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ON WAVES DUE TO ROLLING OF A VERTICAL PLATE

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Waves due to small rolling oscillations of a thin plate either partially immersed or completely submerged in deep water is studied in this paper by an integral equation formulation based on Green's integral theorem in the fluid region. The expressions for the amplitudes of the wave motion at large distances from the plate and the velocity potential are obtained explicitly for both the cases.

1. INTRODUCTION

The problem of generation of waves due to rolling of a thin vertical plate is among the few problems in the linearised theory which admits of closed form solution. Ursell[1] first considered this problem while studying the motion due to rolling of a ship wherein the ship was modelled as a thin vertical plate, partially immersed in deep water and is constrained to oscillate about a horizontal axis through it. He used Havelock's expansion of water wave potential and obtained the amplitude of the radiated waves at infinity. The velocity potential was not given explicitly in Ursell[1] but can be obtained from there (cf. Mandal Banerjea[2]). Later Evans[3] used a tailored version of Green's integral theorem to obtain the amplitude at infinity of the wave motion produced by the general motion of a partially immersed thin vertical plate without obtaining the velocity potential explicitly and deduced Ursell's result as a special case. Using this idea of Evans[3], recently Mandal[4] obtained the amplitude at infinity of the radiated waves due to small oscillations of a thin vertical plate submerged in deep water and deduced as a special case the results for rolling oscillations of the plate. In his analysis, the expression for velocity potential was also not obtained explicitly.

In the present paper the problems of generation of waves due to rolling oscillations of a thin vertical plate which is either partially immersed or completely submerged in deep water are investigated by an integral equation formulation based on Green's integral theorem in the fluid region. For both the problems, the mathematical analysis depends on the solution of a singular integral equation of first kind with a Cauchy type kernel. The expressions for the amplitudes of wave motion at infinity and the velocity potential for both the problems are obtained explicitly. To the best of our knowledge this method was not used in the literature for solving these two problems.

2. Formulation of the Problems

We consider a thin rigid vertical plate $x = 0$, $y \in L$ either partially immersed (for which $L \equiv [0, a]$) or completely submerged (for which $L \equiv [a, b]$) in an incompressible inviscid deep fluid occupying the region $y \geq 0$ with $y = 0$ as the mean free surface. The plate is hinged at $(0, s)$ and is forced to perform simple harmonic oscillations of amplitude $\theta = \text{Re}\,(\theta_\circ\, e^{-i\sigma t})$ about its mean vertical position, σ being the frequency of oscillation. Assuming the motion to be irrotational, it can be described by a velocity potential $\text{Re}\,[\phi(x, y)\, e^{-i\sigma t}]$, where ϕ satisfies

$$\nabla^2\phi = 0 \text{ in the fluid region} \qquad \ldots(2.1)$$

the linearised free surface condition

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \text{ on } y = 0 \qquad \ldots(2.2)$$

where $K = \sigma^2/g$, g being the acceleration due to gravity, the condition on the plate

$$\frac{\partial \phi}{\partial x} = i\sigma\theta_0\,(y - s) \text{ on } x = 0,\ y \in L \qquad \ldots(2.3)$$

the edge condition that

$$r^{1/2}\,\nabla\phi \text{ is bounded as } r \to 0 \qquad \ldots(2.4)$$

where r is the distance from the point $(0, a)$ for $L = [0, a]$ and from the points $(0, a)$ and $(0, b)$ for $L = [a, b]$, the bottom condition

$$\nabla\phi \to 0 \text{ as } y \to \infty. \qquad \ldots(2.5)$$

Also, ϕ is required to satisfy the radiation condition that

$$\phi \sim \begin{cases} A \exp(-Ky + iKx) \text{ as } x \to \infty \\ B \exp(-Ky - iKx) \text{ as } x \to -\infty \end{cases} \qquad \ldots(2.6)$$

where A and B are the amplitudes (unknown) of the wave motion at large distances from the plate on its two sides.

3. Method of Solution

By an appropriate use of Green's integral theorem in the fluid region to $\phi(x, y)$ and the two dimensional source potential $G(x, y; \xi, \eta)$ where G is given by (cf. Thorne[5]).

$$G(x, y; \xi, \eta) = 2\pi i \exp(-K(y+\eta) + iK\,|x - \xi|)$$

$$+ 2\int_0^\infty \frac{M(k, y)\, M(k, \eta)}{k\,(k^2 + K^2)} \exp(-K|x - \xi|)\, dk \qquad \ldots(3.1)$$

$$M(k, y) = k \cos ky - K \sin ky \qquad \ldots(3.2)$$

we find

$$2\pi\phi(\xi,\eta) = \int_L \int_L f(y)\, G_x\,(0, y; \xi, \eta)\, dy \qquad \ldots(3.3)$$

where

$$f(y) = \phi(+0, y) - \phi(-0, y),\ y \in L \qquad \ldots(3.4)$$

is the difference of potential across the plate.

Now using the condition (2.3) and noting that

$$G_{x\xi}\,(0, y; 0, \eta) = G_{\eta\eta}\,(0, y; 0, \eta)$$

we obtain the integral equation for $f(y)$ as

$$\frac{d^2}{d\eta^2} \int_L \int_L f(y)\, G(0, y; 0, \eta)\, dy = 2\pi i\sigma\theta_0\,(\eta - s),\ \eta \in L. \qquad \ldots(3.5)$$

Integrating (3.5) with respect to η and adding K times the result with (3.5) we obtain the following singular integral equation

$$\frac{1}{\pi} \int_L \frac{2y\lambda(y)}{y^2 - \eta^2}\, dy = 2i\sigma\theta_0(\eta) + c,\ \eta \in L \qquad \ldots(3.6)$$

where c is an unknown constant of integration,

$$g(\eta) = \frac{K}{2}\,\eta^2 + \eta(1 - Ks) - s \qquad \ldots(3.7)$$

and

$$\lambda(y) = Kf(y) + f'(y). \qquad \ldots(3.8)$$

We now treat the cases $L = [0, a]$ and $L = [a, b]$ separately.

4. Rolling of a Partially Immersed Plate

In this case, $L = [0, a]$ and the solution of the integral equation (3.6) satisfying

$$\lambda(y) = O((y^2 - a^2)^{-1/2}) \text{ as } y \to a$$

is given by (cf. Mikhlin[6])

$$\lambda(y) = c\lambda_0(y) - \frac{4i}{\pi}\,\lambda_1(y) \qquad \ldots(4.1)$$

where,

$$\lambda_0(y) = y(a^2 - y^2)^{-1/2}$$

$$\lambda_1(y) = y(a^2 - y^2)^{-1/2} \int_0^a \frac{(a^2 - \eta^2)^{1/2}}{\eta^2 - y^2}\, g(\eta)d\eta. \qquad \ldots(4.2)$$

Thus, from (3.8)

$$f(y) = \exp(-Ky) \int_a^y \exp(Kt)\, \lambda(t)\, dt. \qquad \ldots(4.3)$$

To determine the unknown constant c, $f(y)$ from (4.3) is substituted in the original integral equation (3.5). This gives

$$i\pi\sigma\theta_0\ (\eta - s) = \frac{\pi i}{2} K \exp(-K\eta) \int_{-a}^{a} \lambda(y) \exp(-Ky)\, dy$$

$$+ \int_0^\infty \frac{kM(k,\eta)}{k^2 + K^2} \left(\int_0^a \lambda(y) \sin ky\, dy\right) dk, \text{ for } 0<\eta<a. \qquad \ldots(4.4)$$

The y-integral in the second term on the right side of (4.4) can be simplified by using the results obtained by a simple application of Cauchy's integral theorem with an appropriate choice of contour

$$\int_0^a \lambda_0\ (y) \sin ky\, dy = \int_0^a \cos ky\, dy - \int_a^\infty \left(\frac{y}{(y^2 - a^2)^{1/2}} - 1\right) \cos ky\, dy$$

$$\int_0^a \lambda_1\ (y) \sin ky\, dy = -\frac{\pi}{2} \int_0^a \cos ky\, g(y)\, dy - \int_a^\infty \frac{yF(a,y)}{(y^2 - a^2)^{1/2}} \cos ky\, dy \qquad \ldots(4.5)$$

where

$$F(a,y) = \int_0^a \frac{(a^2 - \eta^2)^{1/2}}{\eta^2 - y^2} g(\eta)\, d\eta. \qquad \ldots(4.6)$$

Using these results in (4.4) it is found that after further simplification, the term $\pi i\sigma\theta_0$ $(\eta - s)$ cancels from both sides giving rise to an identity in η as

$$\exp(-K\eta) \left[-iac\,\Delta + \frac{4}{\pi} (\alpha_1 - i\beta_1)\right] = 0, \quad 0 < \eta < a \qquad \ldots(4.7)$$

where

$$\Delta = \pi I_1\ (Ka) + iK_1\ (Ka)$$

$$\alpha_1 = \sigma\theta_0 \left[-\pi a^2 I_1\ (Ka) - 4aI_1\ (Ka) \left(-\frac{K\pi a^2}{8} - a(1 - Ks) + \frac{s\pi}{2}\right)\right.$$

$$\left. + 2(1 - Ks) \frac{\pi a}{K} (I_0\ (Ka)\, \mathbf{L}_1\ (Ka) - \mathbf{L}_0\ (Ka)\, I_1\ (Ka))\right]$$

and

$$\beta_1 = \sigma\theta_0 \left[-a^2 K_2(Ka) + \frac{4a}{\pi} K_1\ (Ka) \left(\frac{-K\pi a^2}{8} - a(1 - Ks) + \frac{s\pi}{2}\right)\right.$$

$$\left. + \frac{\pi}{2K^2} (1 - Ks) + 2(1 - Ks) \frac{a}{K} (K_0(Ka)\mathbf{L}_1(Ka) + \mathbf{L}_0(Ka)K_1(Ka))\right] \qquad \ldots(4.8)$$

I_0 (Ka), I_1 (Ka), K_0 (Ka), K_1 (Ka) being the modified Bessel functions and $\mathbf{L}_0$ (Ka), $\mathbf{L}_1$ (Ka) being modified struve function.

Equation (4.7) results in an equation for c from which it is found that

$$c = -\left(\frac{a\pi\Delta}{4}\right)^{-1}(\beta_1 + i\alpha_1). \qquad \ldots(4.9)$$

To obtain the amplitude of the radiated waves at infinity, we make $\xi \to \pm\infty$ in (3.3) and it is found that

$$A = -B = \frac{1}{2}\int_{-a}^{a} \exp(-Ky)\,\lambda(y)\,dy.$$

Using (4.1), (4.2), (4.8) and (4.9) this is simplified as

$$A = -\frac{\pi\sigma\theta_0 a^2}{\Delta Ka}\left[\frac{1}{2} + \frac{Ks-1}{Ka}\,(I_1\,(Ka) + L_1(Ka))\right]. \qquad \ldots(4.10)$$

This expression coincides with the result given by Evans[3] obtained by a different method. This can also be identified with the result given by Ursell[1] (also see Mandal and Banerjea[2]). Finally from (3.3), the velocity potential is obtained as

$$\phi(\xi,\eta) = A\exp(-K\eta + iK\xi) + \int_0^\infty \frac{\chi\,(k)}{k^2 + K^2}\exp(-k\xi)\,dk,\ \xi > 0$$

$$\phi(\xi,\eta) = -\phi(-\xi,\eta) \text{ for } \xi < 0 \qquad \ldots(4.11)$$

where

$$\chi\,(k) = -\frac{1}{\pi}\int_0^a \lambda(y)\sin ky\,dy. \qquad \ldots(4.12)$$

Using (4.1), (4.2), (4.5) and (4.6), (4.12) can be simplified as (cf Gradshyteyn and Ryzhik[7])

$$\chi\,(k) = -\sigma\theta_0\left[pJ_1\,(ka) - \frac{iKa^2}{2k}J_2\,(ka) + \frac{ia}{k}\,(Ks-1)(\mathbf{H}_0(ka)J_1(ka) - \mathbf{H}_1(ka)J_0(ka))\right] \qquad \ldots(4.13)$$

where,

$$p = (K_2(Ka) + i\pi I_2\,(Ka))\,\frac{a^2}{2\Delta} - \frac{1-Ks}{\Delta K^2} - \frac{a}{\Delta K}\,(1-Ks)\,\mathbf{L}_1(Ka)(K_0(Ka) + i\pi I_0\,(Ka)) + i(1-Ks)\,\frac{a\pi}{K}\,\mathbf{L}_0\,(Ka) \qquad \ldots(4.14)$$

J_0 (ka), J_1 (ka), J_2 (ka) are Bessel function and $\mathbf{H}_0$ (ka), $\mathbf{H}_1$ (ka) are the Struve functions. These results can be identified with the results given by Ursell[1] (cf. Mandal and Banerjea[2]).

5. Rolling of a Submerged Plate

In this case $L = [a, b]$ and the solution of the integral equation (3.5) satisfying

$$\lambda(y) = \begin{cases} 0\,((y^2 - a^2)^{-1/2}) & \text{as } y \to a \\ 0\,((b^2 - y^2)^{-1/2}) & \text{as } y \to b \end{cases}$$

is given by (cf. Mikhlin[6])

$$\lambda(y) = \frac{1}{\rho(y)}\left[\frac{4i}{\pi} F(a, b, y) + D - cy^2\right] \qquad \ldots(5.1)$$

where c and D are unknown constants and

$$F(a, b, y) = \int_a^b \frac{\eta g(\eta)\rho(\eta)d\eta}{y^2 - \eta^2} \qquad \ldots(5.2)$$

$$\rho(y) = (y^2 - a^2)^{1/2}\,(b^2 - y^2)^{1/2}.$$

Hence from (3.7)

$$f(y) = \exp\,(-Ky) \int_a^y \lambda(t) \exp\,(Kt)\, dt. \qquad \ldots(5.3)$$

To find the unknown constants, we make use of the fact that

$$f(b) = 0.$$

This gives a relation between c and D as

$$D\alpha_1\,(-K) - c\alpha_1''(-K) + \frac{4i}{\pi}\,\alpha_1\,(-K, F) = 0 \qquad \ldots(5.4)$$

where

$$\alpha_1\,(K, F) = \int_a^b \frac{\exp\,(-Ku)\, F(a, b, u)}{\rho(u)}\, du$$

$$\alpha_1\,(K) \equiv \alpha_1\,(K, 1) \text{ and } \alpha_1''\,(K) = \frac{d^2}{dK^2}\,\alpha_1\,(K). \qquad \ldots(5.5)$$

Let us write

$$d^2 = \frac{\alpha_1''(-K)}{\alpha_1\,(-K)}. \qquad \ldots(5.6)$$

Then from (5.4)

$$\frac{D}{c} = d^2 - \frac{4i}{\pi c}\,\frac{\alpha_1\,(-K, F)}{\alpha_1\,(-K)}. \qquad \ldots(5.7)$$

Using (5.7) in (5.1) we obtain $\lambda(y)$ as

$$\lambda(y) - c\lambda_0(y) + \frac{4i}{\pi}\lambda_1(y) \qquad \text{...(5.8)}$$

where $\lambda_0(y) = \dfrac{d^2 - y^2}{\rho(y)}$

$$\lambda_1(y) = \left\{F(a, b, y) - \frac{\alpha_1(-K, F)}{\alpha_1(-K)}\right\}\frac{1}{\rho(y)} \qquad \text{...(5.9)}$$

Having expressed D in terms of c by (5.7), it now remains to find c. To find it, $f(y)$ from (5.3) is substituted in (3.5) to obtain for $a < \eta < b$

$$\pi i\sigma\theta_0(\eta - s) = \frac{d^2}{d\eta^2}\left[\pi i \exp(-K\eta)\int_a^b \exp(-Ky)\,\lambda(y)\,dy + \int_0^\infty \frac{M(k, \eta)}{k(k^2 + K^2)}\,dk\int_a^b \sin ky\,\lambda(y)\,dy\right]. \qquad \text{...(5.10)}$$

Using Cauchy's integral theorem with appropriate choice of contour the following results can be derived.

$$\int_a^b \lambda_0(y)\sin ky\,dy = \int_0^a\left(\frac{d^2 - y^2}{R_0(y)} - 1\right)\cos ky\,dy - \int_b^\infty\left(\frac{d^2 - y^2}{R_1(y)} + 1\right)\cos ky\,dy - \int_a^b \cos ky\,dy$$

$$\int_a^b \frac{\sin ky}{\rho(y)}\,dy = \int_0^a \frac{\cos ky}{R_0(y)}\,dy - \int_0^\infty \frac{\cos ky}{R_1(y)}\,dy$$

$$\int_a^b \frac{F(a, b, y)}{\rho(y)}\sin ky\,dy = \int_0^a \frac{F(a, b, y)}{R_0(y)}\sin ky\,dy - \int_b^\infty \frac{F(a, b, y)}{R_1(y)}\sin ky\,dy + \frac{\pi}{2}\int_a^b g(y)\cos ky\,dy \qquad \text{...(5.11)}$$

where

$$R_0(y) = (a^2 - y^2)^{1/2}(b^2 - y^2)^{1/2},\ y < a$$

$$R_1(y) = (y^2 - a^2)^{1/2}(y^2 - b^2)^{1/2},\ y > b. \qquad \text{...(5.12)}$$

Using (5.8), (5.9), (5.11), (5.12), the y-integral in (5.10) is simplified and it is found that the term $\pi i\sigma\theta_0(\eta - s)$ cancels from both sides of the results thus derived to obtain an identity in η for $a < \eta < b$. This identity gives rise to an equation from which the constant c is determined as

$$c = \frac{i}{\Delta}\left(\frac{4}{\pi}\Delta_1 - 2S(K)\right) \qquad \ldots(5.13)$$

where

$$\alpha_0 = \int_{-a}^{a} \frac{d^2 - u^2}{R_0(u)} \exp(-Ku)\, du$$

$$\beta_0 = \int_{b}^{\infty} \frac{d^2 - u^2}{R_1(u)} \exp(-Ku)\, du$$

$$\gamma_0 = \int_{a}^{b} \frac{d^2 - u^2}{\rho(u)} \exp(-Ku)\, du$$

$$\Delta = \alpha_0 - \beta_0 - i\gamma_0 \qquad \ldots(5.14)$$

$$\alpha_2(K, F) = \int_{-a}^{a} \frac{F(a, b, u)}{R_0(u)} \exp(-Ku)\, du$$

$$\alpha_3(K, F) = \int_{b}^{\infty} \frac{F(a, b, u)}{R_1(u)} \exp(-Ku)\, du$$

$$\alpha_i(K) \equiv \alpha_i(K, 1),$$

$$\Delta_{2i} = \begin{vmatrix} \alpha_i(K) & 1 \\ \alpha_i(K, F) & \dfrac{\alpha_1(-K, F)}{\alpha_1(-K)} \end{vmatrix}$$

$$\Delta_1 = \Delta_{22} - \Delta_{23} - i\Delta_{21},\ i = 1, 2, 3,$$

$$S(K) = \int_{a}^{b} \exp(-Kt)\, g(t)\, dt - \sigma\theta_0 \exp(Ka)\left(\frac{a^2}{2} - as\right). \qquad \ldots(5.15)$$

The amplitude at infinity is obtained by making $\xi \to \pm\infty$ in (3.2) and it is found that

$$A = -B$$

where

$$A = \frac{2i\gamma_0}{\Delta} S(K) + \frac{4i}{\pi\Delta_0}\left(\gamma_0(\Delta_{22} - \Delta_{23}) - \Delta_{21}(\alpha_0 - \beta_0)\right). \qquad \ldots(5.16)$$

The above result can be further simplified in terms of complete elliptic functions to coincide with the results of Mandal[4] who however used a different method to obtain it. It may be noted that the function $F(a, b, y)$ in (5.2) with $g(\eta)$ given by (3.7) can be expressed as combination of complete elliptic integrals (cf. Evans[8]). It can be shown by making μ $(= a/b) \rightarrow 0$ in (5.16) that A coincides with the corresponding expression for amplitude at infinity due to rolling of partially immersed plate as given in (4.10).

Finally the expression for velocity potential can be obtained explicitly from (3.2) after using (5.3), (5.8) and (5.9).

6. Conclusion

The method based on Green's integral theorem is used here to obtain the closed form solution of the problem of rolling of a vertical plate which is either partially immersed or submerged in deep water. The amplitude of the radiated waves at infinity for the rolling of a partially immersed plate agrees with the known results obtained by Ursell[1] and Evans[3] earlier using different methods. Explicit expressions for the velocity potential is also obtained. For the submerged plate, similar results are also obtained.

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