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A NOTE ON DETERMINATION OF BAYESIAN THREE-DECISION PLANS USING THYREGOD'S METHOD

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SUMMARY. Three-decision Bayesian plans minimising total average cost of inspection and decision are obtained using Thyregod's method under certain regularity conditions. An example is given to illustrate that the method is well adopted to the case of three-decision plan.

1. Introduction

Three-decision Bayesian plans (n, c_1, c_2) as discussed in Pandey (1974, 1984) can be obtained by using the unidimensional search method of Thyregod (1974). First the decision numbers c_1 and c_2 are obtained analytically as a function of the sample size n under certain conditions of regularity on posterior risk or loss and then a unidimensional search is made over n to obtain optimal plans against various lot sizes systematically. The method has been illustrated by an example. Although the three-decision plans involve two cost differences and complicated decision costs the method is well adopted.

2. Bayesian three-decision ASR plans

Bayesian three-decision ASR plan (n, c_1, c_2) is operated as follows :

From a lot of N items a random sample of n items is selected and number of defective x in the sample is found and the lot is

$$\begin{aligned} &\text{Accepted if } 0 \leqslant x \leqslant c_1 \\ &\text{Screened if } c_1 < x \leqslant c_2 \\ &\text{Rejected if } c_2 < x \leqslant n. \end{aligned} \quad \ldots \quad (2.1)$$

The cost per item for inspection, acceptance, screening and rejection are $k_s(p)$, $k_a(p)$, $k_t(p)$ and $k_r(p)$ respectively and satisfy the four assumptions for costs (AC): (1) All the functions are non-negative and none is identically zero; (2) $k_a(0) < k_t(0) < k_r(0)$; (3) $k_a(1) > k_t(1) > k_r(1)$; and (4) $k_s(p) \geqslant k_m(p)$ where $k_m(p)$ denotes minimum unavoidable cost.

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Assume that the variation of p from lot to lot may be described by the natural conjugate distribution of bernoulli process density i.e. by Beta distribution with parameter (η, δ) where

$$dW(p) = [B(\eta, \delta)]^{-1} p^{\eta-1} (1-p)^{\delta-1}, \; 0 \leqslant p \leqslant 1. \qquad \ldots \quad (2.2)$$

Let $x_i = 1$ or 0 according as i-th item in the sample is defective or non-defective and then $T_n = x_1 + x_2 + \ldots + x_n$ is sufficient for p in conditional density of t given p where density of T_n is

$$g_n(t \mid p) = \binom{n}{t} p^t (1-p)^{n-t} \qquad \ldots \quad (2.3)$$

which has obviously monotone likelihood ratio. Further, under the assumptions made for the cost functions $k_a(p)$, $k_t(p)$ and $k_r(p)$, we note that the cost differences $k_a(p)-k_t(p)$, $k_t(p)-k_r(p)$ have at the most one change of sign. In view of this, the three-decision acceptance sampling problem is monotonely regular following Thyregod (1974). Karlin and Rubin (1956) and Karlin (1968) also give such concepts of monotonely regular problems.

The expected cost for (n, c_1, c_2) is given by

$$K(N, n, c_1, c_2) = nk_s + (N-n)\, G(n, c_1, c_2) \qquad \ldots \quad (2.4)$$

where $\quad G(n, c_1, c_2) = \int \{k_a(p)\, P_a(p) + k_t(p) P_s(p) + k_r(p) P_r(p)\}\, dW(p)$

and $\quad P_a(p) = P(0 \leqslant x \leqslant c_1 \mid p)\,;\; P_s(p) = P(c_1 < x \leqslant c_2 \mid p)\,;$

$$P_r(p) = P(c_2 < x \leqslant n \mid p).$$

Since the three-decision ASR plan is monotonely regular, the following can be shown by trivial proofs in the light of Thyregod's (1974) results :

(a) the cost of optimal three-decision plan $K^0(N) = \inf\limits_{n, c_1, c_2} K(N, n, c_1, c_2)$ is a concave piece-wise linear function of N and the optimal sample size is increasing with lot size.

(b) the optimal decision rule for a given sample size n is: accept if $T_n \leqslant c_1$; screen if $c_1 < T_n \leqslant c_2$ and reject if $T_n > c_2$.

(c) the optimal acceptance numbers $c_1^0(n)$ and $c_2^0(n)$ are solutions to $\lambda_1(n, c_1) = 0$ and $\lambda_2(n, c_2) = 0$ where $\lambda_i(n, t) = \int l_i(p)\; g_n(t \mid p)\; dW(p)/g_n(t)$, $g_n(t) > 0$ $i = 1, 2$, give posterior cost differences with $l_1(p) = k_a(p) - k_t(p)$ and $l_2(p) = k_t(p) - k_r(p)$.

We obtain

$$c_1^0(n) = (n+\eta+\delta)\, p_u - \eta \qquad \ldots \quad (2.5)$$

and

$$c_2^0(n) = (n+\eta+\delta)\, p_v - \eta \qquad \ldots \quad (2.6)$$

where p_u and p_0 are solutions of $k_a(p) = k_l(p)$ and $k_s(p) = k_r(p)$ respectively and are called break-even quality in Pandey (1974). For preference of three-decision plan to two-decision the condition is $p_u < p_0$ which implies $c_1^0(n) < c_2^0(n)$. The smallest sample size for which sampling inspection may be preferred is obtained from $c_1(n) \geqslant 0$ as $n \geqslant \eta\ p_u^{-1} - \eta - \delta$. The expected decision cost for plan (n, c_1, c_2) is given by $G(n, c_1, c_2)$ which is a complicated function involving beta-binomial probabilities.

Define

$$K_{sub}(N, n) = \min_{i \leqslant n} K(N, i, c_1(i), c_2(i)), (n = 0, 1, 2, \ldots) \qquad \ldots \quad (2.7)$$

It can be observed that $K_{sub}(N, n)$ decreases to $K^0(N)$. Further, for some n if

$$K_{sub}(N, n) > K_{sub}(N, n+1)$$

holds for $N = N_0$ then it holds for all $N > N_0$. Thus, we can construct $K_{sub}(N, n+1)$ from $K_{sub}(N, n)$ by simply determining the intersection between $K\{N, n+1, c_1(n+1), c_2(n+1)\}$ and $K_{sub}(N, n)$. Since (2.7) describes piece-wise linear function of N, we may determine this intersection by solving a set of linear equations.

Since $K_{sub}(N, n) = K^0(n)$ for $N \leqslant n$ it suffices to perform successive determinations of $K_{sub}(N, n)$ for $n = 0, 1, 2, \ldots N_1$, in order to tabulate the sampling plans for lot of sizes $N < N_1$. The smallest sample size n satisfying $nk_s > K_{sub}(N_1, n)$ will be an upper limit for sample sizes to be considered.

3. Numerical example

Let $k_a(p) = p$, $k_l(p) = 0.2+0.5p$; $k_r(p) = 0.5$, $k_s(p) = 0.2+0.9p$. It can be noted that these cost functions satisfy the earlier assumptions for costs (AC).

We have $k_s = 0.7625$; $k_m = 0.4896$ for the beta prior distribution of p with parameter $\eta = 2.5$ and $\delta = 1.5$.

From (2.5) and (2.6) we obtain the sample size (n) and the values of optimal decision numbers c_1 and c_2. The values of the optimal decision cost $G(n, c_1, c_2)$ corresponding to the above triplets (n, c_1, c_2) are computed separately by means of a computer programme. The following table gives the optimal decision cost for sample sizes $n = 3, 4, 5, \ldots, 20$.

We shall perform successive determinations of $K_{sub}(N, n)$ for $n = 0, 1, 2, \ldots$

$$K_{sub}(N, 3) = 3\ k_s + 0.5125\ (N-3), (3 \leqslant N) \qquad \ldots \quad (3.1)$$

and

$$K(N, 4, c_1(4), c_2(4)) = 4\ k_s + 0.494922\ (N-4) \qquad \ldots \quad (3.2)$$

TABLE 1. THE OPTIMAL DECISION COST $C(n, c_1, c_2)$ FOR $dW(p) =$ Beta (p ; $\eta = 2.5$, $\delta = 1.5$)

n	c_1	c_2	$C(n, c_1, c_2)\times 10^{-3}$
3	0	1	512.5000
4	1	2	494.9220
5	1	3	476.5625
6	2	3	473.0328
7	2	4	472.2504
8	3	4	471.7254
9	3	5	471.2316
10	4	5	471.0049
11	4	6	470.9983
12	4	7	470.9356
13	5	7	470.9056
14	5	8	470.8806
15	5	9	470.8736
16	6	9	470.8686
17	6	10	470.8666
18	7	10	470.8651
19	7	11	470.8649
20	7	12	470.8648

From intersection of (3.1) and (3.2) we obtain

$$K_{sub}(N, 4) = \begin{cases} 3\,k_s + 0.5125\,(N-3), & (3 \leqslant N \leqslant 19) \\ 4\,k_s + 0.494922(N-4), & (20 \leqslant N). \end{cases}$$

Proceeding on the same line we find

$$K_{sub}(N, 5) = \begin{cases} 3\,k_s + 0.5125\,(N-3), & (3 \leqslant N \leqslant 19) \\ 4\,k_s + 0.494922(N-4), & (20 \leqslant N \leqslant 21) \\ 5\,k_s + 0.4765625(N-5), & (22 \leqslant N) \end{cases}$$

and

$$K_{sub}(N, 6) = \begin{cases} 3\,k_s + 0.5125\,(N-3), & (3 \leqslant N \leqslant 19) \\ 4\,k_s + 0.494922(N-4), & (20 \leqslant N \leqslant 21) \\ 5\,k_s + 0.4765625(N-5), & (22 \leqslant N \leqslant 103) \\ 6\,k_s + 0.4730328(N-6), & (104 \leqslant N). \end{cases}$$

Continuing in this way we get the optimal three-decision Bayesian plans as given in the Table 2.

TABLE 2. OPTIMUM BAYESIAN THREE-DECISION SAMPLING PLANS

N	n	c_1	c_2
3—19	3	0	1
20—21	4	1	2
22—103	5	1	3
104—217	6	2	3
218—551	7	2	4
552—597	8	3	4
598—1757	9	3	5
1758—4386	10	4	5
4387—4661	11	4	6
4662—9731	12	4	7
9732—11677	13	5	7
11678—41674	14	5	8
41675—58341	15	5	9
58342—145832	16	6	9
145833—194440	17	6	10
194441—1458193	18	7	10
1458194—2916371	19	7	11

4. Conclusion

Since the cost-differences $\lambda_1(n, t)$ and $\lambda_2(n, t)$ in case of other three-decision plans also could be shown to satisfy the regularity conditions as in the present case and other conditions for monotone likelihood ratio hold the present method is applicable.

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