \theta^{-1}$ *a.s. as* $k \to \infty$.

Proof of Lemma 4 : Proceeding along similar lines as in Theorem 1, one can show that M_k does not terminate with probability one for fixed k. However, $(b'_k)^{-1} M_k$ does admit a limit as $k \to \infty$. To show that choose ϵ_1 and ϵ_2 both positive s.t.

$$2\epsilon_2 + \Delta_0\|\nu\|^{-1}\epsilon_1 < \Delta_0\|\nu\| - 2a_1 \text{ and } \epsilon_2 < a_1 \quad \ldots \quad (5.29)$$

Define $\quad B'_k = \{|S'T - (2\mu - Y_n - Z_n)'\ (Y_n - Z_n)| < \epsilon_1 \forall k \geqslant k'\} \quad \ldots \quad (5.30)$

Then for given any η and ϵ_1 (as in (5.29)), we can choose k_1 large s.t.

$$P(B_{k_1}) \geqslant 1 - \eta \quad \ldots \quad (5.31)$$

Let k_2 be chosen using (5.10) s.t.

$$|k^{-1}b_k'-a_1| < \epsilon_2 \vee k > k_2. \qquad \ldots \quad (5.32)$$

Let $k_0 = k_1 \vee k_2$. Then choosing suitable stopping times to bound M_k one can show $M_k \to \infty$ as $k \to \infty$ on B_{k_0} and

$$P(B_{k_0}, M_k < \infty \vee k > k_0) > 1-\eta \qquad \ldots \quad (5.33)$$

Thus, we now concentrate on B_{k_0}.

On B_{k_0},

$$W^k_{M_k} > b_k' > W^k_{M_k-1}. \qquad \ldots \quad (5.34)$$

Now $\quad M_k^{-1}W^k_{M_k} = (2k^{-1}+M_k^{-1})\left(Z_0-Z_1+2^{-1}(P-1)\log\dfrac{1+Z_1}{1+Z_0}\right)-2k^{-1}b_k'$

where Z_0, Z_1, (defined in (2.3)) both have M_k in place of n.

$(2k^{-1}+M_k^{-1})\,(Z_0-Z_1) = 2^{-1}\,\sigma_S(Z_0-Z_1)$

$$= 2^{-1}\,\Delta_0\left(\left\|S+\frac{\sigma_S}{\sigma_T}\;T\right\|-\left\|S-\frac{\sigma_S}{\sigma_T}T\right\|\right) \quad \ldots \quad (5.35)$$

$$= 2\,\Delta_0 S'T\left(\left\|\frac{\sigma_T}{\sigma_S}\;S+T\right\|+\left\|\frac{\sigma_T}{\sigma_S}\;S-T\right\|\right)^{-1} \ldots \quad (5.36)$$

Expression in (5.36) is more convenient to handle as $\dfrac{\sigma_T}{\sigma_S}$ is bounded above by 1.

Now, $\quad \left\|\dfrac{\sigma_T}{\sigma_S}\;S+T\right\|+\left\|\dfrac{\sigma_T}{\sigma_S}\,S-T\right\|$

$$= \left\|\frac{\sigma_T}{\sigma_S}\,(S-\nu)+(T-\nu)+\nu\left(1+\frac{\sigma_T}{\sigma_S}\right)\right\|$$

$$+\left\|\frac{\sigma_T}{\sigma_S}\,(S-\nu)-(T-\nu)+\nu\left(\frac{\sigma_T}{\sigma_S}-1\right)\right\| \qquad \ldots \quad (5.37)$$

$$(\text{for } \mu = \mu_1,\ ES = ET = \nu)$$

$$\to 2\|\nu\| \text{ a.s. as } k \to \infty. \qquad \ldots \quad (5.38)$$

For (5.38) add and subtract $2\,\|\nu\|$ to (5.37) and then break up $-2\|\nu\|$ $= -\left(\left(1+\dfrac{\sigma_T}{\sigma_S}\right)\|\nu\|-\left(\dfrac{\sigma_T}{\sigma_S}-1\right)\|\nu\|\right)$. This expression of $-2\|\nu\|$ together with (5.37) can be shown to converge to zero a.s. as $k \to \infty$.

Thus $(2k^{-1}+M_k^{-1})\ (Z_0-Z_1)\to 2\Delta_0\ \nu'\nu(2\|\nu\|)^{-1} = \Delta_0\ \|\nu\|$ a.s. as $k\to\infty$. ... (5.39)
from (5.30) and (5.38). Now

$$(2k^{-1}+M_k^{-1})\left|\ \log\left(\frac{1+Z_1}{1+Z_0}\right)\right| \leqslant (2k^{-1}+M_k^{-1})\log\ (1+|\,Z_1-Z_0|)$$

$$=(2k^{-1}+M_k^{-1})\log\left((2k^{-1}+M_k^{-1})\,(1+|\,Z_1-Z_0|)\right)$$
$$-(2k^{-1}+M_k^{-1})\log\,(2k^{-1}+M_k^{-1})\to 0 \text{ a.s.},$$
as $k\to\infty$ (using (5.39)). ... (5.40)

Thus from (5.39), (5.40), (5.10) and (5.19), we have

$$M_k^{-1}\ W^k_{M_k}\to\Delta_0\ \|\nu\|-2a_1 \text{ a.s. as } k\to\infty. \qquad \text{... (5.41)}$$

Similarly

$$M_k^{-1}\ W^k_{M_k-1}\to\Delta_0\ \|\nu\|-2a_1 \text{ on } B_{k_0} \text{ as } k\to\infty. \qquad \text{... (5.42)}$$

Thus from (5.34), (5.41) and (5.42), we have on B_{k_0},

$$(b_k')^{-1}\ M_k\to(\Delta_0-\|\nu\|-2a_1)^{-1} = \theta^{-1} \text{ as } k\to\infty \qquad \text{... (5.43)}$$

Thus $P\left(\lim_{k\to\infty}(b_k')^{-1}\ M_k=\theta^{-1}\right)\geqslant P\left(\lim_{k\to\infty}(b_k')^{-1}\ M_k=\theta^{-1},\ B_{k_0}\right)\geqslant 1-\eta$ and the fact that η is arbitrary implies Lemma 4.

Lemma 5 : *Let $\{m_k\}$ be any sequence of positive integers s.t. $(b_k')^{-1/2}\ m_k\to\theta^{-1}$ as $k\to\infty$. Then $(b_k')^{-1/2}\left(W^k_{m_k}-m_k\theta_k'\right)$ is asymptotically normal with mean* 0 *and variance $\Delta_0^2\,\theta^{-1}\ (4a_1\ \theta^{-1}+2)$.*

Proof of Lemma 5 :

$$(b_k')^{-1/2}\left(W^k_{m_k}-m_k\theta_k'\right)$$

$$=(b_k')^{-1/2}\left(m_k\frac{\Delta_0\,(S-\nu)'\nu}{\|\nu\|}+(2k^{-1}m_k+1)R_{m_k,k}+(2k^{-1}m_k+1)\left(\frac{p-1}{2}\right)\log\left(\frac{1+Z_1}{1+Z_0}\right)\right)$$

(by 5.19 and 5.24)

$$=(b_k')^{-1/2}m_k\ \frac{\Delta_0(S-\nu)'\nu}{\|\nu\|}+o_P(1) \text{ (by (5.27), (5.28), (5.10) and the choice of } m_k)$$

$$\Longrightarrow N(0,\ \Delta_0^2\theta^{-1}(4a_1\ \theta^{-1}+2))\ \square$$

Lemma 6 : *For given ε and $\eta\ \exists\ k_0$ (large) and c_0 (small) s.t.*

$$\forall k\geqslant k_0, P\left\{\left|\frac{W^k_{m_k}}{m_k}-\frac{W^k_{m'}}{m'}\right|<\varepsilon\ m_k^{-1/2}\ \forall\ m':|\,m'-m_k|<c_0m_k\right\}\geqslant 1-\eta \ \text{...} \quad (5.44)$$

where $\{m_k\}$ *denotes a sequence of positive integers s.t.* $b_k^{-1}m_k \to \theta^{-1}$ *as* $k \to \infty$.

Proof of Lemma 6 : Note, $\dfrac{W^k_{m_k}}{m_k} = \Delta_0 \,\|\nu\|^{-1} (2\bar{X}_k - \bar{Y}_{m_k} - \bar{Z}_{m_k})'\nu$

$$+(2k^{-1}+m_k^{-1})R_{m_k,k}+(k^{-1}+2^{-1}m_k^{-1})(p-1)\log\left(\frac{1+Z_{1,m_k}}{1+Z_{0,m_k}}\right) \quad \ldots \quad (5.45)$$

$$\text{where } Z_{i,m} = \left\| \frac{2\bar{X}_k - \bar{Y}_m - \bar{Z}_m}{4k^{-1}+2m^{-1}} + (-1)^i \; \frac{\bar{Y}_m - \bar{Z}_m}{2m^{-1}} \right\| \text{ for } i = 0, 1 \quad \ldots \quad (5.46)$$

For proving Lemma 6 it is enough to check (5.44) with $t_{j,m}$ in place of $m^{-1}W^k_m$ for each $j = 1, 2, 3$ where $t_{j,m}$ denotes the j-th term on the RHS of (5.45).

Now (5.44) with $t_{1,m}$ (in place of $m^{-1}W^k_m$) follows immediately from Theorem 3 of Anscombe (1952).

(5.44) with $t_{2,m}$ (in place of $m^{-1}W^k_m$) follows from Lemma 7 (given below), (5.26) and Theorem 3 of Anscombe (1952). For $t_{3,m}$ once again Lemma 7 implies the required condition.

Thus the proof of Lemma 6 follows. □

Lemma 7 : *Let* $\{m_k\}$ *be a sequence of integer s.t.* $m_k \to \infty$ *as* $k \to \infty$. *Let* $\{X_{m_k}\}_{k\geq 1}$, $\{Y_{m_k}\}_{k\geq 1}$ *be two sequences of random variable such that the following conditions hold :*

(1) *For all* $\delta > 0$, λ *(depending on* δ*) s.t.* $P(|m_k^{1/2}X_{m_k}| > \lambda) < \delta \;\forall\, k$.

(2) *For given any* ϵ *and* η *(both positive real numbers)* $\exists\, k_0$ *(large) and* c_0 *(small) s.t.* $\forall\, k \geqslant k_0$.

$$P\left\{ |X_{m_k} - X'_m| < \epsilon\, m_k^{-1/2} \;\forall\, m' : |m'-m_k| < c_0 m_k \right\} > 1-\eta$$

(3) $Y_{m_k} \to$ *constant a.s. as* $k \to \infty$.

Then for given any ϵ and η (both positive real numbers) $\exists k_0$ (large) and c_0 (small) s.t. $\forall\, k \geqslant k_0$

$$P\left\{ |X_{m_k} Y_{m_k} - X'_m Y'_m| < \epsilon\, m_k^{-1/2} \;\forall\, m' : |m'-m_k| \leqslant c_0\, m_k \right\} > 1-\eta.$$

Proof of Lemma 7 :

$$|X_{m_k} Y_{m_k} - X_{m'} Y_{m'}| = |X_{m_k} Y_{m_k} - X_{m_k} Y_{m'} + X_{m_k} Y_{m'} - X_{m'} Y_{m'}|$$

$$\leqslant |X_{m_k}|\,|Y_{m_k} - Y_{m'}| + |Y_{m'}|\,|X_{m_k} - X_{m'}| \quad \ldots \quad (5.47)$$

Now for given ε and η, $\exists$ k_0 and c_0 s.t. $\forall$ $k \geqslant k_0$,

$$P\left\{|X_{m_k}||Y_{m_k}-Y_{m'}|<\frac{\varepsilon}{2}\, m_k^{-1/2}\ \forall\ m' : |m'-m_k|<c_0\, m_k\right\}>1-\eta/2 \ldots \quad (5.48)$$

(by (1) and (2)), and

$$P\left\{|Y_{m'}||X_{m_k}-X_{m'}|<2^{-1}\,\varepsilon m_k^{-1/2}\ \forall\ m' : |m'-m_k|<c_0 m_k\right\}>1-\eta/2 \ldots \quad (5.49)$$

(by (2) and (3)). The proof now follows from (5.47), (5.48) and (5.49).

Theorem 5 : *Let N_k be a stopping time such that $k^{-1/2}(N_k-\nu_k)$ converges in distribution to F(F a distribution function). Let m_{0k} denote a real sequence s.t. $k^{-1}m_{0k}\to a$ ($a>o$) and $k^{-1}N_k\to b(b>0)$ a.s. as $k\to\infty$. Then for $b<a$, $N_k\wedge m_{0k}$ has the same limiting distribution F while for $b>a$, $N_k\wedge m_{0k}$ is asymptotically degenerate at m_{0k}.*

Proof of Theorem 5 : *Case* 1 : $a>b$: In this case, we shall show $N_k\wedge m_{0k}-N_k=o_p(1)$.

For that, $P(N_k-N_k\wedge m_{0k}>0)$

$$= P(N_k\wedge m_{0k}<N_k)$$
$$= P(m_{0k}<N_k)$$
$$= P(k^{-1/2}(N_k-\nu_k)>k^{-1/2}(m_{0k}-\nu_k))$$
$$\to 0 \text{ as } k^{-1/2}(m_{0k}-\nu_k)\to\infty \text{ (for } a>b)$$

and $\quad k^{-1/2}(N_k-\nu_k)\Longrightarrow F$ as $k\to\infty$.

Thus $\quad k^{-1/2}(N_k\wedge m_{0k}-\nu_k)\Longrightarrow F$ as $k\to\infty$.

Case 2 : $a<b$: In this case we shall show $N_k\wedge m_{0k}=o_p(1)$.

For that, $\quad P(m_{0k}-N_k\wedge m_{0k}>0)$

$$= p(N_k\wedge m_{0k}<m_{0k})$$
$$= P(N_k<m_{0k})$$
$$= P(k^{-1/2}(N_k-\nu_k)<k^{-1/2}(m_{0k}-\nu_k))$$
$$\to 0 \text{ as } k^{-1/2}(m_{0k}-\nu_k)\to\infty \text{ (for } a<b)$$

and $k^{-1/2}(N_k-\nu_k)\Longrightarrow F$ as $k\to\infty$.

Thus $N_k\wedge m_{0k}$ is asymptotically degenerate at m_{0k}.

Remark 6 : Theorem 5 gives us the asymptotic behaviour of N_1 when truncated. The case $a = b(a, b$ are as in Theorem 5) remains open. Theorem 4 gives the asymptotic distribution of N_1 (untruncated) for the case $\alpha = \beta$. For $\alpha \neq \beta$ one can obtain a similar result.

Remark 7 : For the univariate case, one can obtain a similar asymptotic distribution of the stopping times of the invariant SPRT's proposed by Ghosh and Mukhopadhyay (1980) (for the one-sided case) and Ghosh and Ray Chaudhuri (1984) (for the two-sided case).

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References

Anderson, T. W. (1972): *An Introduction to Multivariate Statistical Analysis*, Wiley Eastern Private Limited, New Delhi.

Anderson, T. W. and Girshick, M. A. (1944): Some extensions of Wishart distribution. *Ann. Math. Statist.*, 15, 345-357.

Anscombe, F. J. (1952): Large sample theory of sequential estimation. *Proc. Camb. Phil. Soc.*, 48, 600-607

Bhattacharya, P. K. and Mallik, A. (1973): Asymptotic normality of the stopping time of sequential procedures. *Ann. Statist.*, 1, 1203-1211.

Dasgupta, Somesh (1974): Probability inequalities and errors in classification. *Ann. Statist.*, 2, No. 4, 751-762.

Ghosh, B. K. (1970): *Sequential Tests of Statistical Hypotheses*, Addison Wesley.

Ghosh, J. K. (1971): A new proof of the Bahadur representation of quantiles and an application. *Ann. Math. Statist.*, 42, No. 6, 1957-1961.

Ghosh, J. K. and Mukhopadhyay, N. (1980): Sequential procedures in identification. *Commun. In Statist.*, Series A, 9, No. 15, 1625-1639.

Ghosh, J. K. and Ray Chaudhuri, A. (1984): An invariant SPRT for identification. *Commun. In Statist. Series C, (Sequential Analysis)*, 3(2), 99-120.

Ghosh, M. and Mukhopadhyay, N. (1975): Asymptotic normality of stopping times in sequential analysis. (*Unpublished*)

Hall, W. J., Wijsman, R. A. and Ghosh, J. K. (1965): The relationship between sufficiency and invariance with application in sequential analysis. *Ann. Math. Statist.*, 36, 575-614.

Lai, T. L. and Siegmund, D. (1977): A non-linear renewal theory with applications to sequential analysis. I. *Ann. Statist.*, 5, 946-954.

Lai, T. L. and Siegmund, D. (1979): A non-linear renewal theory with applications to sequential analysis, II. *Ann. Statist.*, 7, 60-76.

SCHAAFSMA, W. and VAN VARK, G. N. (1977): Classification and discrimination problems with applications, part I, *Statistica Neerlandica*, **31**, no. 1, 25-45.

SITGREAVES, R. (1952): On the distribution of two random matrices used in classification procedures. *Ann. Math. Statist.*, **23**, 263-270.

WALD, A. (1947): *Sequential Analysis*, John Wiley and Sons, Inc. New York.

WHITTAKER, E. T. and WATSON, G. N. (1958): *A Course On Modern Analysis*, Cambridge University Press.

WIJSMAN, R. A. (1967): Cross-sections of orbits and their application to densities of maximal invariants. *Proc. Fifth Berkeley Symp. on Math. Statist. and Prob.*, **1**, 389-400. University of California Press, Berkeley and Los Angeles.

——— (1979): Stopping time of invariant sequential probability ratio tests. *Developments In Statistics*. (Edited by P. R. Krishnaiah) 235-311.

WOODROOFE, M. (1982): *Non-linear Renewal Theory in Sequential Analysis*. SIAM, Philadelphia.

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