Some Contributions to Semidefinite Linear Complementarity Problem

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Dedicated to

Prof. T. Parthasarathy

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Contents

Chapter 0

Plan of the thesis

This thesis is composed of chapters 1 to 5. **In Chapter 1**, we formally define SDLCP and give some examples of SDLCP and show that SDLCP is a special case of variational inequality problem. Also we show that LCP is indeed a special case of SDLCP. Later section of this chapter deals with definitions and notations.

In Chapter 2, we are concerned with the results which have been obtained in an effort to generalize the **P**-matrix condition of LCP to SDLCP. It is known that a matrix M is a **P**-matrix if and only if M does not reverse the sign of any nonzero vector. This notion of the **P**-matrix property has been extended to the SDLCP setup through the **P**-property or the $\mathbf{P_2}$-property. If M is a **P**-matrix, then LCP(M, q) has a unique solution for every $q \in R^n$. But, if L has the **P**-property, it need not imply that, L has the **GUS**-property.

It is known that the **P**-property may not imply the **GUS**-property although it is true in the LCP situation. It has been shown that the $\mathbf{P_2}$-property implies the **GUS**-property. We will also give an example to show the **GUS**-property need not imply the $\mathbf{P_2}$-property. Also from this example, it becomes clear that the $\mathbf{P_2}$-property is stronger than the **P**-property.

If L has the strong monotonicity property, then it has the **GUS**-property and the $\mathbf{P_2}$-property also implies the **GUS**-property. In view of this we would like to know whether there is any relationship between the strong monotonicity property and the $\mathbf{P_2}$-property. Could we say the strong monotonicity property $\Rightarrow$ the $\mathbf{P_2}$-property? We answer this question affirmatively. If a linear transformation $L : S^n \to S^n$ has the strong monotonicity property, then it has the $\mathbf{P_2}$-property.

We provide an example to show that the $\mathbf{P_2}$-property need not imply the strong monotonicity property in general. Finally, in this chapter we derive a set of necessary and sufficient conditions for the linear map M_A to have the strong monotonicity property when $A \in R^{2\times 2}$.

In Chapter 3 we derive results specializing to the Lyapunov, Stein and double-sided multiplication transformation. We prove the equivalence of the strong monotonicity property and the $\mathbf{P_2}$-property for L_A when A is positive definite. When A is normal, the equivalence of the **P**-property and the strong monotonicity property was already established for the transformations L_A and S_A. This result is false for M_A even if we assume A to be normal.

However, we show the equivalence of the **P**-property and the strong monotonicity property for M_A by assuming A to be symmetric matrix. This is the best result one can hope to get on the equivalence of the **P**-, $\mathbf{P_2}$- and the strong monotonicity property for M_A.

We also show that if A is symmetric, L_A or L_{-A} has the **GUS**-property if and only if M_A has the **GUS**-property. In this chapter, we also give a set of sufficient conditions for M_A to have the strong monotonicity property.

It is known that for Lyapunov and Stein transformations, the **P**-property = the **Q**-property. We consider this problem for double-sided multiplication transformation, M_A, **in Chapter 4**. We prove an interesting negative result of independent interest, which identifies a class of the map M_A, that do not enjoy the **Q**-property. Using this, we could prove that the **Q**-property =

the **P**-property, for M_A when $A = A^t$. Also we could show this equivalence when $A \in R^{2\times 2}$. In general, it is not known whether the **Q**-property will imply the **P**-property for M_A.

In Chapter 5 we indicate some open problems relating to SDLCP.

Chapter 1

Introduction

1.1 Variational Inequality Problem

Let f be a continuous function from real n-dimensional space R^n into itself and K be a closed convex set of R^n. The *variational inequality problem* is to find a vector $x^* \in R^n$ such that

$$x^* \in K \quad \text{and} \quad \langle f(x^*), x - x^* \rangle \geq 0 \; \forall \; x \in K. \tag{1.1}$$

We denote the above problem as VI(f, K). It finds extensive application in optimization, economics, traffic equilibrium points, etc. Refer Harker and Pang [14]. This problem has also been studied well in infinite dimensional setting with applications in differential equations, mechanics, etc. Refer Kinderlehrer and Stampacchia [18]. When K is closed convex cone, (1.1) becomes the problem of finding an $x^* \in K$ such that

$$f(x^*) \in K^* \text{ and } \langle x^*, f(x^*)\rangle = 0 \qquad (1.2)$$

where K^* is the dual cone of K and defined as $\{y \in R^n : \langle x, y\rangle \geq 0, \ \forall\, x \in K\}$. This problem is known as *cone complementarity problem.* This has applications in optimization, bi-matrix games, mechanics, economics, etc. Refer Isac [16], Ferris and Pang [6], Ferris, Mangasarian and Pang [5], Murty [22], Pang [24] and Song [30] for more details.

One can see that if $K = R^n_+$ (= nonnegative orthant) and if f is affine, then the problem becomes a *linear complementarity problem.* See Cottle, Pang and Stone [4].

1.2 Semidefinite Linear Complementarity Problem

Let S^n be the space of all symmetric and real $n \times n$ matrices and S^n_+ be the space of symmetric and real $n \times n$ positive semidefinite matrices. Given a linear transformation $L : S^n \to S^n$ and $Q \in S^n$, the *semidefinite linear complementarity problem,* SDLCP(L, Q), is the problem of finding a matrix

$X \in S^n$ such that

$$X \in S^n_+, \;\; Y = L(X) + Q \in S^n_+, \text{ and } \langle X, Y\rangle := tr(XY) = 0 \qquad (1.3)$$

where 'tr' denotes the trace of a matrix. Gowda and Song [9] introduced SDLCP in the above form.

Let $K = S^n_+$ and let $f(X) := L(X)+Q$ be a linear function from $S^n \to S^n$, where, L is also a linear function from $S^n \to S^n$ and $Q \in S^n$. Then, VI(f, K) defined in (1.1), becomes SDLCP(L, Q).

This problem was originally introduced by Kojima, Shindoh and Hara [19] in a slightly different form. The Semidefinite Linear Complementarity Problem (SDLCP) can be considered as a generalization of the linear complementarity problem (LCP), Cottle, Pang and Stone [4].

1.3 Examples of SDLCP

To show the unifying nature of SDCLP, we provide two examples. They are:

1. The standard linear complementarity problem and

2. Geometric SDLCP

1.3.1 The standard linear complementarity problem

Given $M \in R^{n \times n}$ and $q \in R^n$, the linear complementarity problem, LCP(M, q), is to find a vector $x \in R^n$ such that

$$x \in R^n_+, \quad y := Mx + q \in R^n_+ \text{ and } \langle x, y \rangle = 0 \tag{1.4}$$

where $\langle x, y \rangle$ is the usual inner product in R^n.

See Cottle, Pang and Stone [4], Ferris, Mangasarian and Pang [5], Murty [22] and Ferris and Pang [6] for more detailed discussion. Standard linear complementarity problem can be shown as a special case of semidefinite linear complementarity problem as follows (see Song [30]):

Given a standard linear complementarity problem LCP(M, q) as given in (1.4), consider the semidefinite linear complementarity problem, SDLCP$(\mathcal{M}$, Diag$(q))$ where the linear transformation $\mathcal{M} : S^n \to S^n$ is defined by

$$\mathcal{M}(X) := \text{Diag}(M\text{diag}(X))$$

where diag(X) is a vector whose entries are diagonal entries of the matrix X and Diag(u) is a diagonal matrix whose diagonal is the vector u.

The above SDLCP$(\mathcal{M}$, Diag$(q))$ is the problem of finding $X \in S^n$ such that

$$X \in S^n_+, \quad Y := \mathcal{M}(X) + \mathrm{Diag}(q) \in S^n_+, \quad \langle X, Y\rangle = 0.$$

If X solves SDLCP($\mathcal{M}$, Diag(q)), then diag(X) solves LCP(M, q). Conversely, if x is a solution of LCP(M, q), then Diag(x) solves SDLCP($\mathcal{M}$, Diag(q)) and hence these two complementarity problems are equivalent. The following example illustrates the above transformation.

Example 1.3.1 *Let $M \in R^{3\times 3}$ and $q \in R^3$ be given. The linear transformation $\mathcal{M}$ is as follows:*

$$\mathcal{M}(X) = \begin{pmatrix} \sum_{i=1}^3 m_{1i}x_{ii} & 0 & 0 \\ 0 & \sum_{i=1}^3 m_{2i}x_{ii} & 0 \\ 0 & 0 & \sum_{i=1}^3 m_{3i}x_{ii} \end{pmatrix}.$$

Remark 1.3.1 *LCP results do not carry over to SDLCP because of two major reasons: (1) The cone S^n_+ is not polyhedral (where as in LCP, the cone R^n_+ (= nonnegative orthant) is polyhedral) and (2) the matrix product is not commutative.*

1.3.2 Geometric SDLCP

Kojima, Shindoh and Hara [19] introduced the notion of semidefinite linear complementarity problem as a model unifying semidefinite linear programs and problems arising from system and control theory and combinatorial optimization. For more details, see Overton and Wolkowicz [23], Boyd, Ghaoui, Feron and Balakrishnan [3], Wolkowicz, Saigal and Vandenberghe [33] and Vandenberghe and Boyd [32].

The *geometric* SDLCP($\mathcal{F}$) is to find a pair

$$(X, Y) \in \mathcal{F} \cap (S^n_+ \times S^n_+) \ \text{ such that } \ \langle X, Y \rangle = 0$$

where $\mathcal{F}$ be an affine subspace of $S^n \times S^n$ of dimension $\frac{n(n+1)}{2}$. One can easily see that SDLCP(L, Q) is a geometric SDLCP($\mathcal{F}$). (Since SDLCP(L, Q) is given, we can define $\mathcal{F}$ as $\{(X, Y) : Y = L(X) + Q\}$. It is clear that $\mathcal{F}$ is an affine space). To see the converse, we borrow the ideas from Song [30].

Consider the geometric SDLCP($\mathcal{F}$), where $\mathcal{F}$ be an affine subspace of $S^n \times S^n$ of dimension $\frac{n(n+1)}{2}$. $\mathcal{F}$ can be written as:

$$\{(X, Y) \in S^n \times S^n : L_1(X) + L_2(X) = B\}$$

where L_1 and L_2 are linear transformations from S^n to itself and $B \in S^n$. Define $L : S^{3n} \to S^{3n}$ and $Q \in S^{3n}$ by

$$L\left(\begin{bmatrix} X & * & * \\ * & Y & * \\ * & * & Z \end{bmatrix}\right) = \begin{bmatrix} Y & 0 & 0 \\ 0 & L_1(X) + L_2(Y) & 0 \\ 0 & 0 & -L_1(X) - L_2(Y) \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -B & 0 \\ 0 & 0 & B \end{bmatrix}$$

We can verify that if $W = \begin{bmatrix} X & * & * \\ * & Y & * \\ * & * & Z \end{bmatrix}$ solves SDLCP(L, Q), then (X, Y) solves the geometric SDLCP$(\mathcal{F})$. Conversely, if (X, Y) is a solution to geometric SDLCP$(\mathcal{F})$, then $W = \begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & 0 \end{bmatrix}$ solves SDLCP(L, Q). Hence, by solving one problem we can solve the other.

1.4 Preliminaries

1.4.1 Notations

In this section, we summarize the notations that we use repeatedly throughout this thesis. There are some notations, which we have used in this thesis but not mentioned here. They are described at the relevant places.

Spaces: We denote a real n-dimensional Euclidean space by R^n. One-dimensional space, the real line, is denoted by R. The space of all $n \times n$ real matrices is denoted by $R^{n\times n}$. R^n_+ denotes the nonnegative orthant of R^n. The set of all real $n \times n$ symmetric matrices is denoted by S^n and the space of real and symmetric $n \times n$ positive semidefinite matrices is denoted by S^n_+

Vectors: Vectors are usually denoted by lowercase letter such as x in this thesis. A vector $x \in R^n$ is considered to be a column vector. The notation x^t means the transpose of x. The inequality $x \geq 0$ means every component of x is nonnegative and $x \leq 0$ means $-x \geq 0$. The inner product of vectors, x, y in R^n is either denoted by $x^t y$ or $\langle x, y \rangle$.

Matrix Theory: A matrix A with a_{ij} as elements is denoted by $A = (a_{ij})$. Identity matrix is denoted by I. Determinant of a matrix A is represented by $|A|$. When all the entries of a matrix A are nonnegative (positive) we represent it by $A \geq 0$ ($A > 0$). If a square matrix X is symmetric and positive semidefinite (positive definite) we represent it by $X \succeq 0$ ($X \succ 0$). The notation $X \preceq 0$ means the matrix $-X \succeq 0$. Trace of a matrix X is denoted by $\mathrm{tr}(X)$. Transpose of a matrix A is denoted by A^t. The *Kronecker product* of two matrices, $A = (a_{ij})$ of size $m \times n$ and $B = (b_{ij})$ of size $p \times q$ is denoted by the symbol $A \otimes B$ and is defined to be the $mp \times nq$ matrix,

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

Functions: A mapping f with domain D and range R is denoted by $f : D \rightarrow R$. Let $A \in R^{n \times n}$. In this thesis, the following three linear transformations from $S^n \to S^n$ relating to SDLCP are considered:

1. Lyapunov transformation L_A, is defined by $L_A(X) = AX + XA^t$.

2. Stein transformation S_A, is defined by $S_A(X) = X - AXA^t$.

3. Double-sided multiplication transformation M_A, is defined by $M_A(X) = AXA^t$.

The motivations for studying these transformations are given in Chapter 3.

1.4.2 Matrix Theory

We recall the following definitions.

Definition 1.4.1 *Let the matrix A belong to $R^{n\times n}$.*

1. *The trace of A is the sum of all diagonal elements of A.*

2. *A is positive semidefinite (definite) if the quadratic form $x^tAx \geq 0$ (> 0) for all $x \in R^n$ (nonzero $x \in R^n$). A is negative semidefinite (definite) if $-A$ is positive semidefinite (definite).*

3. *A is said to be copositive if $x^tAx \geq 0$ for all $x \in R^n_+$.*

4. *A is positive stable if every eigenvalue of A has positive real part.*

5. *A is orthogonal if* $AA^t = I = A^tA$, *where* I *is the* $n \times n$ *identity matrix.*

6. *A is normal if* $AA^t = A^tA$.

7. *A is said to be a* **P**-*matrix if all its principal minors are positive.*

8. *A diagonal matrix* S *is said to be a signature matrix if its diagonal elements are either -1 or +1.*

Let $A, B \in R^{n \times n}$. Then, $\text{tr}(AB) = \text{tr}(BA)$. The notation $\langle A, B \rangle$ is used to denote the trace of (AB).

We list below some well known matrix theoretic properties. For more details, see Cottle, Pang and Stone [4].

Definition 1.4.2 *A matrix* $A \in R^{n \times n}$ *is said to reverse the sign of a vector* $z \in R^n$ *if* $z_i(Az)_i \leq 0$ *for all* $i = 1, \ldots, n$.

Theorem 1.4.1 *Let* $A \in R^{n \times n}$. *Then, the following statements are equivalent:*

(i) *A is a* **P**-*matrix.*

(ii) *A reverses the sign of no nonzero vector that is* $z_i(Az)_i \leq 0$ *for all* $i = 1, \ldots, n \Rightarrow z = 0$.

(iii) *All real eigenvalues of A and principal submatrices of A are positive.*

For a detailed discussion on **P**-matrix and related topics see Parthasarathy [25].

Theorem 1.4.2 *Let $A \in S^n$. Then, A is positive semidefinite if and only if there exists an orthogonal matrix U such that*

$$A = U^t Diag(\lambda_1, \lambda_2, \ldots, \lambda_n) U$$

where λ_i are nonnegative.

Theorem 1.4.3 *Let $A \in S^n_+$. Then, there exists B such that $B^2 = A$.*

For a proof of Theorem 1.4.2 and Theorem 1.4.3 see Zhang [34].

Since A is real and positive semidefinite, by Theorem 1.4.2, A can be written as $U^t \text{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) U$. Now take $B = U^t \text{Diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}) U$. B is called as *square root* of A and is denoted by $\sqrt{A}$.

In the following theorem, we give some of the important properties of positive semidefinite matrices.

Theorem 1.4.4 *Let the real square matrices X, Y be symmetric. Then, the following results hold:*

(i) $X \succeq 0 \Rightarrow PXP^t \succeq 0$ *for any nonsingular matrix* P.

(ii) $X \succeq 0, Y \succeq 0 \Rightarrow tr(XY) \geq 0$.

(iii) *When* $X \succeq 0, Y \succeq 0$ *then* $tr(XY) = 0$ *if and only if* $XY = 0 = YX$.

(iv) $X \in S^n$ *and* $tr(XY) \geq 0$ *for all* $Y \succeq 0 \Rightarrow X \succeq 0$.

(v) *Given* $X, Y \in R^n$ *with* $XY = YX$, *there exists an orthogonal matrix* U, *diagonal matrices* D *and* E *such that* $X = UDU^t$ *and* $Y = UEU^t$.

(vi) *A matrix* $A \in R^{n\times n}$ *is positive definite if and only if the diagonal of* U^tAU *is nonnegative for every orthogonal* U.

Proof: For proofs of (i), (ii), (iii) and (v) see Song [30]. For completeness sake, we present proof of (iii) here. If $XY = YX = 0$, then $\text{tr}(XY) = 0$. Now we prove that if $\text{tr}(XY) = 0$, then $XY = 0$. Since X and Y are positive definite matrices, we can write by appealing to Theorem 1.4.3 that, $\text{tr}(XY)$ $= \text{tr}(\sqrt{X}\sqrt{X}\sqrt{Y}\sqrt{Y}) = \text{tr}(\sqrt{X}\sqrt{Y}\sqrt{Y}\sqrt{X}) = \text{tr}(AA^t)$, where $A = \sqrt{X}\sqrt{Y}$. Since $X, Y \succeq 0$, $\text{tr}(XY) = \text{tr}(AA^t) = 0$ if and only if $A = 0$. This shows that $\sqrt{X}\sqrt{Y} = 0$ which in turn implies that, $XY = 0$. Proofs of (iv) and (vi) are elementary and are omitted. ■

Remark 1.4.1 *Because of (iii) of Theorem 1.4.4, the SDLCP(L, Q) defined in (1.3) can be restated as:*

$$X \in S_+^n, \quad Y = L(X) + Q \in S_+^n, \quad \textit{and } XY = 0 \tag{1.5}$$

Theorem 1.4.5 *Let $X \succeq 0$ and Y be positive semidefinite (not necessarily symmetric). Then, tr$(XY) \geq 0$.*

Proof: Since X is symmetric and positive semidefinite, it has a square root, $\sqrt{X}$, see Theorem 1.4.3, which is positive semidefinite. So, tr(XY) = tr$(\sqrt{X}Y\sqrt{X}) \geq 0$ because, Y is positive semidefinite. ∎

Theorem 1.4.6 *(Theorem 3, Gowda, Song and Ravindran [9]) Suppose $A \in R^{n\times n}$ is normal. Then, A is positive stable if and only if A is positive definite.*

1.4.3 Basic definitions of linear transformation L

Definition 1.4.3 *Some important definitions (excepting for the* **P**$_3$*-property) pertaining to the linear transformation $L : S^n \to S^n$ are restated below from Gowda and Song [9] and Gowda and Parthasarathy [8]. Definition 1.4.3 say that,*

1. *L has the* **Q**-*property if SDLCP*(L, Q) *has at least one solution for every* $Q \in S^n$.

2. *L has the* **P**-*property if* $XL(X) \preceq 0 \Rightarrow X = 0$.

3. *L has the* $\mathbf{P_1}$-*property if for any* X, *the matrix* $XL(X)$ *is negative semidefinite* $\Rightarrow X = 0$.

4. *L has the* **GUS**-*property* (**Globally Uniquely Solvable**-*property) if for every* $Q \in S^n$, *SDLCP*(L, Q) *has a unique solution.*

5. *L has the* $\mathbf{P_2}$-*property if* $X \succeq 0$ *and* $Y \succeq 0$ *and the matrix* $(X - Y)L(X - Y)(X + Y) \preceq 0 \Rightarrow X = Y$.

6. *L has the* $\mathbf{P_3}$-*property (the* $\mathbf{P_2}$- *in the wider sense) if* $X \succeq 0, Y \succeq 0$ *and the matrix* $(X-Y)L(X-Y)(X+Y)$ *is negative semidefinite* $\Rightarrow X = Y$.

7. *L has the* $\mathbf{R_0}$-*property if the zero matrix is the only solution of SDLCP*$(L, 0)$.

8. *L has the* **monotonicity property** *if* $\langle L(X), X \rangle \geq 0$ *for all nonzero* $X \in S^n$.

9. *L has the* **strict monotonicity property** *if tr*$(XL(X)) > 0$ *for all nonzero* $X \in S^n$.

10. *L has the* **strict semimonotone property** *if* $X \succeq 0$ *and* X *and* $L(X)$ *commute,* $XL(X) \preceq 0 \Rightarrow X = 0$.

11. *L has the* **strong monotonicity property** *if there is an* $\alpha > 0$ *such that* $tr(XL(X)) \geq \alpha tr(X^2)$ *for all* $0 \neq X \in S^n$.

12. *L has the* **cross commutative property** *if for every* $Q \in S^n$ *and solutions* X_1 *and* X_2 *of* $SDLCP(L, Q)$*, the following holds:*

$$X_1 Y_2 = Y_2 X_1 \quad \text{and} \quad X_2 Y_1 = Y_1 X_2$$

where $Y_i = L(X_i) + Q, \quad i = 1, 2.$

Remark 1.4.2 *The statements below follow from the above definitions.*

1. *The* $\mathbf{P_1}$*-property implies the* $\mathbf{P}$*-property.*

2. *In general, the* $\mathbf{P}$*-property does not imply the* $\mathbf{P_1}$*-property, see Example 1.4.1 given below.*

3. *The* $\mathbf{P_3}$*-property implies the* $\mathbf{P_2}$*-property.*

4. *The converse of the above result is not true. That is, the* $\mathbf{P_2}$*-property need not imply the* $\mathbf{P_3}$*-property as shown in the following Example 1.4.2.*

5. *The* $\mathbf{P_2}$*-property implies the* **GUS***-property, see Remark 6, Gowda and Song [9].*

6. *For a linear transformation* L*, strong monotonicity and strict monotonicity properties are equivalent, see page 21 of Song [30].*

7. *Song [30] (Theorem 21) shows that the strong monotonicity property implies the* **GUS***-property. We give an example to show the converse is not true in general.*

Example 1.4.1 *Define* $L : S^2 \to S^2$ *where* $L(X) := \begin{pmatrix} x-y & x \\ x & z+y \end{pmatrix}$ *with* $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$*. We will show that, the map defined as above, has the* $\mathbf{P}$*-property but does not have the* $\mathbf{P_1}$*-property. Define* $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$*. Then,* $L(X) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$*. Clearly,* $XL(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *is negative semidefinite but* $X \neq 0$*. This shows that the map* L *does not have the* $\mathbf{P_1}$*-property. We now show that* L *has the* $\mathbf{P}$*-property. Suppose* $XL(X) = L(X)X \preceq 0$*. We need to show that* $X = 0$*. Observe that,*

$$XL(X) = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \begin{pmatrix} x-y & x \\ x & z+y \end{pmatrix} = \begin{pmatrix} x^2 & x^2+y^2+yz \\ xz+xy-y^2 & xy+z^2+yz \end{pmatrix}.$$

Since $XL(X)$ *is a symmetric negative semidefinite matrix, we have* $x = 0$*, and this implies,* $z^2+yz \leq 0$ *and* $y^2+yz = -y^2$ *(obtained by equating the off-*

diagonal entries) or $2y^2 = -yz$. *If* $y = 0$, *then clearly* $z = 0$; *that is* $X = 0$. *So assume that* $y \neq 0$. *Then, we get* $2y = -z$. *That is,* $z^2 + yz = \frac{z^2}{2} \leq 0$. *This implies that* $z = 0$ *or* $y = 0$ *and this leads to a contradiction. That is,* $X = 0$. *This shows that* L *has the* **P**-*property but does not have the* $\mathbf{P_1}$-*property.*

We give below an example of a linear map $L : S^2 \to S^2$ having the $\mathbf{P_2}$-property but not the $\mathbf{P_3}$-property.

Example 1.4.2 *Let* $A = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$. *Let us consider the double-sided multiplication transformation* $M_A(X) = AXA^t$. *Write* $X = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$. *Then,* $XM_A(X) = \begin{pmatrix} -20 & -16 \\ 24 & -20 \end{pmatrix}$. *Define* $X_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, $X_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. *Then,* $(X_1 - X_2)M_A(X_1 - X_2)(X_1 + X_2) = \begin{pmatrix} -20 & -16 \\ 24 & -20 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -48 & -24 \\ 136 & -128 \end{pmatrix}$. *That is,* $XM_A(X)(X_1 + X_2)$ *is negative definite. Since* $X_1 - X_2 = X \neq 0$, *it follows that the map* M_A *defined as above does not have the* $\mathbf{P_3}$-*property. We will show in the next chapter (see Theorem 2.3.2) that this map has the* $\mathbf{P_2}$-*property.*

We state below some results that we need in the sequel. The following result of Karamardian [17] (Theorem 3.1), specialized to positive semidefinite cone, guarantees a solution to SDLCP(L, Q) for every $Q \in R^n$.

Theorem 1.4.7 *(Theorem 2, Gowda and Song [9]) Consider a linear transformation $L : S^n \to S^n$. If the problems $SDLCP(L, 0)$ and $SDLCP(L, E)$, for some positive definite $E \in S^n$, have unique solutions (namely $X = 0$ is the only solution), then for all $Q \in S^n$, $SDLCP(L, Q)$ has a solution.*

Theorem 1.4.8 *(Theorem 4, Gowda and Song [9]) If L has the* **P***-property, then L has the* **Q***-property.*

Chapter 2

Strong monotonicity property and the P_2-property

2.1 Introduction

Let LCP(M, q) denote the standard linear complementarity problem. It is known that M is a **P**-matrix if and only if M has the sign reversal property, see Parthasarathy [25]. Gowda and Song [9] have extended the notion of sign reversal property in the SDLCP setup through the **P**-property or the $\mathbf{P_2}$-property. Now the following question arises naturally: If $L : S^n \to S^n$ has the **P**-property or the $\mathbf{P_2}$-property, does it imply SDLCP(L, Q) has a unique solution for all Q? The $\mathbf{P_2}$-property can be considered as a generalization of the **P**-matrix condition of the LCP(M, q) since the following two conditions are equivalent for a matrix M:

$$z \in R^n, z * (Mz) \leq 0 \Rightarrow z = 0$$

and

$$x \geq 0, y \geq 0, (x - y) * (Mx - My) * (x + y) \leq 0 \Rightarrow x = y.$$

The notation $z * (Mz)$ denotes the componentwise product of the vectors z and Mz and the inequality is defined componentwise. It is known that the **P**-property may not imply the **GUS**-property. However, it can be shown that the $\mathbf{P_2}$-property implies the **GUS**-property. We will also give an example to show the **GUS**-property need not imply the $\mathbf{P_2}$-property.

Recall that a map $L : S^n \to S^n$ has the strong monotonicity property if trace of $(XL(X))$ is positive for all nonzero $X \in S^n$. Gowda and Song [9] have raised the following question: Does the strong monotonicity property imply the $\mathbf{P_2}$-property? We answer this question affirmatively. We also give a counterexample to show the $\mathbf{P_2}$-property need not imply the strong monotonicity property. In this chapter we study the relationship between $\mathbf{P_2}$, **GUS** and the strong monotonicity properties.

2.2 $\mathbf{P_2}$-property versus GUS-property

In this section, we show the $\mathbf{P_2}$-property implies the **GUS**-property and the strong monotonicity property also implies the **GUS**-property. We also present an example to show that the **GUS**-property need not imply the $\mathbf{P_2}$-property.

Theorem 2.2.1 *(Remark 6, Gowda and Song [9]) Consider a linear transformation $L : S^n \to S^n$. Then, the $\mathbf{P_2}$-property implies the* **GUS***-property.*

We need the following theorem to prove the above Theorem 2.2.1.

Theorem 2.2.2 *(Theorem 7, Gowda and Song [9]) Given a linear transformation $L : S^n \to S^n$, the following are equivalent:*

(i) *For all $Q \in S^n$, $SDLCP(L, Q)$ has at most one solution.*

(ii) *L has the* **P** *and the cross commutative properties.*

(iii) *L has the* **GUS***-property.*

Proof of Theorem 2.2.1: For the sake of completeness, we provide a proof. Assume the $\mathbf{P_2}$-property of L. In view of Theorem 2.2.2, it is enough to show that for any $Q \in S^n$, SDLCP(L, Q) has at most one solution. Let Q be arbitrary and let X_1 and X_2 be two solutions of SDLCP(L, Q). Now letting $Y_i := L(X_i) + Q$ for $i = 1, 2$, and using complementarity conditions, we get

$$(X_1 - X_2)[L(X_1) - L(X_2)](X_1 + X_2) = (X_1 - X_2)(Y_1 - Y_2)(X_1 + X_2)$$

$$= -(X_1Y_2X_1 + X_2Y_1X_2).$$

Since X_i's and Y_i's are symmetric and positive semidefinite, we see that $X_1Y_2X_1$, $X_2Y_1X_2$ are symmetric and positive semidefinite. Therefore, by the $\mathbf{P_2}$-property, $X_1 = X_2$, and thus proves uniqueness.

Converse of the above Theorem 2.2.1 is not always true as illustrated by the following Example 2.2.1. We need the following lemma to show the converse is not true.

Lemma 2.2.1 *(Theorem 9, Gowda and Song, [9]) For a matrix $A \in R^{n\times n}$, consider the Lyapunov transformation L_A. Then the following statements are equivalent:*

(i) *L_A has the* **GUS**-*property.*

(ii) *A is positive stable and positive semidefinite.*

Example 2.2.1 *For* $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, *consider the Lyapunov transformation* L_A. *Since* A *is positive semidefinite and positive stable,* L_A *has the* **GUS**-*property by Lemma 2.2.1. Now let* $X = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ *and* $Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. *Then,* $X \succeq 0$, $Y \succeq 0$ *and* $(X - Y)L_A(X - Y)(X + Y) = 0$. *Since* $X \neq Y$, L_A *does not satisfy the* $\mathbf{P_2}$-*property. Hence, the* **GUS**-*property need not imply in general the* $\mathbf{P_2}$-*property. This example is taken from Parthasarathy, Sampangi Raman and Sriparna [26].*

Next we prove that the strong monotonicity property of a linear transformation always implies the **GUS**-property.

Theorem 2.2.3 *(Theorem 21, Song [30]) Let the linear transformation* $L : S^n \to S^n$ *be given. Assume* L *has the strong monotonicity property. Then,* L *has the* **GUS**-*property.*

Proof: For completeness sake we provide the proof here. Suppose L has the strong monotonicity property. That is, $\text{tr}(XL(X)) > 0$ for all nonzero $X \in R^n$. We will show that L has the **GUS**-property. Let us prove this by contradiction. Let $X \neq Z$ be two solutions to SDLCP(L, Q). Since X, Z are two different solutions there exists $Y \succeq 0$, $W \succeq 0$ such that

$Y = L(X) + Q \succeq 0$, $W = L(Z) + Q \succeq 0$, $XY = 0$ and $WZ = 0$. Now consider the difference, $Y - W$, which is $L(X - Z)$. Since X and Z are two different solutions, $X - Z \neq 0$. Then, $(X-Z)(Y-W) = (X-Z)(L(X-Z)$ or $-XW - ZY = (X-Z)L(X-Z)$. Taking traces on both sides, -tr$(XW+ZY)$ $=$ tr$(X - Z)L(X - Z)$. Since, by hypothesis L has the strong monotonicity property, the right side is positive as $(X \neq Z)$ and the left side is negative leading to a contradiction. This terminates the proof of Theorem 2.2.3. ■

2.3 Strong monotonicity property implies the $\mathbf{P}_2$-property

It is obvious from the definition that the strong monotonicity property implies the **P**-property. Below we prove a stronger result.

Theorem 2.3.1 *(Theorem 4, Parthasarathy, Sampangi Raman and Sriparna [27]) If a linear transformation $L : S^n \to S^n$ has the strong monotonicity property, then it has the $\mathbf{P}_2$-property.*

Proof: We will prove this by contradiction. Suppose there exists an $X \succeq 0$ and $Y \succeq 0$ such that $(X - Y)L(X - Y)(X + Y) \preceq 0$. Assume $X \neq Y$. We see that $X + Y \neq 0$. Then, there exists an orthogonal matrix U, positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ $(1 \leq r \leq n)$ with

$$U^t(X+Y)U = D \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} D$$

where $D = diag(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}, 1, \ldots, 1)$ and I_r is the identity matrix of size $r \times r$. Let $A = (D)^{-1}U^tXUD^{-1}$ and $B = (D)^{-1}U^tYUD^{-1}$. Then, A and B are symmetric positive semidefinite with

$$A + B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$A = \begin{bmatrix} A_r & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} B_r & 0 \\ 0 & 0 \end{bmatrix}$$

where A_r and B_r are $r \times r$ matrices. Now pre multiplying and post multiplying $(X-Y)L(X-Y)(X+Y)$ by $D^{-1}U^t$ and UD^{-1} respectively, and introducing appropriate matrices between the three factors of $(X-Y)L(X-Y)(X+Y)$, we get

$$(A-B)[\hat{L}(A) - \hat{L}(B)](A+B) \preceq 0,$$

where $\hat{L}(Z) = DU^tL(UDZDU^t)UD$. Note that $\hat{L}$ is a strongly monotone linear transformation on S^n. Writing

$$\hat{L}(A) - \hat{L}(B) = \begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix},$$

we get

$$tr\left(\begin{bmatrix} A_r - B_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix}\right) = \langle \hat{L}(A-B), A-B \rangle > 0,$$

i.e, $\mathrm{tr}((A_r - B_r)P) > 0$. On the other hand,

$$(A-B)[\hat{L}(A) - \hat{L}(B)](A+B) \preceq 0$$

gives (after simplification), $\mathrm{tr}((A_r - B_r)P) \leq 0$ leading to a contradiction. Hence, we must have $X = Y$ giving us the $\mathbf{P_2}$-property. ■

Theorem 2.3.1 is false if the transformation is just monotone, see Example 2.2.1. However, one can prove the following proposition.

Proposition 2.3.1 *Let $L : S^n \to S^n$ be monotone. Suppose L has the following property:*

$$X \succeq 0,\ Y \succeq 0,\ \ (X-Y)L(X-Y)(X+Y) = 0 \Rightarrow X = Y.$$

Then L has the $\mathbf{P_2}$-property.

The proof is similar to that of Theorem 2.3.1 and is omitted. This proposition is motivated by Example 2.2.1. In the next section we give a counter example to show that $\mathbf{P_2}$-property need not imply strong monotonicity property. In the next chapter, we show that for the Lyapunov transformation and for double-sided multiplication transformation M_A, when A is symmetric, the strong monotonicity property and the $\mathbf{P_2}$-property are equivalent. Note that in Example 2.2.1, A is positive semidefinite and we have shown that L_A does not satisfy the $\mathbf{P_2}$-property.

2.3.1 A counter example

We have seen some relationship between the $\mathbf{P_2}$-property and the strong monotonicity property. In particular we have shown that in Theorem 2.3.1, for linear maps, the strong monotonicity property implies the $\mathbf{P_2}$-property. We will show in the next chapter that the converse is also true for some special linear transformations (see also Parthasarathy, Sampangi Raman and Sriparna [27]). Now we give an example of a linear map $L : S^2 \to S^2$ which has the $\mathbf{P_2}$-property but does not have the strong monotonicity property.

We need the following theorem to work out the example. First we state and prove a lemma.

Lemma 2.3.1 *Let $X \succeq 0$ and $Y \succeq 0$. Then, there exists a matrix P, ($|P| \neq 0$ and $P \in R^{n\times n}$), such that P^tXP and P^tYP are diagonal matrices with nonnegative entries.*

Proof: Since $X \succeq 0$ and $Y \succeq 0$, $X+Y \succeq 0$. Hence, there exists an orthogonal matrix U such that

$$U^t(X+Y)U = \begin{bmatrix} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & & \vdots \\ 0 & \cdots & \lambda_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.1}$$

where λ_i, $i = 1, \cdots, r$ are the eigenvalues of $X+Y$ with $\text{rank}(X+Y) = r$. We can write RHS of (2.1) as

$$\sqrt{D}\begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}\sqrt{D} = \sqrt{D}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\sqrt{D} \tag{2.2}$$

where $\sqrt{D} = \text{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_r}, 1, \cdots, 1)$. Since $X+Y$ is positive semi-definite, the eigenvalues, λ_i's are positive for all i. From the above equations we can write,

$$(\sqrt{D})^{-1}U^t(X+Y)U(\sqrt{D})^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \tag{2.3}$$

where $(\sqrt{D})^{-1} = \text{diag}(\frac{1}{\sqrt{\lambda_1}}, \cdots, \frac{1}{\sqrt{\lambda_r}}, 1, \cdots, 1)$.

Since, $X \succeq 0$,

$$(\sqrt{D})^{-1}U^tXU(\sqrt{D})^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \succeq 0$$

where the dimension of B_{11} is r. From the above equation we get $B_{12} = 0$, $B_{21} = 0$ and $B_{22} = 0$. This implies that $B_{11} \succeq 0$. Hence, there exists an orthogonal vector T such that $T^tB_{11}T$ is a diagonal matrix will all entries nonnegative.

Now define $P = U(\sqrt{D})^{-1}\begin{pmatrix} T & 0 \\ 0 & I_{n-r} \end{pmatrix}$. Notice that P is real and nonsingular. Then, by simple computations we see

$$P^t(X+Y)P = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, $P^t(X+Y)P$ is a diagonal matrix with all entries nonnegative.

$$P^tXP = \begin{pmatrix} T^tB_{11}T & 0 \\ 0 & 0 \end{pmatrix}$$

is also a diagonal matrix with all entries nonnegative. Finally, we see that, P^tYP=$P^t(X+Y)P$ - P^tXP is also a diagonal matrix with nonnegative entries as $Y \succeq 0$. ■

Theorem 2.3.2 *(Theorem 1, Parthasarathy, Sampangi Raman and Sriparna [28]) Let $A \in R^{2\times 2}$ and consider $M_A(X) = AXA^t$ for $X \in S^2$. Then the following two statements are equivalent:*

(i) *A is positive definite or negative definite.*

(ii) *M_A has the $\mathbf{P_2}$-property.*

Proof: First we prove (ii) implies (i). Suppose A is neither positive definite nor negative definite, then there exists a nonzero vector $x \in R^2$ such that $x^tAx = 0$. Now write $X_1 = xx^t$. Note that $X_1 \geq 0$ and $X_1 \neq 0$. Write $X_2 = 0$. Then, $(X_1 - X_2)A(X_1 - X_2)A^t(X_1 + X_2) = X_1AX_1A^tX_1 = 0$ and X_1 is a nonzero positive semidefinite matrix with $X_1 \neq X_2$. This contradicts the $\mathbf{P_2}$-property of the map M_A. Thus, (ii) implies (i).

We now prove (i) implies (ii). We assume A is positive definite (otherwise take $-A$ which is positive definite). Suppose the $\mathbf{P_2}$-property does not hold for the map M_A. This means, there exists $X_1 \succeq 0, X_2 \succeq 0, (X_1 - X_2)A(X_1 - X_2)A^t(X_1 + X_2) \preceq 0$ and $X_1 \neq X_2$. Since, $X_1 \succeq 0$ and $X_2 \succeq 0$, by Lemma 2.3.1, there exists a nonsingular matrix P such that $X_1 = P^tD_1P$ and $X_2 = P^tD_2P$, where D_1 and D_2 are diagonal matrices with nonnegative diagonal entries. It follows that,

$$P^t(D_1 - D_2)B(D_1 - D_2)B^t(D_1 + D_2)P \preceq 0$$

where $B = PAP^t$. Note B is positive definite as A is positive definite. In other words we have,

$$(D_1 - D_2)B(D_1 - D_2)B^t(D_1 + D_2) \preceq 0. \tag{2.4}$$

Recall that D_1, D_2, B are 2×2 matrices. Now we consider two cases:

(a) $D_1 - D_2$ is a nonnegative or nonpositive diagonal matrix.

(b) $D_1 - D_2$ contains one positive and one negative diagonal entry.

First we dispose off case (a). If $D_1 - D_2 = 0$, trivially, $X_1 = X_2$. So assume $D_1 - D_2 \succeq 0$ and $D_1 - D_2 \neq 0$. From (2.4), it follows $\operatorname{tr}(D_1^2 - D_2^2)B(D_1 - D_2)B^t \preceq 0$. Since, $D_1^2 - D_2^2 \succeq 0$ and $B(D_1 - D_2)B^t \succeq 0$, it follows that $(D_1^2 - D_2^2)B(D_1 - D_2)B^t = 0$ or $(D_1^2 - D_2^2)B(D_1 - D_2) = 0$ (where B is nonsingular). If the rank of $D_1 - D_2 = 2$, we get a contradiction. So assume rank of $D_1 - D_2 = 1$. Without loss of generality (WLOG) assume that the second diagonal entry of $D_1 - D_2$ is positive. Now one can easily verify that the second diagonal entry of $(D_1^2 - D_2^2)B(D_1 - D_2)$ will be positive and this leads to a contradiction. Similarly, we can dispose off when $D_1 - D_2$ is nonpositive. Now we continue the proof for the case (b). That is, we assume one diagonal entry is positive and other entries are negative. Write $(D_1 - D_2) =$ diagonal $(\lambda, -\mu)$ where $\lambda > 0, \mu > 0$. Write $(D_1 + D_2) =$ diagonal(α_1, α_2) and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Recall that B is positive definite.

$$(D_1 - D_2)B(D_1 - D_2)B^t(D_1 + D_2) = \begin{bmatrix} \lambda^2 b_{11} & -\lambda\mu b_{12} \\ -\lambda\mu b_{21} & \mu^2 b_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 b_{11} & \alpha_2 b_{12} \\ \alpha_1 b_{21} & \alpha_2 b_{22} \end{bmatrix} \preceq 0$$

This implies, $\lambda^2 \alpha_1 b_{11}^2 \leq \alpha_1 \lambda\mu b_{12}^2$ and $\alpha_2 \mu^2 b_{22}^2 \leq \alpha_2 \lambda\mu b_{21}^2$. That is, $\lambda b_{11}^2 \leq \mu b_{12}^2$ and $\mu b_{22}^2 \leq \lambda b_{21}^2$ (Note λ, μ and α_1, α_2 are positive numbers). Thus, $b_{11}^2 b_{22}^2 \leq b_{12}^2 b_{21}^2$. Since B is positive definite, $b_{11}b_{22} > b_{12}b_{21}$. Hence, we conclude that $b_{12}b_{21}$ is negative. Now we are going to utilize the symmetry of the matrix $(D_1 - D_2)B(D_1 - D_2)B^t(D_1 + D_2)$ to arrive at a contradiction. By symmetry, the two off diagonal entries of $(D_1 - D_2)B(D_1 - D_2)B^t(D_1 + D_2)$ should be equal. That is, $\lambda^2 \alpha_2 b_{11} b_{21} - \lambda\mu\alpha_2 b_{12} b_{22} = -\lambda\mu\alpha_1 b_{11} b_{21} + \mu^2 \alpha_1 b_{22} b_{12}$ or $b_{11}b_{21}[\lambda^2\alpha_2 + \lambda\mu\alpha_1] = b_{22}b_{12}[\mu^2\alpha_1 + \lambda\mu\alpha_2]$. Since $b_{12}b_{21} < 0$ either $b_{12} > 0$ and $b_{21} < 0$ or $b_{12} < 0$ and $b_{21} > 0$. If $b_{12} > 0$ and $b_{21} < 0$, then $b_{11}b_{21}[\lambda^2\alpha_2 + \lambda\mu\alpha_1] < 0$ and $b_{22}b_{12}[\mu^2\alpha_1 + \lambda\mu\alpha_2] > 0$ and this contradicts the fact that these two expressions are equal. Similar contradiction will arise if $b_{12} < 0$ and $b_{21} > 0$. Thus, the $\mathbf{P}_2$-property must hold when A is positive definite (or negative definite). This proves that (i) imply (ii). ■

Remark 2.3.1 *We have the following observations on this section.*

1. *Gowda, Song and Ravindran [13] prove Theorem 2.3.2 for any n.*

2. *It is easy to see that (ii) implies (i) for every n and the proof is the same as given here.*

3. *If $A \in R^{2\times 2}$ or $A \in R^{3\times 3}$, then M_A has the $\mathbf{P_2}$-property if and only if M_A has the* **GUS**-*property. Also when $A \in R^{n\times n}$ is symmetric, then M_A has the* **GUS**-*property if and only if M_A has the $\mathbf{P_2}$-property, see Theorem 6, Parthasarathy, Sampangi Raman and Sriparna [27].*

We now present an example where M_A has the $\mathbf{P_2}$-property but M_A does not have the strong monotonicity property.

Example 2.3.1 *Let* $A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$. *Observe, A is positive definite. From Theorem 2.3.2, we conclude that the map $M_A(X) = AXA^t$ has the $\mathbf{P_2}$-property. Let* $X = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$. *Then,* $AXA^tX = \begin{bmatrix} -20 & 24 \\ -16 & -20 \end{bmatrix}$. *This implies that $tr(AXA^tX) = -40 < 0$. Thus, the map M_A does not have the strong monotonicity property.*

We will now show that (in this example) the map M_A does not have the $\mathbf{P}_3$*-property. Let* $X_1 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ *and* $X_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. *Then,* $(X_1 - X_2)L(X_1 - X_2)(X_1 + X_2)$ *is (not symmetric but) negative semidefinite with* $X_1 \neq X_2$. *The same example also shows that* M_A *has the* $\mathbf{P}$*-property but* M_A *does not have the* $\mathbf{P}_1$*-property.*

By Theorem 2.3.1, we know that if a linear transformation $L : S^n \to S^n$ has the strong monotonicity property, then it has the $\mathbf{P}_2$-property. The converse of this is not always true as shown in Example 2.3.1. The following results lead to a set of necessary and sufficient conditions for M_A to have the strong monotonicity property in S^2.

2.4 Necessary and sufficient conditions for M_A to have the strong monotonicity property when $A \in R^{2\times 2}$

Motivated by the discussions in the earlier section, we derive necessary and sufficient conditions for the double-sided multiplication transformation M_A to have the strong monotonicity property when $A \in R^{2\times 2}$.

Proposition 2.4.1 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Suppose* $2ad > b^2 + c^2$. *Then,* A *is* *either positive definite or negative definite.*

Proof: We have by hypothesis, $2ad > b^2+c^2$. This implies that signs of a and d are same. Also we have $b^2+c^2 \geq 2bc$ and hence $4ad > b^2+c^2+2bc = (b+c)^2$. Thus, $ad > (\frac{b+c}{2})^2$. That is, determinant of $A + A^t$ is positive. Depending upon the signs of a and d, A is either positive or negative definite. ■

Proposition 2.4.2 *Suppose M_A has the* $\mathbf{P_2}$*-property. Further assume that* $2ad \leq b^2 + c^2$. *Then, M_A does not have the strong monotonicity property.*

Proof: Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$. We will prove the result by contradiction. That is, we will assume that M_A has the strong monotonicity property. Let $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$ be such that $X \neq 0$. Then, $\text{tr}(AXA^tX)$ is given by

$$\text{tr}(M_A(X)X) = a^2x^2+2(ad+bc)y^2+d^2z^2+2axy(b+c)+2yzd(b+c)+xz(b^2+c^2) \tag{2.5}$$

is positive for all $X \neq 0$. Now in particular let $y = 1$ and $z = 0$ in the expression given in (2.5). We get

$$\text{tr}(AXA^tX) = a^2x^2 + 2ax(b+c) + 2(ad+bc) \tag{2.6}$$

which is a quadratic expression in x and it should be positive for all x. Since (2.6) is positive (because of our assumption that M_A has strong monotonicity property), the discriminant $\Delta_x = 4a^2(b+c)^2 - 8(ad+bc)a^2$ is negative. That is,

$$b^2 + c^2 < 2ad.$$

This contradicts the assumption $2ad \leq b^2 + c^2$ and we conclude that M_A does not have the strong monotonicity property. ■

Proposition 2.4.3 *Suppose that $2ad > b^2 + c^2$. Then, M_A has the strong monotonicity property.*

Proof: Since $2ad > b^2 + c^2$, and because of Proportion 2.4.1, A is either positive definite or negative definite. Appealing to Theorem 2.3.2, we conclude M_A has the P_2-property. Now we will show that, $\mathrm{tr}(AXA^tX) > 0$ for all nonzero X. Write $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$, $X \neq 0$. From (2.5) we have $\mathrm{tr}(M_A(X)X)$ as

$$a^2x^2 + 2(ad+bc)y^2 + d^2z^2 + 2axy(b+c) + 2yzd(b+c) + xz(b^2+c^2).$$

Now three cases arise. They are $xz < 0$, $xz = 0$ and $xz > 0$. We will consider the case $xz = 0$ first. For this case, three sub cases are possible. They are,

$(x=0, z\neq 0)$, $(x\neq 0, z=0)$ and $(x=0, z=0)$. We consider all these cases below.

Case: $xz=0$; sub case: $x=0, z=0$ From (2.5), we get $\text{tr}(AXA^tX)=2(ad+bc)y^2$. Since $X\neq 0$, and $x=0, z=0$ implies $y\neq 0$. We have $2ad > b^2+c^2$ by hypothesis. Notice that $b^2+c^2\geq |bc|$ which implies $2(ad+bc)y^2>0$. Hence, M_A has the strong monotonicity property.

Case: $xz=0$; sub case: $x=0, z\neq 0$ Now the expression (2.5) becomes

$$tr(AXA^tX) = 2(ad+bc)y^2 + d^2z^2 + 2yzd(b+c).$$

Observe that the above trace is quadratic in y. The discriminant Δ_y is

$$\begin{aligned} \Delta_y &= 4z^2d^2(b+c)^2 - 8z^2d^2(ad+bc) \\ &= 4z^2d^2(b^2+c^2-2ad). \end{aligned}$$

Clearly, $4z^2d^2(b^2+c^2-2ad) < 0$ as $z>0$ and $d>0$. Since Δ_y is negative, it keeps the sign. When the above trace is evaluated for $y=0$ becomes d^2z^2. Since this quantity is positive, the trace is positive for all y proving that M_A has the strong monotonicity property.

Case: $xz=0$; sub case: $x\neq 0, z=0$ Like the previous case, this case also can be dealt and can be shown that M_A has the strong monotonicity property.

Case: $xz < 0$ The expression for $\text{tr}(M_A(X)X)$ as given in (2.5) is denoted by $(\star)$, and can be written as,

$$(\star) \geq (ax + (b+c)y + dz)^2 + xz(b^2 + c^2 - 2ad).$$

Since $xz < 0$, $(b^2 + c^2 - 2ad) < 0$, $xz(b^2 + c^2 - 2ad) > 0$. Also the first term of the above inequality is nonzero. Hence, $(\star) > 0$. This implies that M_A has the strong monotonicity property.

Case: $xz > 0$ Now treating the $\text{tr}(M_A(X)X)$ as given in (2.5) as a polynomial in y, its discriminant is $\Delta_y = 4(b+c)^2(ax+dz)^2 - 8(ad+bc)\{xz(b^2+c^2) + a^2x^2 + d^2z^2\} = -4(2ad - b^2 - c^2)(a^2x^2 + d^2z^2 - 2bcxz)$. Observing that $a^2x^2 + d^2z^2 - 2bcxz \geq 2adxz - 2bcxz$ (because arithmetic mean of positive values is greater than geometric mean). Also since $xz > 0$ and $(ad - bc) > 0$, $\Delta_y < 0$. The polynomial keeps the sign. Since the trace (as polynomial) is positive for $y = 0$, $\text{tr}(AXA^tX) > 0$ for all y. This implies that M_A has the strong monotonicity property. ■

By propositions 2.4.2 and 2.4.3 we formulate the following theorem.

Theorem 2.4.1 *Let $A \in R^{2\times2}$. Let $M_A(X) = AXA^t$. Then, the following are equivalent:*

(i) $2ad > b^2 + c^2$.

(ii) *M_A has the strong monotonicity property.*

Remark 2.4.1 *Gowda observes that the above result can be generalized to any $A \in R^{n\times n}$. We present below an alternate proof of Theorem 2.4.1 also due to him for the 2×2 case. The symmetric matrix $X \in R^{n\times n}$ can be written as a vector in R^4 through an appropriate matrix E and a vector $u \in R^3$ as shown below:*

$$(x,\ y,\ y,\ z)^t = Eu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We can easily check that,

$$tr(XAXA^t) = < u, E^t(A \otimes A)Eu > .$$

This can be written as,

$$2tr(XAXA^t) = < u, (B + B^t)u >$$

where $B = E^t(A \otimes A)E$. *It follows that,* M_A *is strongly monotone if and only if the leading principal minors of* $(B + B^t)$ *are positive and this yields Theorem 2.4.1.*

Chapter 3

Lyapunov, Stein and double-sided multiplication transformations

3.1 Introduction

In the previous chapter we have studied some properties of linear transformation. In this chapter we will derive results specializing to the following linear transformations, which have been introduced earlier. Let $A \in R^{n\times n}$. In the context of SDLCP, we will consider the following three linear transformations from S^n into S^n:

1. $L_A(X) = AX + XA^t$- called the Lyapunov transformation.

2. $S_A(X) = X - AXA^t$- called the Stein transformation.

3. $M_A(X) = AXA^t$- called the double-sided multiplication transformation.

Remark 3.1.1 *The motivation for studying the Lyapunov transformations as given above comes from the stability analysis of dynamical systems. That is, the dynamical system* $\frac{dx(t)}{dt} = -Ax(t)$ *is asymptotically stable (all trajectories converge to zero as t goes to* ∞*) if and only if there exists a positive definite matrix* X *such that,* $AX + XA^t$ *is negative definite. See Bellman [1] and Lyapunov [20]. Similarly, the motivation for studying the Stein transformation comes from the stability analysis of discrete dynamical systems. See Stein [31].*

For the Lyapunov transformation L_A, Gowda and Song (Theorem 5, [9]) have shown the equivalence of the **P**-property, the **Q**-property and A being positive stable. Gowda and Parthasarathy (Theorem 11, [8]) show that the **P** and **Q**-properties are equivalent for Stein's transformation.

In Section 3.2, we show that the strong monotonicity property and the $\mathbf{P}_2$-property are equivalent in the case of Lyapunov transformation. This result is not true for any general linear transformation. This can be seen from the Example 2.3.1. We also show that if A is symmetric, L_A or L_{-A} has the GUS-property if and only if M_A has the GUS-property. Section 3.3 of this chapter deals with the equivalence of the **P**, **GUS** and $\mathbf{P_2}$-properties of double-sided multiplication transformation for any arbitrary A. When A is normal, the equivalence of the P-property and the strong monotonicity property was established for the transformations L_A and S_A by Gowda, Song and Ravindran [13] and Song [30]. This result is false for M_A even if we assume A to be normal. However, we show the equivalence of the **P**-property and the strong monotonicity property for M_A by assuming A to be a symmetric matrix. This is the best result one can hope for the equivalence of the **P**-property with the strong monotonicity property for M_A. In Section 3.4, we show when SAS is copositive for every signature matrix S and A symmetric, then double-sided multiplication transformation M_A will have the strong monotonicity property.

3.2 Properties of Lyapunov, double-sided multiplication and Stein transformations

In this section, we derive some properties of Lyapunov and double-sided multiplication transformations. We also state some results relating Lyapunov and Stein transformation.

Theorem 3.2.1 *(Theorem 5, Parthasarathy, Sampangi Raman and Sriparna [27]) The following statements are equivalent for a Lyapunov transformation* L_A*:*

(i) *A is positive definite.*

(ii) *L_A has the strong monotonicity property.*

(iii) *L_A has the* $\mathbf{P_2}$*-property.*

Proof: To show (i)$\Rightarrow$ (ii): Suppose A is positive definite. If X is a nonzero matrix in S^n with columns $x_1, x_2, \ldots, x_n$, then $tr(L_A(X)X) = 2tr(XAX) = 2\sum_{i=1}^{n} x_i{}^t Ax_i > 0$. This proves (ii).

(ii) $\Rightarrow$ (iii) has already been established in Theorem 2.3.1.

(iii) $\Rightarrow$ (i): If L_A satisfies P_2, then it has the GUS-property, by Theorem 2.2.1. By invoking Lemma 2.2.1, we get that A is positive stable and positive semidefinite. If A is not positive definite, then there exists an $x \neq 0$ such that $x^tAx = 0$. Take $X = xx^t$ so that X is symmetric and $XL_A(X)X = xx^t(Axx^t + xx^tA^t)xx^t = 0$. But since L_A satisfies $\mathbf{P_2}$, this implies $X = 0$, that is $x = 0$, which is a contradiction. Thus, A is positive definite. ■

Remark 3.2.1 *Note that if we take L_A to be monotone instead of strongly monotone, then it is clear from Example 2.2.1 that the above theorem does not hold good. While the* $\mathbf{P_2}$ *and the* **GUS** *properties are not equivalent in general, they are so for L_A when $(A + A^t)$ is nonsingular.*

Corollary 3.2.1 *If $det(A + A^t) \neq 0$, then the following are equivalent for the Lyapunov transformation L_A:*

(i) *L_A has the* GUS*-property.*

(ii) *L_A satisfies the* $\mathbf{P_2}$*-property.*

Proof: $(ii) \Rightarrow (i)$: Follows from Theorem 2.2.1.

We now show $(i) \Rightarrow (ii)$. Since L_A has the **GUS**-property, A is positive semidefinite from Lemma 2.2.1. Also $|A+A^t| \neq 0$. Thus, $(A+A^t)$ is positive definite or A is positive definite. Now the result follows from Theorem 3.2.1 above. ■

The following theorem due to Gowda, Song and Ravindran [13], show the relation between Lyapunov and Stein transformations.

Theorem 3.2.2 *(Theorem 6, Gowda, Song and Ravindran [13]) Let $A, B \in R^{n\times n}$ with $A + B + AB = I$. Consider the following statements:*

(i) *L_B is strictly monotone.*

(ii) *L_B has the* **GUS**-*property.*

(iii) *S_A has the* **P**-*property and is monotone.*

(iv) *S_A has the* **GUS**-*property.*

(v) *S_A has the* **P**-*property.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). When A is normal the reverse implications hold.

In the case of Lyapunov transformation, the **GUS**-property need not imply A to be positive definite. Take $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Note A is positive stable and positive semidefinite. Then, from Lemma 2.2.1, L_A has the **GUS**-property but A is not positive definite. However, M_A has the **GUS**-property if and only if A is positive definite or negative definite (see Theorem 3.3.2 given below).

Lemma 3.2.1 *If A is symmetric and positive stable, then A is positive definite.*

The above result follows from Theorem 1.4.6.

Theorem 3.2.3 *Suppose A is symmetric. Then, L_A or L_{-A} has the* GUS-*property if and only if M_A has the* GUS-*property.*

Proof: Assume M_A has the GUS-property. Then, by Theorem 3.3.2 (proved in the next section), A is either positive definite or negative definite. If A is positive definite L_A has the GUS-property from Theorem 3.2.1. If A is negative definite, then $-A$ is positive definite and hence L_{-A} has the GUS-property. Conversely, assume L_A has the GUS-property. Then, by Lemma 2.2.1, A is positive stable and positive semidefinite. Since A is symmetric, by appealing to the above Lemma 3.2.1, we see that A is positive definite. From Theorem 3.3.2, it follows that M_A has the GUS-property. Similarly, one can prove when L_{-A} has the GUS-property, then M_A has the GUS-property. ■

3.3 The equivalence of the P-, GUS and P_2-properties for M_A

It is known (when A is normal) through the results of Gowda, Song and Ravindran [13] that for Lyapunov and Stein transformations, the P-property = GUS-property = strong monotonicity property. However, this result is false for the double-sided multiplication transformation M_A, see Example

3.3.1. By assuming the symmetry of A, we show in the following theorem the equivalence of the strong monotonicity property with the **P**-property for M_A.

Theorem 3.3.1 *When A is symmetric, the following statements are equivalent for the double-sided multiplication transformation M_A.*

(i) *A is positive stable or negative stable.*

(ii) *A is positive definite or negative definite.*

(iii) *M_A has the strong monotonicity property.*

(iv) *M_A has the* $\mathbf{P}_2$*-property.*

(v) *M_A has the* **GUS***-property.*

(vi) *M_A has the* **P***-property.*

Proof: $(i) \Leftrightarrow (ii)$: Follows from Theorem 1.4.6 since A is symmetric.

$(ii) \Rightarrow (iii)$: Since $M_A = M_{-A}$, WLOG we assume A to be positive definite. Suppose $tr(M_A(X)X) \leq 0$ for some nonzero $X \in S^n$. Then, $tr(AXAX) \leq 0$. Since A is symmetric and positive definite, $tr(AXAX) \geq 0$ (since XAX

is also positive semidefinite). Thus, $tr(AXAX) = 0 \Rightarrow AXAX = 0$, (from Theorem 1.4.4) $\Rightarrow XAX = 0 \Rightarrow X = 0$ (because A is a positive definite), which is a contradiction. Thus, M_A has the strong monotonicity property.

(iii) $\Rightarrow$ *(iv)*: Follows from Theorem 2.3.1. The proof of the rest of the implications is similar to the one given for Theorem 3.3.2. ■

We can obtain the following theorem, when A is not necessarily a symmetric or normal matrix.

Theorem 3.3.2 *Let $A \in R^{n \times n}$. Then, for the double-sided multiplication transformation M_A the following are equivalent:*

(i) *A is positive definite or negative definite.*

(ii) *M_A has the* $\mathbf{P_2}$*-property.*

(iii) *M_A has the* GUS*-property.*

(iv) *M_A has the* **P***-property.*

(v) *M_A has the* $\mathbf{R_0}$*-property.*

Proof: (i) implies (ii) follows from Corollary 15 of Gowda, Song and Ravindran [13]. (ii) implies (iii) follows from Theorem 2.2.1. (iii) implies (iv) follows from Gowda and Song [9]. (iv) implies (i) can be proved by contradiction: Suppose A is neither positive definite nor negative definite. Then, there exists $x \neq 0$ such that $x^tAx = 0$. Write $X = xx^t$. Now $X \succeq 0$. Since $x^tAx = 0$, we can write $xx^tAxx^t = 0$. This implies $XAX = 0$, which can be written as $XAXA^t = 0$ or $XM_A(X) = 0$. Since $X \neq 0$, M_A does not have the **P**-property. This proves that (iv) imply (i). (iv) implies (v) is trivial. (v) implies (i) is as follows: Suppose M_A has the $\mathbf{R_0}$-property. If n (the order of A) is one, then the result is trivial. So assume $A \in R^{n \times n}, n \geq 2$. Suppose A is neither positive definite nor negative definite. There exists $x \neq 0$ such that $x^tAx = 0$. Define $X = xx^t \neq 0$. Then, $X \succeq 0$, $AXA^t \succeq 0$ and $XAXA = 0$. This implies, by (v), $X = 0$ leading to a contradiction. ■

Remark 3.3.1 *When A is not symmetric, we can only conclude (for M_A) that, the* **P***-property =* **GUS***-property =* $\mathbf{P_2}$*-property. In general, we may not be able to conclude, the* $\mathbf{P_2}$*-property = the strong monotonicity property, unless A is symmetric for the double-sided multiplication transformation, M_A.*

Remark 3.3.2 *The equivalence of (i), (iii) and (iv) was also obtained independently by Bhimasankaram et al. [2].*

Remark 3.3.3 *When A is normal A, is positive stable or negative stable, if and only if any one of the five statements hold in Theorem 3.3.2.*

The following example shows that M_A need not have the strong monotonicity property in general, even if A is normal and positive definite.

Example 3.3.1 *Let* $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. *Observe that* $AA^t = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = A^tA$. *So A is normal. Also A is positive definite and from Theorem 2.3.2 it follows that M_A has the $\mathbf{P_2}$-property. But $tr(XM_A(X)) < 0$, when* $X = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$, *that is, M_A is not even monotone.*

Example 3.3.2 *(Parthasarathy, [25]) Define* $N = \begin{pmatrix} \epsilon & 1 & 0 \\ 0 & \epsilon & 1 \\ 1 & 0 & \epsilon \end{pmatrix}$, *where $\epsilon \in R$. Note that N is not symmetric but it is normal. The eigenvalues of N are $1+\epsilon$, $\epsilon - \frac{1}{2} + \sqrt{\frac{-3}{4}}$ and $\epsilon - \frac{1}{2} - \sqrt{\frac{-3}{4}}$. We can see that N is positive stable if $\epsilon > \frac{1}{2}$, negative stable if $\epsilon < -1$ and neither positive stable nor negative stable if $\epsilon \in [-1, \frac{1}{2}]$. By a remark on Theorem 3.3.2 it follows that M_N has the $\mathbf{P_2}$-property if and only if $\epsilon > \frac{1}{2}$ or $\epsilon < -1$. The transformation does not have the $\mathbf{P_2}$-property if $\epsilon \in [-1, \frac{1}{2}]$. We also note that N is a $\mathbf{P}$-matrix for all $\epsilon > 0$.*

It is known that for Lyapunov and Stein transformations, **P**-property = **Q**-property. We take up this problem for the double-sided multiplication transformation in the next chapter. We could show that for M_A, **P**-property = **Q**-property, when A is symmetric or in $R^{2\times2}$. Also we could show this result when $A \in R^{2\times2}$. In general, it is not known whether the **Q**-property will imply the **P**-property for M_A.

3.4 Sufficient conditions for M_A to have the strong monotonicity property

In this section, we derive a set of sufficient conditions for the double-sided multiplication transformation M_A to have the strong monotonicity property.

Theorem 3.4.1 *Let $A \in R^{n\times n}$. Suppose SAS is a copositive matrix for every signature matrix S and $|A| \neq 0$. Further assume $A = A^t$. Then, M_A is strongly monotone.*

Outline of proof: First we provide the outline of the proof and give detailed arguments for each of the points in the outline.

(a) We show that M_A is strictly semimonotone.

(b) Then, we show that the map M_A has the $\mathbf{R_0}$-property.

(c) We invoke a theorem due to Bhimasankaram et al. [2] that if M_A has the $\mathbf{R_0}$-property, then A is positive definite or negative definite.

(d) Finally, we will appeal to Theorem 3.3.1 to conclude that M_A is strongly monotone.

Proof of (a): Suppose there exists $X \succeq 0$, $X \neq 0$ and $M_A(X)X \preceq 0$. Since $X \neq 0$, we assume that there exists a nonzero row. WLOG we assume that to be the first row and we will denote it by, $x^t = (x_{11}, x_{12}, \ldots, x_{1n})$. If x^t is not a nonzero vector, there exists a signature matrix S such that Sx is a nonnegative vector. Since $AXAX \preceq 0$ with $X \succeq 0$, $\text{tr}(AXAX) \leq 0$. Since A is symmetric and $X \succeq 0$, $AXA \succeq 0$, $\text{tr}(AXAX) \geq 0$. Thus, $\text{tr}(AXAX) = 0$. By appealing to Theorem 1.4.4, we see that, $AXAX = 0$ implies $XAX = 0$ (because $|A| \neq 0$) and $XSSASSX = 0$. The first diagonal entry of $XSSASSX$ is equal to $x^tSSASSx$. Write $u = Sx$. Then, $u^tSASu = 0$. Since SAS is copositive, symmetric and $u \geq 0$ it follows from a known result of Gowda [7] that $SASu = 0$. That is, $ASu = 0$ or $Au' = 0$, where $u' = Su \neq 0$. This contradicts the fact that A is nonsingular. This proves (a).

Proof of (b): Follows from (a).

Proof of (c): Suppose M_A has the $\mathbf{R}_0$-property. If the order of n is one, then the result is trivial. So assume $A \in R^{n\times n}, n \geq 2$. Suppose A is neither positive definite nor negative definite. Then there exists $x \neq 0$ such that $x^t Ax = 0$. Define $X = xx^t \neq 0$. Then, $XAXA = 0$ and this implies that $X = 0$ from (b), and this leads to a contradiction.

Proof of (d): We know from (c) that A is positive definite or negative definite. Since A is copositive, it follows that A is positive definite. By appealing to Theorem 3.3.1, we see that M_A has the strong monotonicity property. ∎

Remark 3.4.1 *Symmetry is essential to conclude A is positive definite in the above result as the following example illustrates.*

Example 3.4.1 *Let* $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. *Then, SAS is copositive for every signature matrix $S \in R^{2\times 2}$. But A is not positive definite as* $A + A^t = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ *is a singular positive semidefinite matrix.*

Chapter 4

Equivalence of the P- and Q-properties for M_A

4.1 Introduction

If a linear transformation L has the P-property, then L also has the Q-property; this is a consequence of Theorem 1.4.7, due to Karamardian [17]. In general, the converse of the above result need not be true. That is, if L has the Q-property, then L need not posses the **P**-property. However, for the Lyapunov transformation L_A and for the Stein transformation S_A, the **P**-property is equivalent to the Q-property, see Gowda and Parthasarathy [8] and Gowda and Song [9]. In this chapter, we want to address the following question for the transformation M_A: For the transformation M_A, can we assert the **P**-property is equivalent to the Q-property? We answer this question partially in the affirmative, in two cases: (i) when A is symmetric and (ii) when $A \in R^{2\times2}$. In section 4.2, we introduce a class of M_A for which the Q-property fails. In section 4.3, we prove that for double-sided multiplication transformation M_A, the **P**-property and the Q-property are equivalent when A is symmetric. In section 4.4, we show the equivalence of the P and Q-properties for M_A for arbitrary $A \in R^{2\times2}$. In general, it is not known whether the Q-property will imply the **P**-property for M_A.

4.2 A class of double-sided transformations not enjoying the Q-property

In this section, we prove a very interesting result. That is, we identify a class of the map, M_A, which does not admit the Q-property. Though we use this result in the subsequent sections, we note that this is of independent interest.

Example 4.2.1 *Let $M_S(X) = SXS$ where $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. We will show that this map does not have the Q-property. Take $Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Now we will show that SDLCP(M_S, Q) does not have a solution. Assume the contrary. That is, there exists $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \succeq 0$ and $SXS + Q \succeq 0$ with*

$$X(SXS + Q) = 0. \tag{4.1}$$

(4.1) can be written as

$$\begin{pmatrix} x^2 - y^2 - y & -xy + yz - x \\ xy - yz - z & -y^2 + z^2 - y \end{pmatrix} = 0. \qquad (4.2)$$

From the main diagonals of the above matrix we get

$$x^2 - y^2 = y = z^2 - y^2.$$

This implies, $x^2 = z^2 \Rightarrow x = z$ *as* $X \succeq 0$. *Substituting* $x = z$ *in any one of the off diagonal entries of the matrix in (4.2) we get,* $x = 0$. *Since* $x = z$, $z = 0$. *Consequently,* $x = 0, z = 0$ *and* $X \succeq 0$ *implies* $y = 0$ *or* $X = 0$. *Note that by our assumption* X *is a solution. That is,* $SXS + Q \succeq 0$. *But for* $X = 0$, *we get* $SXS + Q = Q \not\succeq 0$, *contradicting our assumption that* X *is a solution. Thus, the map* M_S *does not have the* **Q**-*property.*

Taking a clue from the above example, we derive the following result.

Theorem 4.2.1 *Let* S *be a signature matrix of order* n. *Assume* $S \neq \pm I$. *Then,* $M_S(X) = SXS$ *does not enjoy the* **Q**-*property.*

Proof: Since $S \neq \pm I$, it must contain both negative and positive diagonal entries. WLOG assume S is of the form $\text{diag}[-1, -1, -1, 1, \ldots, 1]$ where the first k diagonal entries are -1 and the remaining $n-k$ diagonal entries are +1. (Note $1 \le k < n$). Let

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1k} & x_{1\ k+1} & \cdots & x_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} & x_{k\ k+1} & \cdots & x_{kn} \\ x_{k+1\ 1} & \cdots & x_{k+1\ k} & x_{k+1\ k+1} & \cdots & x_{k+1\ n} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n1} & \cdots & x_{nk} & x_{n\ k+1} & \cdots & x_{nn} \end{pmatrix}.$$

Then

$$SXS = \begin{pmatrix} x_{11} & \cdots & x_{1k} & -x_{1\ k+1} & \cdots & -x_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} & -x_{k\ k+1} & \cdots & -x_{kn} \\ -x_{k+1\ 1} & \cdots & -x_{k+1\ k} & x_{k+1\ k+1} & \cdots & x_{k+1\ n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -x_{n1} & \cdots & -x_{nk} & x_{n\ k+1} & \cdots & x_{nn} \end{pmatrix}.$$

Define

$$Q = \begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & -1 & \ldots & 0 \\ 0 & \ldots & -1 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

That is, take Q as $q_{k\ k+1} = -1 = q_{k+1\ k}$ and $q_{ij} = 0$ otherwise. Claim: SDLCP(M_S, Q) has no solution for the Q defined as above. We will prove this by contradiction. Suppose there is a solution. Then, there exists $X \succeq 0$

such that $SXS + Q \succeq 0$ and $X(SXS+Q) = 0$. Then,

$$X(SXS+Q) = \begin{pmatrix} x_{11} & \cdots & x_{1k} & x_{1\ k+1} & \cdots & x_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} & x_{k\ k+1} & \cdots & x_{kn} \\ x_{k+1\ 1} & \cdots & x_{k+1\ k} & x_{k+1\ k+1} & \cdots & x_{k+1\ n} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n1} & \cdots & x_{nk} & x_{n\ k+1} & \cdots & x_{nn} \end{pmatrix} \times$$

$$\begin{pmatrix} x_{11} & \cdots & x_{1k} & -x_{1\ k+1} & \cdots & -x_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} & -x_{k\ k+1} - 1 & \cdots & -x_{kn} \\ -x_{k+1\ 1} & \cdots & -x_{k+1\ k} - 1 & x_{k+1\ k+1} & \cdots & x_{k+1\ n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -x_{n1} & \cdots & -x_{nk} & x_{n\ k+1} & \cdots & x_{nn} \end{pmatrix} = 0.$$

Let us consider the following two entries of the above product:

<u>$(k,\ k+1)$th element:</u>

$$-x_{k1}x_{1\ k+1} \ldots - x_{kk}(x_{k\ k+1}+1) + x_{k\ k+1}x_{k+1\ k+1} + \ldots + x_{kn}x_{n\ k+1} = 0 \quad (4.3)$$

<u>$(k+1,\ k)$th element:</u>

$$x_{k+1\ 1}x_{1\ k} \ldots + x_{k+1\ k}x_{k\ k} - x_{k+1\ k+1}(x_{k+1\ k}+1) - \ldots - x_{k+1\ n}x_{n\ k} = 0 \quad (4.4)$$

Adding the equations (4.3) and (4.4) we get

$$-x_{kk} - x_{k+1\ k+1} = 0. \tag{4.5}$$

This implies, that $x_{kk} = x_{k+1\ k+1} = 0$. Since X is symmetric and positive semidefinite, all the principal submatrices of X are positive semidefinite. In particular, the principal submatrix $\begin{pmatrix} x_{kk} & x_{k\ k+1} \\ x_{k+1\ k} & x_{k+1\ k+1} \end{pmatrix}$ is symmetric and positive semidefinite. Since $x_{kk} = x_{k+1\ k+1} = 0$, we have, $x_{k\ k+1} = x_{k+1\ k} = 0$. As $X \succeq 0$ is a solution to SDLCP(M_S, Q), $SXS + Q \succeq 0$. The 2×2 principal submatrix of $SXS + Q$ with elements from $k, k+1$ rows and $k, k+1$ columns, $\begin{pmatrix} x_{kk} & -x_{k\ k+1} - 1 \\ -x_{k+1\ k} - 1 & x_{k+1\ k+1} \end{pmatrix}$, should also be positive semidefinite. Since, $x_{kk} = x_{k+1\ k+1} = 0 = x_{k\ k+1} = x_{k+1\ k}$, the above 2×2 matrix is not positive semidefinite. Hence, X can not be a solution. That is, SDLCP(M_S, Q) does not have a solution. ∎

4.3 The equivalence of the P- and Q-properties of M_A when A is symmetric

We need the following lemmas to show the equivalence of the **P**- and **Q**-properties of M_A when A is symmetric.

Lemma 4.3.1 *Let $M_A(X) = AXA^t$. Then, the following statements are equivalent:*

(i) *A is positive definite or negative definite.*

(ii) *M_A has the $\mathbf{R_0}$-property.*

(iii) *For every orthogonal U, diagonal entries of UAU^t are different from zero.*

Proof: $(i) \Leftrightarrow (ii)$: Follows from Theorem 3.3.2.

$(i) \Rightarrow (iii)$: Follows from hypothesis that A is positive definite or negative definite.

$(iii) \Rightarrow (i)$: Assume that the matrix A is neither positive definite nor negative definite. Then, there exists a vector $x \neq 0$ such that $x^tAx = 0$. WLOG assume that $||x|| = 1$ and construct an orthogonal matrix $U = (x, \star, \cdots, \star)$. Then, $(U^tAU)_{11} = x^tAx = 0$. Contradicts (iii). ■

Lemma 4.3.2 *Let $D = diag[d_1, d_2, \ldots, d_n]$ where $d_i \neq 0$ for all i. Write $D = D_+S$ where $D_+ = diag[|d_1|, |d_2|, \ldots, |d_n|]$ and S the corresponding signature matrix S. If M_D has the* Q*-property, then M_S has the* Q*-property.*

Proof: To show that for every $Q \in S^n$, SDLCP(M_S, Q) has a solution. Given Q, define $\bar{Q} = \sqrt{D_+}Q\sqrt{D_+}$ where $\sqrt{D_+} = \begin{pmatrix} \sqrt{|d_1|} & 0 & \ldots & 0 \\ 0 & \sqrt{|d_2|} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{|d_n|} \end{pmatrix}$.

Let X be a solution to SDLCP$(M_D, \bar{Q})$. Then, there exists $Z = M_D(X) + \bar{Q}$ such that $XZ = 0$. We have,

$$Z = D_+ SXSD_+ + \sqrt{D_+}Q\sqrt{D_+}. \tag{4.6}$$

Pre and post multiplying (4.6) by $\sqrt{D_+}^{-1}$ we get,

$$\begin{aligned}\sqrt{D_+}^{-1} Z \sqrt{D_+}^{-1} &= \sqrt{D_+}SXS\sqrt{D_+} + Q \\ &= S\sqrt{D_+}X\sqrt{D_+}S + Q.\end{aligned}$$

Define $Y = \sqrt{D_+}X\sqrt{D_+}$ and $W = SYS + Q$. Clearly, $Y \succeq 0$ and $W \succeq 0$. Note $W = \sqrt{D_+}^{-1} Z \sqrt{D_+}^{-1}$ and $YW = \sqrt{D_+}X\sqrt{D_+}\sqrt{D_+}^{-1}Z\sqrt{D_+}^{-1} = \sqrt{D_+}XZ\sqrt{D_+}^{-1} = 0$ as $XZ = 0$ and hence the lemma. ■

Lemma 4.3.3 *The linear map M_A has the* **Q**-*property if and only if the map M_{UAU^t} has the* **Q**-*property.*

Proof: If part: Assume that M_A has the **Q**-property. Fix an orthogonal matrix U. We will show that M_{UAU^t} has the **Q**-property. In particular, fix a matrix $Q \in S^n$. Consider $\bar{Q} = U^tQU$. The problem SDLCP$(M_A, \bar{Q})$ has a solution $\bar{X}$, that is, $\bar{X} \succeq 0$, $A\bar{X}A^t + \bar{Q} \succeq 0$ and $\bar{X}(A\bar{X}A^t + \bar{Q}) = 0$. Define $X = U\bar{X}U^t$. Then, we see that X solves SDLCP(M_{UAU^t}, Q).

Only if part: Is obvious if we let $U = I$. Hence, the lemma. ■

Lemma 4.3.4 *If the linear map M_A has the* Q*-property, then* $|A| \neq 0$.

Proof: Since M_A has the Q-property, SDLCP(M_A, Q) has a solution for all Q. Take Q to be any negative definite matrix and let X be a solution. That is,

$$AXA^t + Q \succeq 0 \tag{4.7}$$

$$-Q \succ 0 \tag{4.8}$$

Adding (4.7) and (4.8) we get $AXA^t \succ 0$. This implies that $|A| \neq 0$. ■

Theorem 4.3.1 *Let $M_A = AXA^t$ with $A = A^t$. Then, the following statements are equivalent:*

(i) *M_A has the* Q*-property.*

(ii) *M_A has the* **P***-property.*

(iii) *A is positive definite or negative definite.*

Proof: *(ii)* $\Leftrightarrow$ *(iii)*: Follows from Theorem 3.3.2.

(ii) $\Rightarrow$ *(i)*: Follows from (Karamardian result) Theorem 1.4.8.

To prove *(i)* $\Rightarrow$ *(iii)*: Since A is symmetric, there exists orthogonal U such that $A = UDU^t$ where $D = \text{diag}[d_1, d_2, \ldots, d_n]$, $d_i \neq 0$ for all i and are eigenvalues of A. Since M_A has the Q-property, M_{UAU^t} has the Q-property by Lemma 4.3.3 and consequently M_D has the Q-property. From Lemma 4.3.4, $|D| \neq 0$. Note that $d_i \neq 0$ implies $d_i > 0$ for all i or $d_i < 0$ for all i or $d_i < 0$ for some i and $d_i > 0$ for the remaining i's. If $d_i > 0$ for all i, then A is positive definite and hence the theorem. If $d_i < 0$ for all i, then A is negative definite and hence the theorem. If $d_i < 0$ for some i and $d_i > 0$ for the remaining i's, we have to arrive at a contradiction. M_A has the Q-property implies that M_D has the Q-property which in turn implies that M_S has the Q-property by Lemma 4.3.2. But our hypothesis is that $S \neq \pm I$ and Theorem 4.2.1 says that if $S \neq \pm I$, then M_S does not have the Q-property, which leads to a contradiction. Thus, A is positive definite or negative definite. ■

Remark 4.3.1 (i) *When A is symmetric, and if M_A has the Q-property, then M_{A^t} has the Q-property.*

(ii) *If M_{A+A^t} has the Q-property, then M_A has the Q-property.*

4.4 The equivalence of the P- and Q-properties of M_A when $A \in R^{2\times 2}$

In this section, we will bė proving that the Q-property of M_A is equivalent to the P-property when $A \in R^{2\times 2}$. Here, A need not be a symmetric matrix. First we explain the ideas behind the proof.

Ideas behind the proof: Assume M_A has the Q-property. We show that the diagonals of A and UAU^t are different from zero. Then, Lemma 4.3.1 implies, A is positive definite or negative definite and consequently M_A has the P-property. To be precise, we show that if any one of the diagonals of A is zero then M_A does not have the Q-property.

Lemma 4.4.1 *If M_A has the* Q*-property, then $M_{\lambda A}$ also has the* Q*-property for any nonzero $\lambda \in R$.*

Proof of the above Lemma is straightforward and we omit it.

Lemma 4.4.2 *Let* $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ *and assume* $|A| \neq 0$. *Then, M_A does not have the* Q*-property.*

Proof: By contradiction. Assume M_A has the Q-property. Since $|A| \neq 0$, it implies that $bc \neq 0$. If $b = c$, then A is a symmetric matrix with the Q-property and from Theorem 4.3.1, A is positive definite or negative definite and this leads to a contradiction.

We now give the proof when $b = -c$. WLOG assume $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $Q^\circ = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. We will prove SDLCP($M_A, Q^\circ$) does not have a solution. Suppose X is a solution. Write $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$. Then, $X(M_A(X) + Q^\circ) = 0$. That is $\begin{pmatrix} x & y \\ y & z \end{pmatrix}\begin{pmatrix} z & -y-1 \\ -y-1 & x \end{pmatrix} = 0$. From this, it follows that $-x(1+y) + xy = 0$ implies $x = 0$ and $yz + z(-y-1) = 0$ implies $z = 0$ and consequently $X = 0$ leading to a contradiction.

We now consider the case $b \neq \pm c$. If M_A has the Q-property, then there exists $X \succeq 0$, $AXA^t + Q^\circ \succeq 0$ with $X(AXA^t + Q^\circ) = 0$. Let $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$. Then,

$$X(AXA^t + Q^\circ) = \begin{pmatrix} b^2xz + bcy^2 - y & bcxy - x + c^2xy \\ b^2yz + bcyz - z & bcy^2 - y + c^2xz \end{pmatrix} = 0. \tag{4.9}$$

From the diagonals we get,

$$b^2xz + bcy^2 - y = 0 = bcy^2 - y + c^2xz.$$

This implies that, $(b^2 - c^2)xz = 0$. Since $b \neq \pm c$ it implies $xz = 0$. Now $xz = 0$ implies $x = 0$ or $z = 0$. If $x = 0$, then $y = 0$, as X is positive semidefinite. Now let us compute with $x = 0$, $y = 0$

$$AXA^t + Q^\circ = \begin{pmatrix} b^2z & bcy-1 \\ bcy-1 & c^2x \end{pmatrix} = \begin{pmatrix} b^2z & -1 \\ -1 & 0 \end{pmatrix} \not\geq 0.$$

This is a contradiction as $AXA^t + Q^\circ$ is positive semidefinite. This implies $x \neq 0$. Similarly, we can show $z \neq 0$. It is easy to show that $\text{rank}(X) = 1$. So $xz = y^2$. Hence we have $y \neq 0$. From (4.9) we have, $bcxy + c^2xy = x$ or

$$y = \frac{1}{c(b+c)}. \tag{4.10}$$

Also from (4.9) we have $b^2xz + bcy^2 = y$ or $(b^2 + bc)y^2 = y$ or

$$y = \frac{1}{b(b+c)}. \tag{4.11}$$

From (4.10) and (4.11) we get

$$y = \frac{1}{b(b+c)} = \frac{1}{c(b+c)}.$$

This implies, $b = c$ and this leads to a contradiction. ■

We have essentially shown that when $A \in R^{2\times 2}$ and when M_A has the **Q**-property atleast one diagonal entry of A must b different from zero. Now we show both the diagonal entries of A must be different from zero when M_A has the **Q**-property.

Lemma 4.4.3 *Let* $A = \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$ *where* $bc \neq 0$. *Then* M_A *does not enjoy the* Q*-property.*

Proof: We prove this result by contradiction. There are two cases to be analysed. i) $bc < 0$ and ii) $bc > 0$. **Case i).** If $b = c$ then Lemma 4.3.1 gives the necessary contradiction. So we assume that $b \neq c$. WLOG assume $c < b$ (if $c > b$ a similar proof can be given). Let $Q^\circ = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$ be a solution to SDLCP(M_A,Q°). Then, $X(M_A(X) + Q^\circ) = 0$. That is,

$$\begin{pmatrix} x & y \\ y & z \end{pmatrix} \begin{pmatrix} x + 2by + b^2 z & cx + bcy - 1 \\ cx + bcy - 1 & c^2 x \end{pmatrix} = 0 \tag{4.12}$$

It is easy to check that rank of X is one. That is $xz = y^2$. If $x = 0$, then $y = 0$. This means $M_A(X) + Q^\circ = \begin{pmatrix} * & -1 \\ -1 & 0 \end{pmatrix} \not\succeq 0$ and this leads to a contradiction. So $x \neq 0$. We will now show that $z \neq 0$. Since $X \neq 0$, $|M_A(X) + Q^\circ| = 0$. Suppose $z = 0$. Then, $\begin{vmatrix} x & cx - 1 \\ cx - 1 & cx \end{vmatrix} = 0$. That is, $c^2x^2 - (cx - 1)^2 = 0$ or $2cx - 1 = 0$ or $x = \frac{1}{2c}$. Since $c < 0$, x is negative and this leads to a contradiction. That is $x \neq 0$ and $z \neq 0$. Thus, $x \neq 0$, $z \neq 0$ and $y \neq 0$. From (4.12) we have, $x^2 + 2bxy + b^2xz = c^2xz$ or

$$x + 2by + (b^2 - c^2)z = 0 \tag{4.13}$$

and $cx^2 + bcxy - x + c^2xy = 0$ or $cx + bcy + c^2y = 1$ or

$$x + by + cy = \frac{1}{c} \tag{4.14}$$

Since $c < 0$, it follows that $by + cy \neq 0$ or $b + c \neq 0$. Subtracting (4.14) from (4.13) we get, $y(b-c) + (b^2 - c^2)z = \frac{-1}{c}$ or

$$z = -\frac{y}{b+c} - \frac{1}{c(b^2 - c^2)}.$$

Note $b \neq -c$. If $b = -c$, then from (4.14) $x = \frac{1}{c} < 0$ leading to a contradiction. Also from (4.14) we get $x = \frac{1}{c} - y(b+c)$. We also know, $xz = y^2$. (Because of rank of X is one.) Now substituting for x and z in $xz = y^2$ we get,

$$y^2 = xz = (\frac{1}{c} - y(b+c))(-\frac{y}{b+c} - \frac{1}{c(b^2 - c^2)}).$$

After simplification we get, $y = \frac{1}{2c^2}$ and from (4.14) we get $x = \frac{1}{c} - \frac{(b+c)}{2c^2} = \frac{c-b}{2c^2} < 0$ and this leads to a contradiction.

Case ii) $bc > 0$. We assume WLOG $b < 0$, $c < 0$ and $c < b$. (If b and c are positive we will look at M_{SAS} where S is a signature matrix with diagonal entries +1 and -1 instead of M_A). Now we repeat the proof as in Case i), except that the case $b = -c$ can not occur when $bc > 0$. ∎

Theorem 4.4.1 *Let* $A = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ *with* $|A| \neq 0$. *Then,* M_A *does not enjoy the* **Q***-property.*

Proof: Define $\tilde{A} = \frac{1}{a}A = \begin{pmatrix} 1 & \frac{b}{a} \\ \frac{c}{a} & 0 \end{pmatrix}$, $a \neq 0$. By Lemma 4.4.3, $\tilde{A}$ does not enjoy the Q-property. Hence, we can restate the above theorem as: *if* $A = \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$ *with* $bc \neq 0$, *then* M_A *does not enjoy the* **Q***-property.* ■

Theorem 4.4.2 *Let* $A \in R^{2\times 2}$. *Then, the following two statements are equivalent:*

(i) M_A *has the* **Q***-property.*

(ii) M_A *has the* **P***-property.*

Proof: **(ii)** $\Rightarrow$ (i) : Follows from Theorem 1.4.8.

(i) $\Rightarrow$ **(ii)** : If M_A has the Q-property, then M_{UAU^t} has the Q-property for all orthogonal matrices $U \in R^{2\times 2}$ by Lemma 4.3.3. By Theorem 4.4.1, the diagonal elements of UAU^t should be nonzero. Hence, by Lemma 4.3.1, A is positive or negative definite and thus M_A has the P-property. ■

We end this section with the following open problem: Suppose A is normal matrix of order n. Could we then assert the Q-property of M_A will imply the P-property of M_A?

Chapter 5

Open Problems

5.1 Introduction

In this chapter, we pose a few interesting problems, which had arisen naturally, during the course of this thesis work.

5.2 Problems

1. Suppose the double-sided multiplication transformation, M_A, has the strong monotonicity property. Does this imply that A is symmetric? Or, if A is not symmetric will it imply that M_A does not have the strong monotonicity property?

2. For any linear transformation L, the **P**-property of L implies the **Q**-property which follows from a result due to Karamardian [17]. For

Lyapunov and Stein transformations, **P**-property = **Q**-property, see Gowda and Parthasarathy [8] and Gowda and Song [9]. In the case of the double-sided multiplication transformation, we could prove the equivalence of these two properties only when **i)** A is symmetric (Theorem 4.3.1) and **ii)** $A \in R^{2\times 2}$ (Theorem 4.4.2). In general, it is not known whether the **Q**-property of M_A will imply the **P**-property for M_A. In fact, the answer is not known even when A is a normal matrix.

Note added:

We answer Problem 1 by means of the following counter example.

Example 5.2.1 *Let* $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and let* $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$. *Then* $tr(XM_A(X))$ $= x^2 + 2xy + 2y^2 + z^2 + xz + 2yz = (\frac{x}{\sqrt{2}} + \sqrt{2}y + \frac{z}{\sqrt{2}})^2 + \frac{x^2}{2} + \frac{y^2}{2}$. *It is easy to see that* $tr(XM_A(X)) > 0 \ \forall \ X \neq 0$. *Note that A is not a symmetric matrix. Strong monotonicity of* M_A *can also be seen from Theorem 2.4.1.*

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