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ON CHARACTERISTIC FUNCTIONS SATISFYING A FUNCTIONAL EQUATION AND RELATED CLASSES OF SIMULTANEOUS INTEGRAL EQUATIONS

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SUMMARY. The characteristic functions satisfying (2) below are proved to be essentially the semi-stable laws. Auxiliary simultaneous integral equations are studied in Theorems 1, 2, 2′ and 3. The final form of results due to B. Ramachandran-C. R. Rao and to R. Shimizu is obtained.

Ramachandran and Rao (1970) considered characteristic functions (ch.f.'s) non-vanishing on $\mathcal{R}$ and satisfying a functional equation

$$f(t) = \prod_{j=1}^{\infty} \{f(a_j t)\}^{\gamma_j} \prod_{j=1}^{\infty} \{f(-b_j t)\}^{\delta_j}, \; t \in \mathcal{R} \qquad \dots \; (1)$$

where $0 < a_j, b_j < 1$ and the γ_j, δ_j are positive constants. Under certain assumptions on these parameters, which later turned out to be superfluous, they showed that such an f is infinitely divisible (i.d.) and derived the forms that the Lévy spectral functions M and N in the Lévy representation $L(a, \sigma^2, M, N)$ for $\phi = ln\, f$ take—these results are also to be found reproduced in Kagan, Linnik and Rao (1973), Theorems 5.4.2, 5.6.1 and 5.6.3. On the basis of this analysis, such ch.f.'s were called "generalized stable laws" (but turn out to be not too different from the "semi-stable" laws as pointed out by Shimizu : see below). Earlier, Shimizu (1968) had considered (1) in the case where $\gamma_j = \delta_j = 1$ for all j and only finite products appear on the R.H.S. of (1). Both papers used variants of complex analysis arguments involving the Laplace transform introduced by Yu.V. Linnik. Then, Davies and Shimizu (1976) obtained closed-form formulas for f satisfying (1), directly,

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i.e., without first attempting to prove that f is i.d.; also, they used real analysis arguments: that f is i.d. now became a *consequence* of their analysis; however they still had to impose certain restrictions on the parameters. Shimizu (1978) generalized and streamlined these arguments to cover the more comprehensive form of (1) given by

$$\phi(t) = \int_{(0,1]} \phi(tu) d\mu(t) + \int_{(0,1]} \phi(-tu) d\nu(t) \qquad \ldots \quad (2)$$

where μ and ν are σ-finite measures on (the Borel subsets of) $(0, 1]$, to obtain essentially the same conclusions on ϕ satisfying (2) as had been obtained earlier for (1), but still under the restriction that there exist α and β such that

$$1 = \int u^{\alpha} d(\mu+\nu)(u) < \int u^{\beta} d(\mu+\nu)(u) < \infty \qquad \ldots \quad (3)$$

The result, stated briefly, is that for some real c, $f(t)\, e^{ict}$ is a 'semi-stable' ch.f. with exponent α (and in particular i.d.).

In the light of recent work on the integrated Cauchy functional equation (ICFE), it has now become possible to do away with the assumption (3) in solving (2): the existence of an α satisfying the equality in (3) no longer has to be postulated as an assumption, it now becomes part of our assertion: for, the non-negative even function $\psi = -\operatorname{Re}\phi$ satisfies the functional equation (a "multiplicative version" of the usual ICFE):

$$\psi(t) = \int_{(0,1]} \psi(ut)\, d(\mu+\nu)(u)$$

and hence we conclude that there exists a unique α such that $\int_{(0,1]} u^{\alpha} d(\mu+\nu)(u) = 1$ and that $\psi(t) = |t|^{\alpha}\, \Gamma(ln|t|)$, where Γ has every member of $S(\mu) \bigcup S(\nu)$ as period and $0 < \alpha \leqslant 2$ necessarily. The rest of the argument for Theorem 4 of Shimizu (1978) remains in force and the above analysis obviates the need for assumption (3). Theorem 3(a) below and its Corollary are needed for the above (supplementary) argument, and we provide below new proofs of these results, again made possible by recent work on the ICFE.

Theorems 1, 2 and 2′ are related results on simultaneous integral equations. They generalize earlier results obtained by Shimizu and by Ramachandran and Rao in the above context (to obtain the forms of the Lévy spectral functions for ϕ). We also provide a proof of Theorem 3 based on Theorem 2, and Theorem 3 is in turn used to obtain Theorem 2′, which is a sharper version of Theorem 2.

Theorem 1 : *Let g and h be non-negative Borel-measurable functions on* $[0, \infty)$, *locally integrable w.r.t. Lebesgue measure. Let μ and ν be σ-finite measures on (the Borel σ-algebra of)* $[0, \infty)$ *and let for all $x \geqslant 0$ (may be relaxed to: for almost all (Leb. meas.) $x \geqslant 0$)* :

$$\left.\begin{aligned} g(x) &= \int g(x+y)d\mu(y) + \int h(x+y)d\nu(y) \\ h(x) &= \int h(x+y)d\mu(y) + \int g(x+y)d\nu(y) \end{aligned}\right\} \qquad \ldots \quad (4)$$

If $\lambda = \mu + \nu$, then either

(i) there exists no real α such that $\int e^{\alpha y}d\lambda(y) = 1$ in which case we must have $g = h = 0$ a.e., or

(ii) there exists a real α, necessarily unique, such that $\int e^{\alpha y}\, d\lambda(y) = 1$, in which case we must have :

$$g(x) = \{p(x)+q(x)\}e^{\alpha x} \text{ a.e.}, \; h(x) = \{p(x)-q(x)\}e^{\alpha x} \text{ a.e.} \qquad \ldots \quad (5)$$

where, $S(\sigma)$ denoting the support of the measure σ,

$$\left.\begin{aligned} p(x+y) &= \quad p(x) \text{ for all } y \in S(\mu) \cup S(\nu) \\ q(x+y) &= \begin{cases} q(x) \text{ for } y \in S(\mu) \\ -q(x) \text{ for } y \in S(\nu) \end{cases} \end{aligned}\right\} \qquad \ldots \quad (6)$$

Proof : By arguments which are by now standard, we may confine our discussion to the case where g and h are continuous. Let $k = g+h$, so that $k = k * \lambda$—writing $(f * \sigma)(x) = \int f(x+y)d\sigma(y)$. Then, from LR(1982)—also R(1982)— it follows that either (i) holds or there exists an α, unique, such that $\int e^{\alpha y}\, d\lambda(y) = 1$ and that $k(x) = r(x)e^{\alpha x}$, where r has every element of $S(\lambda)$ as period. Writing

$$g_1(x) = g(x)e^{-\alpha x}, \; d\tilde{\lambda}(y) = e^{\alpha y}d\lambda(y)$$

(so that $\tilde{\lambda}$ is a p.m.) and defining $h_1, \tilde{\mu}, \tilde{\nu}$ analogously, we have

$$g_1 = g_1 * \tilde{\mu} + h_1 * \tilde{\nu}, \; h_1 = h_1 * \tilde{\mu} + g_1 * \tilde{\nu}$$

where

$0 \leqslant g_1, h_1 \leqslant r$, and p, q given by

$$2p = g_1 + h_1, \; 2q = g_1 - h_1$$

are such that

$$p = p * \tilde{\lambda}, \; q = q * (\tilde{\mu} - \tilde{\nu})$$

It follows from the Choquet-Deny-Shimizu theorem that

$$p(x+y) = p(x) \quad \text{for all} \quad y \in S(\tilde{\lambda}) = S(\lambda)$$

and from Theorem 2 of Shimizu (1978)—also see Theorem 3 of this paper—that q satisfies (6) ($S(\tilde{\mu}) = S(\mu)$, $S(\tilde{\nu}) = S(\nu)$). Hence the theorem.

Theorem 2 : *Let g and h be non-negative locally integrable functions on $\mathcal{R}$ and let μ and ν be σ-finite measures on $\mathcal{R}$ such that the relations below hold for all (or almost all) real x :*

$$g = g * \mu + h * \nu, \; h = h * \mu + g * \nu. \qquad \ldots \quad (7)$$

Let $\lambda = \mu + \nu$. Then either

(i) there exists no real α such that $\int e^{\alpha y} \, d\lambda(y) = 1$, in which case $g = h = 0$ *a.e.*, or

(ii) there exists a unique real α such that $\int e^{\alpha y} \, d\lambda(y) = 1$, in which case (5) holds *a.e.* on $\mathcal{R}$, with p and q satisfying (6), or

(iii) there exist two real numbers $\alpha < \beta$ such that $\int e^{\alpha y} \, d\lambda(y) = \int e^{\beta y} \, d\lambda(y) = 1$, in which case we must have

$$\left.\begin{aligned} g(x) &= r_1(x)e^{\alpha x} + s_1(x)e^{\beta x} \\ h(x) &= r_2(x)e^{\alpha x} + s_2(x)^{\beta x} \end{aligned}\right\} \text{a.e.} \qquad \ldots \quad (8)$$

where the r_i and s_i, $i = 1, 2$, have every element of $S(\mu)$ as period : in fact, they have every element of $S(\sigma)$ as period where σ is the measure defined by (10) below.

Proof : As before, we need only to consider the cases where g and h are continuous. If $g+h = k$, then $k = k * \lambda$ and we have, from LR(1984)—also R(1984), RP(1984) and R(1987)—that

$$k(x) = r(x)e^{\alpha x} + s(x)e^{\beta x} \qquad \ldots \quad (9)$$

where $\int e^{\alpha y} d\lambda(y) = \int e^{\beta y} d\lambda(y) = 1$ (note that if no such α, β exist then (i) holds ; if there is a unique such α, we may imitate the proof of Theorem 1 to obtain statement (ii) ; thus we need consider below only the case (iii) and r and s are periodic with every element of $S(\lambda)$ as period. Then, iterating the relations (7), we have for every positive integer n,

$$g = g * \left\{ \mu + \nu^{2*} * \left(\sum_{r=0}^{n-1} \mu^{r*} \right) \right\} + h * \mu^{n*} * \nu.$$

If $\mu(\mathcal{R}) = 0$ or $\nu(\mathcal{R}) = 0$, the above reduces to : $g = g{*}\nu^{2*}$ or $g = g{*}\mu$ respectively ; in the complementary cases,

$$(h{*}\mu^{n*} * \nu)(x) \leqslant (k{*}\mu^{n*} * \nu)(x)$$

$$= \int \{r(x+y_1+y_2)e^{\alpha(x+y_1+y_2)} + s(x+y_1+y_2)e^{\beta(x+y_1+y_2)}\}.\ d\mu^{n*}(y_1)d\nu(y_2)$$

$$\leqslant M\{\int e^{\alpha y} d\mu(y)\}^n . \{\int e^{\alpha y} d\nu(y)\} e^{\alpha x} + M\{\int e^{\beta y} d\mu(y)\}^n . \{\int e^{\beta y} d\nu(y)\} e^{\beta x} \to 0 \text{ as } n \to \infty$$

since every { } above is strictly < 1 and M being an upper bound for (the periodic and hence bounded) functions r and s. Writing (in all cases, whether or not $\mu(\mathcal{R}) = 0$ or $\nu(\mathcal{R}) = 0$)

$$\sigma = \mu + \nu^{2*} * \left(\sum_{r=0}^{\infty} \mu^{r*} \right) \qquad \dots \quad (10)$$

it then follows that $g = g{*}\sigma$, similarly $h = h{*}\sigma$, and therefore $k = k{*}\sigma$ as well, showing that we must also have

$$k(x) = r^*(x)e^{\gamma x} + s^*(x)e^{\delta x}$$

where $\qquad \int e^{\gamma y} d\sigma(y) = \int e^{\delta y} d\sigma(y) = 1$: take $\gamma \leqslant \delta$.

Since, however, k already has the representation (9), it follows that we must have $\gamma = \alpha$, $\delta = \beta$ (the cases where r or s/r^* or s^* is zero identically are easily disposed of), and (since $g = g{*}\sigma$, $h = h{*}\sigma$) g and h must have the representations (8) with the r_i, s_i having every element of $S(\sigma)$ and so of $S(\mu)$ as period.

We pass on to (new) proofs of Theorems 2 and 3 of Shimizu (1978) and of extensions thereof to $\mathcal{R}$ from $[0, \infty)$. As already stated, the Corollary below finds vital application in solving equations (1) and (2).

Theorem 3 : (a) *Let μ and ν be (strict) sub–p.m.'s on $[0, \infty)$ such that $\mu+\nu$ is a p.m., and let L be a real bounded continuous function on $[0, \infty)$ such that for $x \geqslant 0$*

$$L(x) = \int L(x+y)\, d(\mu-\nu)\,(y). \qquad \dots \quad (11)$$

Then $L(x+u) = L(x)$ for $u \in S(\mu)$ and $= -L(x)$ for $u \in S(\nu)$

(b) *The same assertion holds if L, μ and ν are defined on $\mathcal{R}$ and* (11) *holds for all real x.*

Corollary : *The conclusion of the theorem holds if $L(x+y)-L(x)$ is bounded for every fixed $y > 0$ while* (11) *holds (L itself need not be bounded).*

Remark 1 : Part (a) is Theorem 2 of Shimizu (1978) and the Corollary is Theorem 3 thereof. Part (b) is of independent interest and also needed to enable us to pass from Theorem 2 to Theorem 2′.

Remark 2 : The theorem and (provided $\int x\,d\mu(x) \neq 0$ if $\nu \equiv 0$) corollary remain in force if μ or ν is not a strict sub-p.m., i.e., if $\nu \equiv 0$ or $\mu \equiv 0$, only minor modifications being needed in the proofs below.

Proof : We shall consider the set-up in Part (b) ; the proofs are essentially the same for both parts. Writing (11) in the form $L = L*(\mu-\nu)$ and iterating it, we see that, for every positive integer n,

$$\begin{aligned} L &= L*\mu - L*\nu \\ &= L*\mu-(L*\mu-L*\nu)*\nu \\ &= L*(\mu+\nu^{2*})-L*\mu*\nu \\ &= \ldots \\ &= L*\left\{\mu+\nu^{2*}*\left(\sum_{r=0}^{n-1}\mu^{r*}\right)\right\}-L*\mu^{n*}*\nu. \end{aligned}$$

As in the proof of Theorem 2, we see that $(L*\mu^{n*}*\nu)(x)\to 0$ for every x, L being bounded and μ being a strict sub-p.m., so that, with σ being defined by (10), $L = L*\sigma$. Straightforward computation shows that σ is a p.m. in our present situation ($\mu+\nu$ being a p.m.) Hence, by the Choquet-Deny-Shimizu theorem, $L(x) = L(x+y)$ for $y \in S(\sigma)$ and in particular for $y \in S(\mu)$. Then we have from (11) that for all real x

$$\nu(\mathcal{R})L(x) = -\int L(x+y)d\nu(y).$$

If $u \in S(\nu)$, then for any $y \in S(\nu)$, $u+y \in S(\nu^{2*}) \subset S(\sigma)$ and hence $L(x+u+y) = L(x)$. It follows that

$$\nu(\mathcal{R})\, L(x+u) = -\int L(x+u+y)d\nu(y) = -L(x).\,\nu(\mathcal{R}).$$

Hence the theorem.

The corollary is needed for the solution of (2) after establishing the form of Re $\phi(t) = ln|f(t)|$. We refer the reader to Shimizu (1978) for the detaiils. We may obtain a short proof of the Corollary as follows. Part (a) of the theorem applied to $L(x+y)-L(x)$, for fixed $y > 0$, implies that

$$L(x+y+u)-L(x+u) = \begin{cases} L(x+y)-L(x) & \text{if } u \in S(\mu), \\ -\{L(x+y)-L(x)\} & \text{if } u \in S(\nu). \end{cases} \qquad \ldots \quad (12)$$

Fix $u \in S(\mu)$. Then the first relation in (12) yields that for all $y > 0$

$$L(x+y+u)-L(x+y) = L(x+u)-L(x).$$

Choose now a $y \in S(\nu)$. Then the second relation in (12) implies that the LHS of the preceding relation is also $= -\{L(x+u)-L(x)\}$ so that $L(x+u)-L(x) = -\{L(x+u)-L(x)\} = 0$. Using this fact in (11), we have

$$L(x) = -\int_{[0,\infty)} L(x+y)d\nu^*(y) \qquad \dots \quad (13)$$

where $\nu^* = \nu/\nu([0,\infty))$ is a p.m. If now $u \in S(\nu)$, then (13) implies that

$$\begin{aligned} L(x)+L(x+u) &= -\int_{[0,\infty)} \{L(x+y)+L(x+u+y)\}d\nu^*(y) \\ &= -\int \{L(x+u)+L(x)\}d\nu^*(y) \text{ from (12)} \\ &= -\{L(x)+L(x+u)\} \end{aligned}$$

whence the corollary follows.

It is of some interest to note that Theorem 2 may be used to prove Theorem 3(b), which in turn is needed to prove Theorem 2′ below : Let K be a positive constant such that $|L(x)| \leqslant K$. Then $g = K+L$ and $h = K-L$ are both non-negative and, further, satisfy (7). Hence, by Theorem 2, they admit the representations (8). Both being bounded, it follows that $\alpha = \beta = 0$ in this particular case in (8)—in other words, g and h have every element of $S(\sigma)$ ($\supset S(\mu)$) as period. Using this fact in (7), we see that

$$g(x) = \int h(x+y)d\nu^*(y), \; h(x) = \int g(x+y)d\nu^*(y)$$

where $\nu^* = \nu/\nu\,(\mathcal{R})$ is a p.m. As before, we then conclude that, since $S(\sigma) \supset S(\nu^{2*})$, if $u \in S(\nu)$, then $g(x+u) = h(x)\nu^*(\mathcal{R}) = h(x)$, so that $L(x+u) = -L(x)$ for such u.

We are now in a position to postulate a sharper version of Theorem 2, namely,

Theorem 2′ : *In case* (iii) *of Theorem* 2,

$$\left.\begin{aligned} g(x) &= \{p_1(x)+q_1(x)\}\, e^{\alpha x}+\{p_2(x)+q_2(x)\}\, e^{\beta x} \\ h(x) &= \{p_1(x)-q_1(x)\}\, e^{\alpha x}+\{p_2(x)-q_2(x)\}\, e^{\beta x} \end{aligned}\right\} \qquad \dots \quad (14)$$

a.e. (Leb. meas.), where p_i, q_i, $i = 1, 2$, *are such that*

$$p_i(x+y) = p_i(x) \qquad \text{for all } y \in S(\mu) \cup S(\nu)$$

$$q_i(x+y) = \begin{cases} q_i(x) & \text{for all } y \in S(\mu) \\ -q_i(x) & \text{for all } y \in S(\nu) \end{cases} \qquad \dots \quad (15)$$

Proof: As before, we need only consider the case where g and h are continuous.

By Theorem 2, g and h admit the representations (8). Substituting in (7) and recalling that every element of $S(\mu)$ is a period for the r_i, s_i, taking 1 as a positive element of $S(\mu)$ w.l.g. and defining

$$g_1(x) = g(x)\, e^{-\beta x},\; g_2(x) = g_1(x)-g_1(x+1),\; g_3(x) = g_2(x)\, e^{(\beta-\alpha)x},$$

and h_1, h_2, h_3 similarly, we have that

$$g_3 = g_3 * \tilde{\mu}+h_3 * \tilde{\nu},\; h_3 = h_3 * \tilde{\mu}+g_3 * \tilde{\nu}$$

where $d\tilde{\mu}(y) = e^{\alpha y}\, d\mu(y), d\tilde{\nu}(y) = e^{\alpha y}\, d\nu(y)$ and $\tilde{\sigma} = \tilde{\mu}+\tilde{\nu}$ is theorefore a p.m. But (8) also implies that

$$g_3 = cr_1,\; h_3 = cr_2 \text{ where } c = 1-e^{\alpha-\beta}.$$

(If $\alpha = \beta$, the proof is similar to that of Theorem 1, so we shall consider only the case $\alpha < \beta$). Thus p and q given by

$$2p = g_3+h_3,\; 2q = g_3-h_3$$

satisfy the equations $p = p * (\tilde{\mu}+\tilde{\nu}),\; q = q * (\tilde{\mu}-\tilde{\nu})$.

Applying the Choquet-Deny-Shimizu theorem to p and Theorem 3(b) to q, or Theorem 3(b) in the light of Remark 2 to both p and q, we see that p and q satisfy (6). It follows that r_1 and r_2 are of the forms p_1+q_1 and p_1-q_1 respectively, where p_1, q_1 have the properties (15). The assertions re. s_1 and s_2 follow dually. Hence the theorem.

This joint paper is the common outcome of the technical papers Lau-Gu (1986) and Ramachandran (1986). The referee of the latter is to be thanked for suggestions which led to improvements in the presentation as well as in arguments. As pointed out by the referee of this joint paper, a slight improvement in presentation would have resulted if (iii) of Theorem 2 had been replaced by the statement of Theorem 2′ with appropriate attendant alterations.

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