Inequalities for the q-Permanent. II

A. K. Lal*
Indian Statistical Institute
New Delhi-110016, India

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ABSTRACT

For a complex number q, the q-permanent of an $n \times n$ complex matrix $A = ((a_{ij}))$, written $\text{per}_q(A)$, is defined as

$$\text{per}_q(A) = \sum_{\sigma \in \mathscr{S}_n} q^{l(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $\mathscr{S}_n$ is the symmetric group of degree n, and $l(\sigma)$ the number of inversions of σ [i.e., the number of pairs i, j such that $1 \leqslant i < j \leqslant n$ and $\sigma(i) > \sigma(j)$]. The function is of interest in that it includes both the determinant and the permanent as special cases. It is known that if A is positive semidefinite and if $-1 \leqslant q \leqslant 1$, then $\text{per}_q(A) \geqslant 0$. We obtain results for the q-permanent, a few of which are generalizations of some results of Ando.

1. INTRODUCTION

Let $\mathscr{S}_n$ denote the symmetric group on n symbols. If $\sigma \in \mathscr{S}_n$, then $l(\sigma)$ will denote the number of inversions of σ. Recall that an inversion of σ is a pair (i, j) such that $1 \leqslant i < j \leqslant n$ and $\sigma(i) > \sigma(j)$.

*The author thanks the University Grants Commission of India for a research fellowship.
Address correspondence to: A. K. Lal, Dept. of Maths, IIT Kanpur, Kanpur, India 208016.

For a square matrix A, we write $A \geqslant 0$ to indicate that A is Hermitian positive semidefinite. Also, $A \geqslant B$ means that $A \geqslant 0$, $B \geqslant 0$, and $A - B \geqslant 0$.

For a complex number q, the *q-permanent* of an $n \times n$ matrix $A = ((a_{ij}))$, denoted by $\operatorname{per}_q(A)$, is defined as

$$\operatorname{per}_q(A) = \sum_{\sigma \in \mathscr{S}_n} q^{l(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Observe that $\operatorname{per}_{-1}(A) = \det A$, $\operatorname{per}_0(A) = \prod_{i=1}^n a_{ii}$, and $\operatorname{per}_1(A) = \operatorname{per} A$, where det and per denote determinant and permanent, respectively. Here we have made the usual convention that $0^0 = 1$. The q-permanent thus provides a parametric generalization of both the determinant and the permanent. The main purpose of this paper is to prove inequalities for the q-permanent for $q \in [-1, 1]$ which generalize the results in [2].

We introduce further notation. If $m \geqslant 2$ is an integer and q a complex number, then we define the generalized factorial of a positive integer m as

$$m!! = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{m-1}).$$

We also set $0!! = 1!! = 1$. For convenience our notation here does not involve q. Note that if $q = -1$ then $m!! = 0$ for $m \geqslant 2$. For convenience we will make the convention that $0/0 = 0$ throughout. This saves us the trouble of considering the case $q = -1$ separately at each stage. We will also use the fact that $l(\sigma) = l(\sigma^{-1})$.

For an $n \times n$ matrix A and for $\sigma \in \mathscr{S}_n$ we define $d_A(\sigma) = \prod_{i=1}^n a_{i\sigma(i)}$, the diagonal product of A corresponding to the permutation σ. If A is an $n \times n$ matrix, then the Schur power matrix of A, denoted by $\Pi(A)$, has been defined as follows (see, for example, [8]). The rows and columns of $\Pi(A)$ are indexed by $\mathscr{S}_n$. If $\sigma, \tau \in \mathscr{S}_n$, then the (σ, τ) entry of $\Pi(A)$ is

$$d_{\tau\sigma^{-1}}(A) = \prod_{i=1}^{n} a_{\sigma(i)\tau(i)}.$$

It can be seen that $\Pi(A)$ is a principal submatrix of the Kronecker product $\otimes^n A = A \otimes A \otimes \cdots \otimes A$ taken n times, and therefore, if $A \geqslant 0$ then $\Pi(A) \geqslant 0$.

If q is a complex number, define the $n! \times n!$ matrix $\Gamma_{n,q}$ as follows. The rows and columns of $\Gamma_{n,q}$ are indexed by $\mathscr{S}_n$. If $\sigma, \tau \in \mathscr{S}_n$, then the (σ, τ)

entry of $\Gamma_{n,q}$ is $q^{l(\tau\sigma^{-1})}$. We set

$$\Pi_q(A) = \Pi(A) \circ \Gamma_{n,q}$$

where $\circ$ denotes the Hadamard product.

Let V be the Euclidean space of all complex-valued functions on $\mathcal{S}_n$. Then V is of dimension $n!$. The canonical Kronecker basis for V is $\{\delta_\sigma : \sigma \in \mathcal{S}_n\}$ which is ordered lexicographically. Then it is clear that the space $L(V)$ of linear endomorphisms on V can be identified with $n! \times n!$ complex matrices. For $\sigma \in \mathcal{S}_n$, we define, $U_\sigma \in L(V)$ by $U_\sigma \delta_\tau = \delta_{\sigma\tau}$. Then $U : \mathcal{S}_n \to L(V)$ is known as the left regular representation on $\mathcal{S}_n$. It is easily seen that we also have

$$\Pi(A) = \sum_{\sigma \in \mathcal{S}_n} d_A(\sigma) U_{\sigma^{-1}},$$

and hence we get

$$\Pi_q(A) = \sum_{\sigma \in \mathcal{S}_n} q^{l(\sigma)} d_A(\sigma) U_{\sigma^{-1}}.$$

Observe that

$$\mathrm{per}_q(A) = \frac{1}{n!} \langle \Pi_q(A)\mathbf{1}, \mathbf{1} \rangle \tag{1}$$

where $\mathbf{1}$ is the column vector of all ones. It has been proved by Bożejko and Speicher [4] that if $q \in [-1, 1]$ then $\Gamma_{n,q} \geqslant 0$. It then follows from (1) that if $A \geqslant 0$ and $q \in [-1, 1]$ then $\mathrm{per}_q(A) \geqslant 0$.

2. GRAM'S INEQUALITY FOR THE SCHUR POWER MATRIX

We need to develop some preliminaries before coming to the main result in this section. The following notation will be used. If $\pi \in \mathcal{S}_n$, then P^π will denote the permutation matrix corresponding to π. Thus the (i, j) entry of P^π is 1 if $j = \pi(i)$ and 0 otherwise. We denote by $\Omega(r, n)$ the set of all r-tuples $\mathbf{k} = (k_1, \ldots, k_r)$ of nonnegative integers $k_1, \ldots, k_r$ such that $\sum_{i=1}^r k_i = n$.

Let $B = (B_1, B_2, \ldots, B_r)$ be an $n \times r$ matrix, where B_i denotes the ith column of B. If $\mathbf{k} \in \Omega(r, n)$, then we denote by $B(\mathbf{k})$ the $n \times n$ matrix

$$\Big(\underbrace{B_1, \ldots, B_1}_{k_1}, \underbrace{B_2, \ldots, B_2}_{k_2}, \ldots, \underbrace{B_r, \ldots, B_r}_{k_r}\Big).$$

We will deal with partitions of $\{1, 2, \ldots, n\}$. For $\mathbf{k} \in \Omega(r, n)$ let $\mathscr{P}(\mathbf{k})$ denote the collection of all partitions

$$\mathscr{U} = (U_1, \ldots, U_r)$$

of $\{1, 2, \ldots, n\}$ such that the cardinality of U_i is k_i for $i = 1, 2, \ldots, r$.

Choose and fix $\mathbf{k} \in \Omega(r, n)$. Let $I_1 = \{1, \ldots, k_1\}$, $I_2 = \{k_1 + 1, \ldots, k_1 + k_2\}, \ldots,$ $I_r = \{n - k_r + 1, \ldots, n\}$. We call $\mathscr{I} = (I_1, \ldots, I_r)$ the *standard partition* in $\mathscr{P}(\mathbf{k})$.

If $\mathscr{U} \in \mathscr{P}(\mathbf{k})$ and if $\sigma \in \mathscr{S}_n$, then we say that $\sigma \ll \mathscr{U}$ if σ maps U_j onto I_j for $j = 1, 2, \ldots, r$.

Let $\mathscr{U} \in \mathscr{P}(\mathbf{k})$, and let B be an $n \times r$ matrix. We define

$$e_B(\mathscr{U}) = \prod_{j=1}^{r} \prod_{i \in U_j} b_{ij}.$$

Suppose $-1 < q \leqslant 1$. We construct matrices G, H as follows. Both G and H are square matrices whose order is the same as the cardinality of $\mathscr{P}(\mathbf{k})$ (which equals $n!/k_1! \cdots k_r!$). The rows and columns of G, H are indexed by $\mathscr{P}(\mathbf{k})$. If $\mathscr{U}$, $\mathscr{V} \in \mathscr{P}(\mathbf{k})$, then the $(\mathscr{U}, \mathscr{V})$ entry of G and H is defined as

$$g_{\mathscr{U},\mathscr{V}} = \frac{1}{k_1! \cdots k_r!} z_{\mathscr{U},\mathscr{V}} = \frac{1}{k_1! \cdots k_r!} \sum_{\sigma \ll \mathscr{U}} \sum_{\tau \ll \mathscr{V}} q^{l(\tau\sigma^{-1})} \tag{2}$$

and

$$h_{\mathscr{U},\mathscr{V}} = \frac{1}{k_1!! \cdots k_r!!} \sum_{\sigma \ll \mathscr{U}} \sum_{\tau \ll \mathscr{V}} q^{l(\sigma) + l(\tau)}.$$

Lemma 1. *With the above notation, $G \geqslant H$. Equality holds if $q = 1$.*

Proof. See Lemma 2 in Bapat and Lal [3]. ■

The main result of this section is *Gram's inequality for the Schur power matrix*, which in turn gives us Gram's inequality for the q-permanent. The result generalizes Theorem 1 in Bunce [5].

THEOREM 2. *Let $A \geqslant 0$ be an $n \times n$ matrix, and suppose $A = BB^*$, where $B = (B_1, \ldots, B_r)$ is $n \times r$. Then for $q \in [-1, 1]$,*

$$\Pi_q(A) \geqslant \sum_{\mathbf{k} \in \Omega(r,n)} \frac{1}{k_1!! \cdots k_r!!} \Pi_q(B[1,2,\ldots,n \mid \mathbf{k}])$$

$$\times \Pi_q(B^*[\mathbf{k} \mid 1,2,\ldots,n]). \qquad (3)$$

Equality holds in (3) *if* $q = \pm 1$.

Proof. First suppose $-1 < q \leqslant 1$. Using the fact that the q-permanent is a multilinear function of each column, we have

$$\Pi_q(A) = \Pi_q(BB^*)$$

$$= \Pi_q\left(\sum_{j=1}^{r} \overline{b_{1j}} B_j, \ldots, \sum_{j=1}^{r} \overline{b_{nj}} B_j \right)$$

$$= \sum_{\mathbf{k} \in \Omega(r,n)} \frac{1}{k_1! \cdots k_r!} \sum_{\pi \in \mathcal{S}_n} \overline{d_{B(\mathbf{k})}(\pi)} \, \Pi_q(B(\mathbf{k}) P^{\pi}).$$

Thus the result will be proved if we show that for any $\mathbf{k} \in \Omega(r, n)$

$$\frac{1}{k_1! \cdots k_r!} \sum_{\pi \in \mathcal{S}_n} \overline{d_{B(\mathbf{k})}(\pi)} \, \Pi_q(B(\mathbf{k}) P^{\pi})$$

$$\geqslant \frac{1}{k_1!! \cdots k_r!!} \Pi_q(B[1,2,\ldots,n \mid \mathbf{k}]) \Pi_q(B^*[\mathbf{k} \mid 1,2,\ldots,n]). \quad (4)$$

Fix $\mathbf{k} \in \Omega(r, n)$, and let $C = B(\mathbf{k})$. Then

$$\sum_{\pi \in \mathcal{S}_n} \overline{d_{B(\mathbf{k})}(\pi)}\, \Pi_q(B(\mathbf{k})P^{\pi}) = \sum_{\sigma \in \mathcal{S}_n} \overline{d_C(\sigma)}\, \Pi_q(C_{\sigma(1)}, \dots, C_{\sigma(n)})$$

$$= \sum_{\sigma, \tau \in \mathcal{S}_n} \overline{d_C(\sigma)}\, d_C(\tau) q^{l(\tau\sigma^{-1})} U_\sigma U_{\tau^{-1}}$$

$$= \sum_{\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathbf{k})} \overline{e_B(\mathcal{U})}\, e_B(\mathcal{V}) z_{\mathcal{U}\mathcal{V}} U_\tau U_{\sigma^{-1}}, \quad (5)$$

where $z_{\mathcal{U}\mathcal{V}}$ has been defined in (2). Also,

$$\pi_q(B[1, 2, \dots, n \mid \mathbf{k}])\Pi_q(B^*[\mathbf{k} \mid 1, 2, \dots, n])$$

$$= \left(\sum_{\sigma \in \mathcal{S}_n} q^{l(\sigma)} d_C(\sigma) U_{\sigma^{-1}} \right) \overline{\left(\sum_{\tau \in \mathcal{S}_n} q^{l(\tau)} d_C(\tau) U_{\tau^{-1}} \right)}$$

$$= \sum_{\sigma, \tau \in \mathcal{S}_n} \overline{d_C(\tau)}\, d_C(\sigma) q^{l(\sigma) + l(\tau)} U_\tau U_{\sigma^{-1}}$$

$$= \sum_{\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathbf{k})} \overline{e_B(\mathcal{U})}\, e_B(\mathcal{V}) U_\tau U_{\sigma^{-1}} \sum_{\sigma \ll \mathcal{U}} \sum_{\tau \ll \mathcal{V}} q^{l(\sigma) + l(\tau)}. \quad (6)$$

Lemma 1, (5), and (6) together imply (4). If $q = 1$, then the expressions (5), (6) are equal and (4) holds with equality. If $q = -1$, then again, keeping in mind our convention that $0/0 = 0$, it is easily seen that equality holds in (4). That completes the proof. ■

COROLLARY 3. *Let $A \geqslant 0$ be an $n \times n$ complex matrix, and suppose that $A = BB^*$, where $B = (B_1, \dots, B_r)$ is $n \times r$. Then for $q \in [-1, 1]$*

$$\operatorname{per}_q(A) \geqslant \sum_{\mathbf{k} \in \Omega(r, n)} \frac{1}{k_1!! \cdots k_r!!} |\operatorname{per}_q(B(\mathbf{k}))|^2. \quad (7)$$

Equality holds in (7) if $q = \pm 1$.

Proof. From (3), we have

$$\langle \Pi_q(A)\mathbf{1}, \mathbf{1}\rangle \geqslant \left\langle \sum_{\mathbf{k}\in\Omega(r,n)} \frac{1}{k_1!!\cdots k_r!!}\Pi_q(B[1,2,\dots,n \mid \mathbf{k}]) \times \Pi_q(B^*[\mathbf{k} \mid 1,2,\dots,n])\mathbf{1}, \mathbf{1}\right\rangle.$$

which gives us the result. ∎

3. INEQUALITIES USING INDUCED MATRICES

We now define the kth *induced matrix* $P_{k,q}(A)$ of an $n \times n$ complex matrix A. It is the $\binom{n+k-1}{k}$-square matrix whose entries are

$$\frac{\operatorname{per}_q(A[\alpha \mid \beta])}{\sqrt{\mu_q(\alpha)\mu_q(\beta)}}$$

arranged lexicographically in α, $\beta \in G_{k,n}$. Recall that $G_{k,n}$ denotes the set of all nondecreasing sequences $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of integers, $1 \leqslant \alpha_i \leqslant n$, $i = 1, 2, \dots, k$, that $A[\alpha \mid \beta]$ is the $k \times k$ matrix whose (i, j) entry is $a_{\alpha_i\beta_j}$, and that $\mu_q(\alpha)$ is the product of the generalized factorials of the multiplicities of the distinct integers in α. We will write n_k in place of $\binom{n+k-1}{k}$.

THEOREM 4. *For $q \in [-1, 1]$ the map $A \mapsto P_{k,q}(A)$ possesses the following properties*:

(i) $P_{k,q}(I_n) = I_{n_k}$, *I_n being the identity matrix of order n.*
(ii) $P_{k,q}(cA) = c^k P_{k,q}(A)$ *and* $P_{k,q}(A^*) = P_{k,q}(A)^*$.
(iii) $P_{k,q}(A)$ *is positive semidefinite if A is so.*
(iv) $P_{k,q}(A + B) \geqslant P_{k,q}(A) + P_{k,q}(B)$ *if A, B are positive semidefinite.*
(v) $P_{k,q}((A + B)^{1/k}) \geqslant P_{k,q}(A^{1/k}) + P_{k,q}(B^{1/k})$ *if A, B are positive semidefinite.*

Proof. (i) and (ii) can be easily verified. For (iii) we proceed as follows.

For each $\alpha, \beta \in G_{k,n}$ consider the matrix $A[\alpha \mid \beta]$, which is a $k \times k$ matrix. Then $\Pi(A[\alpha \mid \beta])$ is a $k! \times k!$ matrix. If $\sigma, \tau \in \mathscr{S}_k$ then the (σ, τ) entry of $\Pi(A[\alpha \mid \beta])$ is

$$d_{A[\alpha \mid \beta]}(\tau\sigma^{-1}) = \prod_{i=1}^{k} a_{\alpha_{\sigma(i)}\beta_{\tau(i)}}.$$

Note that any two rows (columns) of $\Pi(A[\alpha \mid \beta])$ have same elements except for the positions that they appear at. We now construct a square matrix $\hat{A}$ of order $k!n_k$ from A. The matrix $\hat{A}$ is partitioned into blocks of size $k! \times k!$. That is, $\hat{A}$ has $n_k \times n_k$ blocks with each block of size $k! \times k!$. We index these blocks using $\alpha, \beta \in G_{k,n}$ and arrange them lexicographically. Then for a fixed $\alpha, \beta \in G_{k,n}$, we put the matrix $\Pi(A[\alpha \mid \beta])$ in the (α, β) block. Note that in this way we have indexed the rows and columns of $\hat{A}$ by $\hat{\alpha}_\sigma$'s arranged lexicographically.

Since $A \geqslant 0$, we have $\otimes^k A \geqslant 0$, implying $(\otimes^k A) \otimes J_{k!} \geqslant 0$, where $J_{k!}$ denotes the $k! \times k!$ matrix with each entry 1. Observe that $\hat{A}$ is a principal submatrix of $(\otimes^k A) \otimes J_{k!}$ corresponding to the rows and columns $\hat{\alpha}_\sigma$'s. Hence $\hat{A} \geqslant 0$.

Let $x_\mu^t = (x_1, x_2, \dots, x_{n_k})$, where $x_i = 1/\sqrt{\mu_q(\alpha^i)}$ for $i = 1, 2, \dots, n_k$, and $\alpha^1, \alpha^2, \dots, a^{n_k}$ are the totality of elements of $G_{k,n}$. Thus if $\mathscr{M}_{n_k} = x_\mu x_\mu^t$, then $\mathscr{M}_{n_k} \geqslant 0$ and is of order $n_k \times n_k$. Thus $\mathscr{M}_{n_k} \otimes J_{k!} \geqslant 0$. Also, $J_{n_k} \otimes \Gamma_{k,q} \geqslant 0$, as $\Gamma_{k,q} \geqslant 0$. Hence $\hat{A} \circ (J_{n_k} \otimes \Gamma_{k,q}) \geqslant 0$. Therefore

$$\hat{A} \circ (J_{n_k} \otimes \Gamma_{k,q}) \circ (\mathscr{M}_{n_k} \otimes J_{k!}) \geqslant 0.$$

Let A_Γ be the matrix obtained by adding the elements of each block of $\hat{A} \circ (J_{n_k} \otimes \Gamma_{k,q}) \circ (\mathscr{M}_{n_k} \otimes J_{k!})$. Then it is easily seen that

$$A_\Gamma = k! P_{k,q}(A).$$

Hence $P_{k,q}(A) \geqslant 0$, and thus we have proved (iii).

Reasoning as in case (iii), with the fact that for positive semidefinite matrices A and B we have $\otimes^k(A + B) \geqslant \otimes^k A + \otimes^k B$, gives us (iv). Part (v) follows from the result proved by Lieb [6] and Ando [1], that the map $A \mapsto \otimes^k A^{1/k}$ is concave and positively homogeneous, hence superadditive—that is, if $A, B \geqslant 0$ then

$$\otimes^k (A + B)^{1/k} \geqslant \otimes^k A^{1/k} + \otimes^k B^{1/k}$$

—and from the reasoning used to prove (iii). ∎

COROLLARY 5. *Suppose $n = mk$, and let each $n \times n$ matrix A be partitioned as follows:*

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix},$$

in which each A_{ij} is $k \times k$. Then the map Φ_q from M_n, the set of all $n \times n$ matrices, to M_m defined by

$$\Phi_q(A) = \begin{pmatrix} \operatorname{per}_q(A_{11}) & \operatorname{per}_q(A_{12}) & \cdots & \operatorname{per}_q(A_{1m}) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{per}_q(A_{m1}) & \operatorname{per}_q(A_{m2}) & \cdots & \operatorname{per}_q(A_{mm}) \end{pmatrix}$$

possesses the following properties:

(i) $\Phi_q(I_n) = I_m$.
(ii) $\Phi_q(cA) = c^k \Phi_q(A)$ *and* $\Phi_q(A^*) = \Phi_q(A)^*$.
(iii) $\Phi_q(A)$ *is positive semidefinite if A is so.*
(iv) $\Phi_q(A + B) \geqslant \Phi_q(A) + \Phi_q(B)$ *if A, B are positive semidefinite.*
(v) $\Phi_q((A + B)^{1/k}) \geqslant \Phi_q(A^{1/k}) + \Phi_q(B^{1/k})$ *if A, B are positive semidefinite.*

Proof. This follows from the observation that $\Phi_q(A)$ is the principal submatrix of the induced matrix $P_{k,q}(A)$ on fixed indices. Hence there is a positive (i.e., order preserving) linear map ϕ from M_{n_k} to M_m such that $\phi(I_{n_k}) = I_m$ and

$$\Phi_q(A) = \phi\big(P_{k,q}(A)\big) \qquad \text{for every } A. \qquad \blacksquare$$

THEOREM 6. *The map Φ_q is completely positive in the sense that if A_{ij} $(i, j = 1, 2, \dots, p)$ are $n \times n$ matrices such that*

$$\begin{pmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pp} \end{pmatrix} \geqslant 0,$$

then

$$\begin{pmatrix} \Phi_q(A_{11}) & \cdots & \Phi_q(A_{1p}) \\ \vdots & \ddots & \vdots \\ \Phi_q(A_{p1}) & \cdots & \Phi_q(A_{pp}) \end{pmatrix} \geqslant 0.$$

Proof. Apply Corollary 5 with pn instead of n. ■

COROLLARY 7. *The map Φ_q possesses the following properties:*

(vi) $\Phi_q(A)^*\Phi_q(A) \leqslant \Phi_q(A^*A)$.

(vii) $\Phi_q(A)^{-1} \leqslant \Phi_q(A^{-1})$ *if A is positive definite.*

Proof. (vi) follows from Theorem 6 on observing that

$$\begin{bmatrix} A^*A & A^* \\ A & I_n \end{bmatrix} \geqslant 0.$$

Similarly (vii) follows from the inequality

$$\begin{bmatrix} A & I_n \\ I_n & A^{-1} \end{bmatrix} \geqslant 0.$$

for positive definite A. ■

With $k = n$, $\Phi_q(A)$ reduces to $\mathrm{per}_q(A)$; hence we have the following consequence of Theorem 6.

COROLLARY 8. *Let A, B, C be $n \times n$ matrices. If*

$$\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$$

is positive semidefinite, then

$$|\mathrm{per}_q(C)|^2 \leqslant \mathrm{per}_q(A)\,\mathrm{per}_q(B).$$

Proof. By Theorem 6, we get

$$\begin{bmatrix} \operatorname{per}_q(A) & \operatorname{per}_q(C) \\ \overline{\operatorname{per}_q(C)} & \operatorname{per}_q(B) \end{bmatrix} \geqslant 0;$$

hence its determinant is nonnegative. ■

4. INEQUALITIES OF SCHWARTZ TYPE

Let A_i, B_i $(i = 1, 2, \dots, p)$ be $n \times n$ matrices.

THEOREM 9. *For any* $0 \leqslant \lambda_i \leqslant 1$ $(i = 1, 2, \dots, p)$ *and* $-1 \leqslant q \leqslant 1$,

$$\left|\operatorname{per}_q\left(\sum_i B_i^* A_i\right)\right|^2 \leqslant \operatorname{per}_q\left(\sum_i B_i^*(A_i A_i^*)^{\lambda_i} B_i\right) \operatorname{per}_q\left(\sum_i (A_i^* A_i)^{1-\lambda_i}\right).$$

Proof. Since

$$\begin{bmatrix} \Sigma_i B_i^*(A_i A_i^*)^{\lambda_i} B_i & \Sigma_i B_i^* A_i \\ \Sigma_i A_i^* B_i & \Sigma_i (A_i^* A_i)^{1-\lambda_i} \end{bmatrix}$$

$$= \sum_i \begin{bmatrix} B_i^* & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} (A_i A_i^*)^{\lambda_i} & A_i \\ A_i^* & (A_i^* A_i)^{1-\lambda_i} \end{bmatrix} \begin{bmatrix} B_i & 0 \\ 0 & I_n \end{bmatrix},$$

by Corollary 8 it suffices to prove that

$$\Delta_i := \begin{bmatrix} (A_i A_i^*)^{\lambda_i} & A_i \\ A_i^* & (A_i^* A_i)^{1-\lambda_i} \end{bmatrix} \geqslant 0.$$

Let $|A_i| = (A_i^* A_i)^{1/2}$. Then there exists a unitary matrix U_i such that $A_i = U_i |A_i|$. Then obviously $(A_i A_i^*)^{\lambda_i} = U_i |A_i|^{2\lambda_i} U_i^*$ and $(A_i^* A_i)^{1-\lambda_i} =$

$|A_i|^{2(1-\lambda_i)}$; hence

$$\Delta_i = \begin{bmatrix} U_i|A_i|^{\lambda_i} \\ |A_i|^{1-\lambda_i} \end{bmatrix} \left[|A_i|^{\lambda_i} U_i^*, |A_i|^{1-\lambda_i} \right] \geqslant 0. \qquad \blacksquare$$

COROLLARY 10. *For $q \in [-1, 1]$ we have*

$$\left| \mathrm{per}_q \left(\sum_i B_i^* A_i \right) \right|^2 \leqslant \mathrm{per}_q \left(\sum_i B_i^* B_i \right) \mathrm{per}_q \left(\sum_i A_i^* A_i \right).$$

Proof. This follows from Theorem 9 with $\lambda_i = 0$ for each i. ■

REMARK. When $p = 1$ this inequality can be written as

$$|\mathrm{per}_q(B^*A)|^2 \leqslant \mathrm{per}_q(B^*B)\,\mathrm{per}_q(A^*A),$$

which is the *Cauchy-Schwartz inequality* for the q-permanent and was proved in Bapat and Lal [3].

COROLLARY 11. *If $|A_i| = (A_i^* A_i)^{1/2}$ then*

$$\left| \mathrm{per}_q \left(\sum_i A_i \right) \right|^2 \leqslant \mathrm{per}_q \left(\sum_i |A_i^*| \right) \mathrm{per}_q \left(\sum_i |A_i| \right).$$

In particular, if all A_i are normal then

$$\left| \mathrm{per}_q \left(\sum_i A_i \right) \right| \leqslant \mathrm{per}_q \left(\sum_i |A_i| \right).$$

Proof. The first inequality follows from Theorem 9 with $\lambda_i = 0$ and $B_i = I_n$ for $i = 1, 2, \ldots, p$. ■

COROLLARY 12. *If A is positive semidefinite and doubly stochastic, then for any $0 \leqslant \lambda \leqslant 1$*

$$\left|\operatorname{per}_q\left(\frac{1}{n}J_n - \lambda\left(A - \frac{1}{n}J_n\right)\right)\right| \leqslant \operatorname{per}_q(A).$$

Proof. Since $(1/n)J_n$ is an orthogonal projection such that $(1/n)AJ_n = (1/n)J_n A = (1/n)J_n$, it follows that $A - (1/n)J_n \geqslant 0$. Then by Corollary 10 and Corollary 5(iv)

$$\begin{aligned}\left|\operatorname{per}_q\left(\frac{1}{n}J_n - \lambda\left(A - \frac{1}{n}J_n\right)\right)\right| &\leqslant \operatorname{per}_q\left(\frac{1}{n}J_n + \lambda\left(A - \frac{1}{n}J_n\right)\right) \\ &\leqslant \operatorname{per}_q\left(\frac{1}{n}J_n + A - \frac{1}{n}J_n\right) \\ &= \operatorname{per}_q(A).\end{aligned}$$

■

REMARK. Certain inequalities of Minkowski type in Ando [2] can also be generalized to inequalities for the q-permanent.

We now apply the previous results to obtain more inequalities for the q-permanent. The next results partially generalize results of Marcus and Minc [7].

THEOREM 13. *If $A = ((a_{ij}))$ is a positive semidefinite $n \times n$ matrix with row sums $r_1, r_2, \dots, r_n$, then*

$$\operatorname{per}_q(A) \geqslant \frac{n!!}{s(A)^n} \prod_{i=1}^{n} |r_i|^2,$$

provided $s(A) = \sum_{i=1}^{n} r_i \neq 0$.

Proof. Since A is positive semidefinite, there exists a matrix C such that $A = C^*C$. Suppose $C = (x_1, x_2, \dots, x_n)$ where for each i, $i = 1, 2, \dots, n$, $x_i^t = (x_{i1}, x_{i2}, \dots, x_{in})$. Thus

$$a_{ij} = (C^*C)_{ij} = \sum_{t=1}^{n} x_{it}\bar{x}_{jt} \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n}\sum_{t=1}^{n} x_{it}\bar{x}_{jt} = r_i.$$

Since $s(A) \neq 0$, let $u = [1/\sqrt{s(A)}]\sum_{i=1}^{n} x_i$, i.e., $u_j = [1/\sqrt{s(A)}]\sum_{t=1}^{n} x_{tj}$ for $j = 1, 2, \ldots, n$. It is easily seen that $\langle u, u\rangle = 1$. Let us define an $n \times n$ matrix D as $D = (u, u, \ldots, u)$. Then it can easily be shown that

$$\operatorname{per}_q(D^*D) = \operatorname{per}_q(J_n) = n!!.$$

We also have

$$C^*D = \frac{1}{\sqrt{s(A)}}\begin{pmatrix} r_1 & r_1 & \cdots & r_1 \\ r_2 & r_2 & \cdots & r_2 \\ \vdots & \vdots & \ddots & \vdots \\ r_n & r_n & \cdots & r_n \end{pmatrix}.$$

Thus

$$\operatorname{per}_q(C^*D) = \frac{n!!}{[s(A)]^{n/2}}\prod_{i=1}^{n} r_i.$$

Now again, by the remark on Corollary 10, we get the required result. ■

COROLLARY 14. *If all the rows of a positive definite $n \times n$ matrix A are equal to k, then*

$$\operatorname{per}_q(A) \geqslant n!!\left(\frac{k}{n}\right)^n.$$

Proof. Easily follows from Theorem 13. ■

THEOREM 15. *If A is a positive semidefinite matrix with eigenvalues λ_1, $\lambda_2, \ldots, \lambda_n$, then*

$$\operatorname{per}_q(A) \geqslant n!!\sum_{t=1}^{n} |\xi_t|^2 \lambda_t^n,$$

where ξ_t is the product of the coordinates of the unit eigenvector corresponding to λ_t.

Proof. Since A is positive semidefinite, we have $A = U^*DU$, where U is unitary and $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Let $D^{1/2} = \operatorname{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i = \lambda_i^{1/2}$. Using Gram's inequality, we have

$$\operatorname{per}_q(A) = \operatorname{per}_q(U^*D^{1/2}(U^*D^{1/2})^*)$$

$$\geqslant \sum_{\mathbf{k}\in\Omega(n,n)} \frac{1}{k_1!!\cdots k_n!!}\left|\operatorname{per}_q(U^*D^{1/2}(\mathbf{k}))\right|^2. \tag{8}$$

If we retain only those n-tuples $\mathbf{k} = (k_1, \dots, k_n)$ for which one of the k_i's is equal to n and others zero, we get

$$\operatorname{per}_q(A) \geqslant \sum_{t=1}^{n} \frac{1}{n!!}\left|\operatorname{per}_q\Big(U^*D^{1/2}(0,\dots,0,\underbrace{n}_{t\text{th}},0,\dots,0)\Big)\right|^2$$

$$= \frac{1}{n!!}\sum_{t=1}^{n}\left|n!!\prod_{j=1}^{n} u_{tj}\alpha_t^n\right|^2$$

$$= n!!\sum_{t=1}^{n}|\xi_t|^2\lambda_t^n.$$

Hence the result. ■

THEOREM 16. *If A is a positive semidefinite matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then*

$$\operatorname{per}_q(A) \geqslant n!!\sum_{t=1}^{n}|\xi_t|^2\lambda_t^n + |\operatorname{per}_q(U)|^2 \det A,$$

where the ξ_t's are defined as in the above theorem.

Proof. In addition to the n-tuples for which one of the k_i's is equal to n and others zero, we also retain on the RHS of (8) the n-tuple $(1, 1, \dots, 1)$. Then (8) yields the required result. ■

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REFERENCES

1 T. Ando, Concavity of certain maps among positive definite matrices and its application to Hadamard products, *Linear Algebra Appl.* 26:203–241 (1979).

2 T. Ando, Inequalities for permanents, *Hokkaido Math. J.* 10:18–36 (1981).

3 R. B. Bapat and A. K. Lal, Inequalities for the q-permanent, Second Conference of the International Linear Algebra Society (Lisbon, 1992), *Linear Algebra Appl.*, 197/198:397–409 (1994).

4 M. Bożejko and R. Speicher, An example of a generalized Brownian motion, *Comm. Math. Phys.* 137:519–531 (1991).

5 John W. Bunce, The Binet-Cauchy theorem and inequalities for the Schur power matrix, *Linear and Multilinear Algebra* 30:5–11 (1991).

6 E. H. Lieb, Convex trace functions and the Wigner-Yanase-Dyson conjecture, *Adv. Math.* 11:267–288 (1973).

7 M. Marcus and H. Minc, Generalized matrix functions, *Trans. Amer. Math. Soc.* 116:316–329 (1965).

8 R. Merris, The permanental dominance conjecture, in *Current Trends in Matrix Theory* (Frank Uhlig and Bob Grone, Eds.), Elsevier, 1987, pp. 213–223.