A geometrical formulation of Abelian gauge structure in non-Abelian gauge theories and disconnected gauge group

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A geometrical formalism of non-Abelian gauge theory with a topological term is constructed here with special reference to the role of Abelian gauge structures in non-Abelian theories. It is shown that when fermionic currents are written in chiral forms, we can take into account the disconnected gauge group which helps us to formulate a non-Abelian gauge structure so that the theory becomes asymptotically free.

I. INTRODUCTION

In a recent paper,[1] Wu and Zee have pointed out that the inclusion of topological Lagrangian in non-Abelian gauge theories introduces certain topologically nontrivial Abelian background gauge fields in the configuration space of these theories. In particular the θ-term and the topological mass term leads, respectively, to a vortex and a monopole in gauge orbit space in $3+1$ and $2+1$ dimensions. In view of this the θ-vacuum may be considered to arise from a kind of Bohm–Ahranov effect. The nature of the Abelian gauge field in a $3+1$ dimension is found to be the field of a vortex line and in a $2+1$ dimension it is the field of a monopole.

In this paper we have tried to study the geometrical interpretation of the analysis of Wu and Zee. We shall show that in $3+1$ and $2+1$ dimensions the same geometrical feature is responsible for the generation of topological terms in the non-Abelian gauge field Lagrangian. This specific geometrical feature is responsible for the realization of the Berry phase which is a generalized version of the Bohm–Ahranov phase in quantum mechanics.

It is found that an inherent anisotropic feature which is responsible for the quantization of a fermion[2] in a $3+1$ dimension appears as the main geometrical feature underlying the topological term in the non-Abelian gauge field Lagrangian and this anisotropy is caused by attaching a "direction vector" or "vortex line" to a space–time point. The motion of a particle in such an anisotropic space is equivalent to that of a charged particle in presence of a magnetic monopole. The added bonus in this formalism is that when fermions are taken in chiral forms the theory is free from chiral anomaly. It also suggests the introduction of disconnected gauge group for chiral fermions and helps to develop quantum electrodynamics involving chiral fermions in an asymptotically free way.

In Sec. II, we shall recapitulate the geometrical and topological feature arising in a $(3+1)$-dimensional space–time when a direction vector or vortex line is attached to a space–time point and its implication in $(2+1)$-dimensional system. In Sec. III, we shall discuss certain features associated with the gauge orbit space and in Sec. IV we shall discuss about asymptotic freedom and some features of a non-Abelian gauge field arising out of a disconnected Abelian gauge group.

II. TOPOLOGICAL ORIGIN OF FERMION NUMBER, VORTEX LINE AND NON-ABELIAN GAUGE FIELD

Different quantization procedures suggest that to have quantum probability from a classical system we have to introduce classical probability in a specific geometrical setup. In a recent paper[2]

Nelson's stochastic quantization procedure has been generalized to have a relativistic framework and quantization of a Fermi field taking into account Brownian motion processes in the internal space also apart from that in the external space. For the quantization of a Fermi field the introduction of an anisotropy in the internal space is necessary so that the internal variable appears as a direction vector attached to the external space–time point. The opposite orientations of the direction vector correspond to particle and antiparticle, respectively. To be equivalent to the Feynman path integral we have to take into account a complexified space–time when the coordinate is given by $z_\mu = \chi_\mu + i\xi_\mu$ where ξ_μ corresponds to a direction vector attached to the space–time point x_μ.[3] Since for quantization, we have to introduce Brownian motion processes both in the external and internal space, after quantization, for an observational procedure we can think of the mean position of the particle q_μ in the external observable space with a stochastic extension determined by the internal stochastic variable ξ_μ. The nonrelativistic quantum mechanics is obtained in the sharp point limit.[4] It has been shown that when we consider the internal space anisotropic in nature so that ξ_μ appears as a direction vector, we can generate two internal helicities in terms of two spinorial variables giving rise to fermion and antifermion.[3] This helps us to have a gauge theoretic extension of a relativistic quantum particle when the gauge group is given by SL(2,c). This inherent gauge structure seems to be the major ingredient of quantization procedure. Recently Klauder[5] has formulated a quantization procedure that has the explicit property of coordinate independence. This is very significant in the sense that it specifies the role of geometry in quantum probability. This geometrical approach to the phase space quantization can be interpreted in terms of a universal magnetic field acting on a free particle in a higher dimensional configuration space when quantization corresponds to freezing the particles to its first Landau level. The crucial role of gauge field then becomes transparent in this formalism and paves the way to the equivalence of stochastic and Klauder quantization. The geometric quantization[6] then appears as a natural consequence of these two formalisms since the Hermitian line bundle introduced there finds a physical meaning in terms of the inherent gauge field in stochastic phase space formulation or in interaction with the magnetic field in Klauder quantization.

This analysis of the stochastic quantization procedure suggests that we can write the position and momentum variable of a quantum particle as

$$Q_\mu = q_\mu + i\hat{Q}_\mu; \quad P_\mu = p_\mu + i\hat{P}_\mu, \tag{1}$$

where $q_\mu(p_\mu)$ denotes the mean position (momentum) in the external observable space and $\hat{Q}_\mu(\hat{P}_\mu)$ is given by the internal variable denoting the stochastic extension. The observation of Heisenberg's uncertainty relation from stochastic mechanics along with the customary time energy uncertainty relation helps us to formulate the commutation relations

$$[Q_\mu, P_\nu] = i\hbar g_{\mu\nu}, \quad [Q_\mu, Q_\nu] = 0 = [P_\mu, P_\nu]. \tag{2}$$

This suggests

$$[\hat{Q}_\mu, \hat{P}_\nu] = i\hbar g_{\mu\nu}, \quad [\hat{Q}_\mu, \hat{Q}_\nu] = 0 = [\hat{P}_\mu, \hat{P}_\nu]. \tag{3}$$

However, as we have pointed out, the quantization of a Fermi field is achieved when we introduce an anisotropic feature in the internal space so that we can have two opposite internal helicities corresponding to fermion and antifermion, the components of the internal variables in this case will not commute, so in this case, the relations (2) and (3) will not be satisfied. Indeed, we will have

$$\begin{gathered}[Q_\mu, Q_\nu] \neq 0, \quad [P_\mu, P_\nu] \neq 0, \\ [\hat{Q}_\mu, \hat{Q}_\nu] \neq 0, \quad [\hat{P}_\mu, \hat{P}_\nu] \neq 0.\end{gathered} \tag{4}$$

Introducing a new constant $\omega=\hbar/lmc$, where m is the mass of the particle, the quantum uncertainty relation can now be written in terms of the dimensionless variables, where we replace Q_μ by Q_μ/l, and P_μ by P_μ/mc

$$[Q_\mu,P_\nu]=i\omega g_{\mu\nu}, \quad [\hat{Q}_\mu,\hat{P}_\nu]=i\omega g_{\mu\nu}. \tag{5}$$

As has been shown by Brooke and Prugovecki,[7] these relativistic canonical commutation relations admit the following representation of Q_μ/ω and P_μ/ω:

$$Q_{\mu/\omega}=-i(\partial/\partial p_\mu+\phi_\mu), \quad P_{\mu/\omega}=i(\partial/\partial q_\mu+\psi_\mu), \tag{6}$$

where ϕ_μ and ψ_μ are complex-valued functions. Now, when we introduce an anisotropy in the internal space giving rise to the internal helicity to quantize a fermion, ϕ_μ and ψ_μ became matrix-valued functions due to the noncommutativity character of the components $Q_\mu(P_\mu)$.

When we consider that the two opposite orientations of the direction vector ξ_μ attached to the space–time point x_μ in complexified Minkowski space–time having the coordinate $z_\mu=x_\mu+i\xi_\mu$ give rise to two opposite internal helicities corresponding to fermion and antifermion, we can formulate the internal helicity in terms of the two component spinorial variable $\theta(\bar{\theta})$.[8] In fact for a massive spinor, we can choose the chiral coordinate in this space as

$$z^\mu=x^\mu+(i/2)\lambda_\alpha^\mu\theta^\alpha \quad (\alpha=1,2), \tag{7}$$

where we identify the coordinate in the complex manifold $z^\mu=x^\mu+i\xi^\mu$ with $\xi^\mu=\frac{1}{2}\lambda_\alpha^\mu\theta^\alpha$. We can replace the chiral coordinate by the matrices

$$z^{AA'}=x^{AA'}+(i/2)\lambda_\alpha^{AA'}\theta^\alpha, \quad x^{AA'}=\begin{bmatrix} x^0-x^1 & x^2+ix^3 \\ x^2-ix^3 & x^0+x^1 \end{bmatrix}, \tag{8}$$

and

$$\lambda_\alpha^{AA'}\in \mathrm{SL}(2,c).$$

With these relations, the twistor equation is now modified as

$$\bar{z}_a z^a+\lambda_\alpha^{AA'}\theta^\alpha\bar{\pi}_A\pi_{A'}=0, \tag{9}$$

where $\bar{\pi}_A$ $(\pi_{A'})$ is the spinorial variable corresponding to the four momentum variable p^μ the conjugate of x^μ and is given by the matrix representation

$$p^{AA'}=\bar{\pi}^A\pi^{A'} \tag{10}$$

and

$$z^a=(\omega^A,\pi_{A'}), \quad \bar{z}_a=(\bar{\pi}_A,\bar{\omega}^{A'}),$$

with

$$\omega^A=i[x^{AA'}+(i/2)\lambda_\alpha^{AA'}\theta^\alpha]\pi_{A'}.$$

Equation (9) now involves the helicity operator

$$s=-\lambda_\alpha^{AA'}\theta^\alpha\bar{\pi}_A\pi_{A'}, \tag{11}$$

which we identify as the internal helicity of the particle and it corresponds to the fermion number. It may be noted that we have taken the matrix representation of p^μ, the conjugate of x^μ in the complex coordinate $z^\mu = x^\mu + i\xi^\mu$ as $p^{AA'} = \bar{\pi}^A \pi^{A'}$ implying $p_\mu^2 = 0$ and so the particle will have mass due to the nonvanishing character of the quantity ξ_μ^2. It is observed that the complex conjugate of the chiral coordinate will give rise to a massive particle with opposite internal helicity corresponding to antifermion. In the null plane we can write the chiral coordinates as follows:[9]

$$z^{AA'} = x^{AA'} + (i/2)\bar{\theta}^A \theta^{A'}, \tag{12}$$

where the coordinate ξ^μ is replaced by $\xi^{AA'} = \frac{1}{2}\bar{\theta}^A \theta^{A'}$. In this case the helicity operator is given by

$$s = -\bar{\theta}^A \theta^{A'} \bar{\pi}_A \pi_{A'} = -\bar{\epsilon}\epsilon, \tag{13}$$

where $\epsilon = i\theta^{A'}\pi_{A'}$, $\bar{\epsilon} = -i\bar{\theta}^A\bar{\pi}_A$. The corresponding twistor equation describes a massless spinor field. The state with the helicity $+\frac{1}{2}$ is the vacuum state of the fermion operator

$$\epsilon|s = +\tfrac{1}{2}\rangle = 0. \tag{14}$$

Similarly the state with the internal helicity $-\frac{1}{2}$ is the vacuum state of the fermion operator

$$\epsilon|s = -\tfrac{1}{2}\rangle = 0. \tag{15}$$

In the case of a massive spinor we can define a plane D^- where for the coordinate $z_\mu = x_\mu + i\xi_\mu$, ξ_μ belongs to the interior of the forward light cone $\xi \gg 0$ and represents the upper half plane. The lower half plane D^+ is given by the set of all coordinates Z_μ with ξ_μ in the interior of the backward light cone ($\xi \ll 0$). The map $Z \to Z^*$ sends the upper half plane to the lower half plane. The space M of null plane ($\xi_\mu^2 = 0$) is the Shilov boundary so that a function holomorphic in $D^-(D^+)$ is determined by its boundary values. Thus if we consider that any function $\phi(z) = \phi(x) + i\phi(\xi)$ is holomorphic in the whole domain, the helicity $+\frac{1}{2}(-\frac{1}{2})$ in the null plane may be taken to be the limiting value of internal helicity in the upper (lower) half plane.

In the sense of Minkowski space–time the domain having the characteristics $\xi \gg 0$ and $\xi \ll 0$ in the upper and lower half planes indicates that the domain is disconnected in nature. This points out that the behavior of the angular momentum operator in such a region will be similar to that of a charged particle moving in the field of a magnetic monopole. In fact the wave function $\phi(z_\mu) = \phi(x_\mu) + i\phi(\xi_\mu)$ can be treated to describe a particle moving in the external space–time having the coordinate x_μ with an attached direction vector ξ_μ. Thus $\phi(z_\mu)$ should take into account the polar coordinates r, θ, ϕ along with the angle χ specifying the rotational orientation around the direction vector ξ_μ. The eigenvalue of the operator $i\partial/\partial\chi$ just corresponds to the internal helicity $+\frac{1}{2}(-\frac{1}{2})$. For an extended body represented by the DeSitter group SO(4,1), θ, ϕ, χ just represent the three "Euler angles". The angular momentum in this space is given by $\mathbf{J} = \mathbf{r} \times \mathbf{p} - \mu \mathbf{r}$ where μ is the eigenvalue of $i\partial/\partial\chi$ and can take the value $\pm\frac{1}{2}$. This suggests that in such a space a particle can move with $l = \frac{1}{2}$. The fact that in such an anisotropic space the angular momentum can take the value $\frac{1}{2}$ is found to be analogous to the result that a monopole charged particle composite representing a dyon satisfying the condition $e\mu = \frac{1}{2}$ have their angular momentum shifted by $\frac{1}{2}$ unit and their statistics shift accordingly.[10]

In the complexified space–time exhibiting the internal helicity states we can now write the metric as $g_{\mu\nu}(x, \theta, \bar{\theta})$. It has been shown elsewhere[11] that the metric structure gives rise to the SL(2,c) gauge theory of gravitation and generates the field strength tensor $F_{\mu\nu}$ given in terms of gauge fields B_μ which are matrix valued having the SL(2,c) group structure and is given by

$$F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + [B_\mu, B]. \tag{16}$$

So from relations (6), we can identify ϕ_μ with B_μ and can associate another gauge field C_μ with ψ_μ satisfying the relation (16). This suggests that for a relativistic quantum particle which is taken as a stochastically extended one, the fermionic character of a particle associates the functions defined on stochastic phase space with matrix-valued non-Abelian gauge fields having the SL(2,c) group structure. That is, we write

$$\frac{Q_\mu}{\omega}=-i\left(\frac{\partial}{\partial p_\mu}+B_\mu\right),\quad \frac{P_\mu}{\omega}=i\left(\frac{\partial}{\partial q_\mu}+C_\mu\right). \tag{17}$$

The asymptotic zero curvature condition $F_{\mu\nu}=0$ implies that we can write the non-Abelian gauge field on the boundary as

$$B_\mu=U^{-1}\partial_\mu U,\quad U\in \mathrm{SL}(2,e). \tag{18}$$

With this substitution, we note that the term $F_{\mu\nu}F^{\mu\nu}$ in the Lagrangian gives rise to the skyrme term $\mathrm{Tr}[\partial_\mu UU^+,\partial_\nu UU^+]^2$ so that we can write the skyrme Lagrangian

$$L=M^2\,\mathrm{Tr}(\partial_\mu U^+\partial_\mu U)+\mathrm{Tr}[\partial_\mu UU^+,\partial_\nu UU^+]^2, \tag{19}$$

where the first term can be derived from the term like $M^2B_\mu B^\mu$ where M is a suitable constant having the dimension of mass. Thus we find that the quantization of a Fermi field considering an anisotropy in the internal space leading to an internal helicity description corresponds to the realization of a nonlinear α-model where the skyrme term ($L_{\mathrm{skyrme}}=\mathrm{Tr}[\partial_\mu UU^+,\partial_\nu UU^+]^2$) introduced for stabilization of the soliton automatically arises here as an effect of quantization.

From this analysis it appears that massive fermions appear as solitons and the fermion number is of topological origin. Indeed for the Hermitian representation, we can take the group manifold as SU(2) and this leads to a mapping from the space three sphere s^3 to the group space $s^3[sU(2)=s^3]$ and the corresponding winding number is given by

$$q=\frac{1}{24\pi^2}\int ds_\mu\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}(U^{-1}\partial_\nu UU^{-1}\partial_\alpha UU^{-1}\partial_\beta U). \tag{20}$$

Evidently q can be taken to represent the fermion number.

It is to be noted that the simplest Lagrangian density which is invariant under SL(2,c) transformation in spinor affine space is given by

$$L=-\tfrac{1}{4}\,\mathrm{Tr}\,\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}. \tag{21}$$

Following Carmeli and Malin[12] if we apply the usual procedure of variational calculus, we get the field equations

$$\partial_\delta(\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta})-[B_\delta,\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}]=0. \tag{22}$$

Taking the infinitesimal generators of the group SL(2,c) in the tangent space[12] as

$$g^1=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad g^2=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad g^3=\begin{bmatrix}0&1\\0&0\end{bmatrix}. \tag{23}$$

We can write

$$B_\mu=l_\mu^a g^a=\mathbf{b}_\mu\cdot\mathbf{g},\quad F_{\mu V}=f_{\mu\nu}^a g^a=\mathbf{f}_{\mu\nu}\cdot\mathbf{g}. \tag{24}$$

Thus to describe a matter field in this geometry the total Lagrangian will be modified by the introduction of this SL(2,c) invariant Lagrangian density. Hence, for a massless Dirac field we write for the Lagrangian

$$L=-\bar{\psi}\gamma_\mu D_\mu\psi-\tfrac{1}{4}\,\mathrm{Tr}\,\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}, \tag{25}$$

where $D_\mu=\partial_\mu-igB_\mu$, being a suitable coupling strength. From this a conserved current is constructed[7]

$$\mathbf{J}_\mu=\bar{\psi}\gamma_\mu\psi+\epsilon^{\mu\nu\alpha\beta}\mathbf{b}_\nu\times\mathbf{f}_{\alpha\beta}=\mathbf{J}_x^\mu+\mathbf{J}_0^\mu. \tag{26}$$

From (2) it follows that

$$\epsilon^{\mu\nu\alpha\beta}(\partial_\nu\mathbf{F}_{\alpha\beta}-\mathbf{b}_\nu\times\mathbf{f}_{\alpha\beta})=0. \tag{27}$$

This suggests that

$$\mathbf{J}_\theta^\mu=\epsilon^{\mu\nu\alpha\beta}\mathbf{b}_\nu\times\mathbf{f}_{\alpha\beta}=\epsilon^{\mu\nu\alpha\beta}\partial\nu\mathbf{f}_{\alpha\beta}. \tag{28}$$

However, in (1) if we split the Dirac massless spinor in chiral forms and identify the internal helicity with left (right) chirality corresponding to $\theta(\bar{\theta})$ we have the following conservation laws[13]

$$\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_R\gamma_\mu\psi_R)+J_\mu^1]=0,$$

$$\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L)+ig\bar{\psi}_R\gamma_\mu\psi_R+J_\mu^2]=0, \tag{29}$$

$$\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L)+J_\mu^3]=0.$$

These three equations represent a consistent set of equations if we choose

$$J_\mu^1=-J_\mu^2/2,\quad J_\mu^3=+J_\mu^2/2, \tag{30}$$

which evidently guarantees the vector current conservation. Thus we can write

$$\partial_\mu(\bar{\psi}_R\gamma_\mu\psi_R+J_\mu^2)=0,\quad \partial_\mu(\bar{\psi}_L\gamma_\mu\psi_L-J_\mu^2)=0. \tag{31}$$

From (10) we get,

$$\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)=\partial_\mu J_\mu^5=-2\partial_\mu J_\mu^2. \tag{32}$$

Thus the chiral anomaly is expressed here in terms of the second SL(2,c) component of the gauge field current J_μ^2. We note that chiral currents are modified by the introduction of J_μ^2 and the anomaly vanishes.

We note that the charge corresponding to the gauge field part is of the form

$$q=\int J_v^2 d^3x=\int_{\text{surface}}\epsilon^{ijk}d\sigma_i f_{jk}^{(2)}\quad (i,j,k=1,2,3). \tag{33}$$

Visualizing $f_{jk}^{(2)}$ to be magnetic field-like components for the vector potential $b_i^{(2)}$ we see that q is actually associated with the magnetic strength. It may be added here that the association of chiral anomaly with the Berry phase suggests that this topological phase $e^{i\phi_\beta}$ is given by the relation $\phi_\beta=2\pi\mu$ where $q=2\mu$ and is related to the fermion number.[14]

Thus we find that the quantization of a Fermi field associates a background magnetic field corresponding to $f_{ij}^{(2)}$ and the charge corresponding to the gauge field effectively represents a magnetic charge.

The term $\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{Tr}\,F_{\alpha\beta}F_{\gamma\delta}$ in the Lagrangian (25) can be actually expressed as a four divergence $\partial_\mu\Omega_\mu$ where

$$\Omega^\mu=-\frac{1}{16\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}[B_\nu F_{\sigma\beta}-\tfrac{2}{3}(B_\nu B_\alpha B_\beta)]. \tag{34}$$

We recognize that the gauge field Lagrangian is related to the Pontryagin density

$$P=-\frac{1}{16\pi^2}\,\mathrm{Tr}\,{}^*F_{\mu\nu}F^{\mu\nu}=\partial_\mu\Omega^\mu, \tag{35}$$

where Ω^μ is the Chern–Simons secondary characteristic class. The Pontryagin index $q=\int Pd^4x$ is a topological invariant.

The introduction of the Chern–Simons term modifies the axial vector current as $\tilde{J}^5_\mu=J^5_\mu+i\hbar\Omega\mu$ where $\partial_\mu\tilde{J}^5_\mu=0$ though $\partial_\mu J^5_\mu\neq 0$. We find from Eq. (32) that the Chern–Simons term is effectively represented by the current constructed from the SL(2,c) gauge field. Thus we have Chern–Simons term effectively in built in the system and is associated with the topological aspects of the fermion arising out of the quantization procedure. From this analysis, we find that the non-Abelian gauge field associates a fictitious magnetic field in 3+1 dimension, and the introduction of the Pontryagin term $(\theta/32\pi^2)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ effectively takes care of the anisotropic feature of the space–time when a direction vector (vortex line) is attached to the space–time point.

In a 2+1 dimension the Hopf invariant is defined as

$$H=-\frac{1}{4\pi}\int d^3x\,\epsilon^{\hat{\mu}\hat{\nu}\hat{\lambda}}A_{\hat{\mu}}F_{\hat{\nu}\hat{\lambda}}. \tag{36}$$

Now if μ denotes a four-dimensional index then

$$\partial\rho\,\epsilon^{\rho\mu\nu\lambda}A_\mu F_{\nu\lambda}=\tfrac{1}{2}\epsilon^{\rho\mu\nu\lambda}F_{\rho\mu}F_{\nu\lambda}$$

connects the Hopf invariant to the chiral anomaly. So from the above analysis we note that the Pontryagin term is associated with the Hopf term in a (2+1)-dimensional system. Non-Abelian gauge theories in 2+1 dimension with the incorporation of the Chern–Simons term

$$\frac{\mu}{2}\,\mathrm{Tr}\,\epsilon^{\mu\nu\lambda}[A_\mu F_{\nu\lambda}-\tfrac{2}{3}(A_\mu A_\nu A_\lambda)], \tag{37}$$

represent the effect of a magnetic monopole field with the pole strength μ. The fact that a three-dimensional manifold B can be considered as a boundary of the four-dimensional manifold $M(\partial M=B)$ suggests that the topological action viz., the Pontryagin term which arises as an effect of quantization in a four dimension has its counterpart in a three dimension the Chern–Simons action and we have the relation

$$\int_{M_4}F\wedge F=\int_{M_3}(A\wedge dA+\tfrac{2}{3}A\wedge A\wedge A). \tag{38}$$

This implies that the same geometrical property which is responsible for the Pontryagin term in 3+1 dimension induces the Chern–Simons term in a 2+1 dimension.

Thus we find that both in 3+1 and 2+1 dimensions the origin of the Pontryagin term and the Chern–Simons term, respectively, associates an inherent magnetic monopole like behavior which manifests its properties through the anisotropic feature of space–time when a direction vector (vortex line) is attached to a space–time point.

III. GAUGE ORBIT SPACE

We have shown above that there is a hidden Abelian gauge field in a non-Abelian gauge theory with Pontryagin term in 3+1 and Chern–Simons term in 2+1 dimension. Then as argued by Wu and Zee[1] the field theory under consideration may be regarded as a particle whose position is indicated by A (non-Abelian gauge potential) moving in the space U of non-Abelian gauge potentials under the influence of an Abelian electromagnetic potential A. In the language of differential forms we can write

$$A=g^{-1}\partial g+g^{-1}ag, \tag{39}$$

where

$$A=A_i\partial x^i, \quad a=a_i dx^i.$$

The space of gauge orbits U/G where G denotes the space of local gauge transformation $g(x)$ consists of the points $a(x)$.

Following Wu and Zee[1] we can consider a homotopic analysis of this gauge-orbit space. Recalling that $\pi_3(g)=Z$ for all simple non-Abelian groups G and $\pi_2(G)=0$, $\pi_n(U)=0$ for all n, this helps to write

$$\pi_n(U/G)=\pi_{n-1}(G) \quad \text{for } n>1. \tag{40}$$

In 3+1 dimension

$$\pi_1(U/G)=\pi_0(G)=\pi_3(G)=Z. \tag{41}$$

The equality $\pi_0(G)=\pi_3(G)$ follows from the condition that the gauge transformation $g(x)$ approaches a constant independent of the direction of x as $x\rightarrow\alpha$. Thus U/G is multiply connected and has the topology of a ring and the corresponding field strength corresponds to a vortex line. However in 2+1 dimension

$$\pi_1(U/G)=\pi_0(G)=\pi_2(G)=0. \tag{42}$$

Considering the second homotopy instead

$$\pi_2(U/G)=\pi_1(G)=\pi_3(G)=Z.$$

This suggests that U/G has a topology similar to that of a sphere and the corresponding field strength corresponds to a magnetic monopole.

From the quantum geometry as developed here we observed that in 3+1 dimension the Pontryagin term arises out of SL(2,c) invariance in spinor affine space where the gauge fields belong to SL(2,c). These SL(2,c) gauge fields may be traced back to be generated from the spinorial variables in the metric $g_{\mu\nu}(x,\theta,\bar{\theta})$ and corresponds to the direction vector (vortex line) attached to the space-time point.

We have shown that this gives rise to the Pontryagin index

$$q=\int J_0^2 d^3x=\int_{\text{surface}} \epsilon^{ijk} d\sigma i f_{jk}^{(2)} \quad (i,j,k=1,2,3),$$

which suggests $f_{jk}^{(2)}$ to be magnetic field like components for the background vector potential $b_i^{(2)}$, q is actually associated with the magnetic pole strength.

In this way the introduction of a direction vector or vortex line attached to a space–time point effectively associates a background magnetic field and the charge corresponding to the gauge field effectively represents magnetic charge. Thus the geometry involved in a vortex line may be associated with that of a charged particle moving in the field of a magnetic monopole. The fact that in 3+1 dimension the gauge orbit space U/G has the topology of a ring indicates that there is a hole in it. So the magnetic flux through the hole in the gauge orbit space is nonzero. In view of this the θ vacuum may be taken to arise from the Bohm–Ahranov type of effect in ordinary space. In 2+1 dimension the topology of the gauge orbit space U/G corresponding that of a sphere representing a magnetic monopole may thus be taken to arise from the same geometrical feature.

We may point out here that the situation is more general in this geometrical formalism. Indeed, the Pontryagin term is associated with the Berry phase[14] through the relationship with chiral anomaly and this Berry phase may be viewed as a generalized version of Bohm–Ahranov effect. In fact, it has been shown that the Berry phase is given by the relation $e^{i\phi_B}=e^{i2\pi\mu}$ where

$$2\mu=q=\int J_\nu^2 d^3x=\int \partial_\mu J_\mu^2 d^4x=-\frac{1}{2}\int \partial_\mu J_\mu^5 d^4x. \tag{44}$$

Thus the background Abelian gauge field when the θ-term is introduced in a non-Abelian gauge theory is effectively responsible for the Berry phase.

IV. QED IN SUPERSPACE AND ASYMPTOTIC FREEDOM

In a recent paper[13] it has been pointed out that the chiral description of the fermions in 3+1 dimension in terms of the spinorial variables $\theta,\bar{\theta}$ in the metric tensor $g_{\mu\theta}(x,\theta,\bar{\theta})$ giving rise to SL(2,c) gauge field current necessiates the introduction of a disconnected gauge group for the external Abelian field interacting with the matter field in a chiral symmetric way. In case, the external Abelian gauge field is the electromagnetic field (photon), the Lagrangian density is given by

$$L=-\bar{\psi}\gamma_\mu D_\mu\psi-\tfrac{1}{4}(\epsilon^{\alpha\beta\gamma\delta}\tilde{F}_{\alpha\beta}\tilde{F}_{\gamma\delta})-\tfrac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})+\mathrm{Tr}(J_\mu A^\mu). \tag{45}$$

Here, D_μ is the SL(2,c) gauge covariant derivative and considering the order of $\psi-B_\mu$ coupling to be negligible compared to matter current electromagnetic field coupling we can replace it by ∂_μ. We have

$$\tilde{F}_{\alpha\beta}=\partial_\alpha B_\beta-\partial_\beta B_\alpha+[B_\alpha,B_\beta],\quad B_\alpha\in[\mathrm{SL}(2e)]\,,$$

$$F_{\mu\nu}=\partial_\nu A_\mu-\partial_\mu A_\nu. \tag{46}$$

A_μ being the electromagnetic gauge potential and J_μ is the matter current matrix given by

$$J_\mu=\begin{bmatrix}\bar{\psi}_R\gamma_\mu\psi_R+J_\mu^2 & 0\\ 0 & \bar{\psi}_L\gamma_\mu\psi_L-J_\mu^2\end{bmatrix}. \tag{47}$$

J_μ^2 being the second component of the SL(2,c) gauge field current as discussed in the previous section. It is evident that this matrix structure of J_μ exhibiting chiral form suggests that for A_μ we should take the disconnected gauge group[15] $U_{1L}\otimes U_{1R}=U_1\times\{1,d\}$ where 'd' is the orientation reversing operation. Evidently in such an interaction the field strength and current are not gauge invariant but only gauge covariant each changing sign under 'd'. This is similar to the non-Abelian theories where field strengths and currents are only gauge covariant even under gauge transformations connected to the identity.

The internal symmetry group here is $O(2)$ which is given by the relation

$$O(2)=\mathrm{SO}(2)\times\{1,d\}=U_1\times\{1,d\}. \tag{48}$$

Indeed, we can take

$$A_\mu=\begin{bmatrix} A_{\mu+} & 0 \\ 0 & A_{\mu-} \end{bmatrix}. \tag{49}$$

Following KisKis,[15] we can think of a large system of observers each responsible for a small open region of the connected space–time manifold M. Let us consider that all the frames in M have the same orientation. Physically this means that the space is simply connected and the observer can give an unambiguous definition of positive charge everywhere. This suggests that we can introduce the connection (gauge field) in the Lagrangian

$$L^i=L_g^i+L_M^i, \tag{50}$$

where i identifies quantities associated with the region U_i, L_M^i is the matter field Lagrangian and L_g^i is the kinetic energy term for the connection. The gauge symmetry of the L_g^i is given by

$$A\rightarrow g^{-1}(\partial+A)g, \tag{51}$$

with g a smooth map

$$g=U_i\rightarrow O(2), \tag{52}$$

which may lie in either component of $O(2)$. A transformation that reverses the orientation at each point can be written as

$$g=dg_0,\quad g_0=U_i\rightarrow \mathrm{SO}(2),\quad d=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{53}$$

This gives

$$A=g_0^{-1}(\partial-A)g_v. \tag{54}$$

We see that it is a combination of charge conjugation and orientation preserving gauge rotation. Evidently in this formalism the chiral currents interact with the gauge field in a disconnected form. Indeed, writing

$$A_\mu=\begin{bmatrix} A_{\mu+} & 0 \\ 0 & A_{\mu-} \end{bmatrix},$$

we find the interaction term is given by

$$\begin{bmatrix} (\bar\psi_R\gamma_\mu\psi_R+J_\mu^2)A_{\mu+} & 0 \\ 0 & (\bar\psi_L\gamma_\mu\psi_L-J_\mu^2)A_{\mu-} \end{bmatrix}. \tag{55}$$

Evidently there is no term like $A_{\mu+}A_{\mu-}$ in the Lagrangian.

It is now noted that in this disconnected gauge field formalism the basic current is $\bar{\psi}_R\gamma_\mu\hat{\psi}_R+J_\mu^2$ or $\bar{\psi}_L\gamma_\mu\hat{\psi}_L-J_\mu^2$.

The background J_μ^2 current arising out of the anisotropic feature (the fictitious magnetic monopole) effectively can be represented by a fictitious chiral current. Indeed from the relation (32)

$$\partial_\mu J_\mu^2=-\tfrac{1}{2}\partial_\mu J_\mu^5,$$

we have the general solution

$$-J_\mu^2\sim J_\mu^5+J_\mu^V\sim\bar{\tilde{\psi}}_L\gamma_\mu\tilde{\psi}_L, \tag{56}$$

where $\tilde{\psi}_L$ is a left handed fictitious spinor. Similarly we can take

$$J_\mu^2=\tilde{J}_\mu^2\sim\bar{\tilde{\psi}}_R\gamma_\mu\tilde{\psi}_R.$$

This helps us to formulate SU(2) doublets $\binom{\psi_L}{\tilde{\psi}_L},\binom{\psi_R}{\tilde{\psi}_R}$ and we can associate the gauge field interaction in terms of the SU(2) triplet

$$\begin{pmatrix} A_{\mu+} \\ A_{\mu0} \\ A_{\mu-} \end{pmatrix},$$

with the following Lagrangians:

$$\begin{gathered}(\bar{\psi}_R\gamma_\mu\psi_R+\bar{\tilde{\psi}}_R\gamma_\mu\tilde{\psi}_R)A_{\mu+},\\ \tfrac{1}{2}(\bar{\psi}_R\gamma_\mu\psi_R-\bar{\psi}_L\gamma_\mu\psi_L+\bar{\tilde{\psi}}_R\gamma_\mu\tilde{\psi}_R-\bar{\tilde{\psi}}_L\gamma_\mu\tilde{\psi}_L)A_{\mu0},\\ (\bar{\psi}_L\gamma_\mu\psi_L+\bar{\tilde{\psi}}_L\gamma_\mu\tilde{\psi}_L)A_{\mu-},\end{gathered} \tag{57}$$

where

$$A_{\mu0}=\frac{1}{\sqrt{2}}\,(A_{\mu+}+A_{\mu-}).$$

This "effective non-Abelian nature" associated with the disconnected gauge fields helps us to suggest that the corresponding interactions are now asymptotically free.

It may be noted that the standard minimal electromagnetic interactions with the background magnetic field can be written as

$$L=ie\bar{\psi}\gamma_\mu\psi A_\mu+ie\bar{\tilde{\psi}}\gamma_\mu\tilde{\psi}A_\mu. \tag{58}$$

The fictitious spinor will give rise to counter terms which will cancel the divergences at each order in perturbation theory as these are of the same form as those appearing in the original Lagrangian. This is not surprising as we have shown earlier that the chiral anomaly which is generally considered to arise from regularization of short distance singularities has been removed in this formalism.

V. DISCUSSION

In this paper we have discussed the geometrical and topological feature of the inherent Abelian gauge field structure in a nonAbelian gauge field theory in 3+1 and in 2+1 dimensions. It has been observed that in 3+1 and in 2+1 dimensions the same geometrical feature is responsible for the generation of topological terms in the non-Abelian gauge field Lagrangian. This specific geometrical feature is responsible for the realization of the Berry phase. The quantization of a Fermi field in 3+1 dimension involves an inherent anisotropic feature associated with the introduction of a direction vector or vortex line attached to a space–time point. The motion of a charged particle in such an anisotropic space is analogous to that of a charged particle in the field of a magnetic monopole. The geometry of the anisotropic space and quantization of fermions in such a space ultimately gives rise to a topological index which is analogous to the magnetic pole strength.

This formalism suggests that the fermion current when split in chiral form helps us to formulate the electromagnetic interactions in disconnected form which suggests that in this case the theory becomes asymptotically free.

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