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Association in time of a vector valued process

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Abstract

Consider a vector valued process $\underline{X}$ given by $\underline{X} = \{\{X_i(m), m \geqslant 1\}, 1 \leqslant i \leqslant k\}$ which takes values on a finite set E^k where $E: \{1, 2, \ldots, n\}$. We derive sufficient conditions under which such a stochastic process is associated in time. An illustrative example wherein such a process is useful is also provided.

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1. Introduction

Consider a vector valued process $\underline{X}$ given by $\underline{X} = \{\{X_i(m), m \geqslant 1\}, 1 \leqslant i \leqslant k\}$ that takes values on a finite set E^k where $E: \{1, 2, \ldots, n\}$. The array can be written as

$$\begin{array}{ccccccc}
X_1(1) & X_1(2) & \cdots & X_1(b-1) & X_1(b) & \cdots & X_1(m) \\
X_2(1) & X_2(2) & \cdots & X_2(b-1) & X_2(b) & \cdots & X_2(m) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
X_a(1) & X_a(2) & \cdots & X_a(b-1) & X_a(b) & \cdots & X_a(m) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
X_k(1) & X_k(2) & \cdots & X_k(b-1) & X_k(b) & \cdots & X_k(m).
\end{array}$$

We assume that for fixed b, variables in the *bth column* $X_1(b), X_2(b), \ldots, X_k(b)$ are dependent random variables and random vectors across rows form a stochastic process.

Motivation for the model comes from data on an oral hygiene study. Dentists recorded the reduction in the amount of plaque on teeth. Each individual in the data was monitored for a couple of days. Two teeth were identified, one on left lower canine which is in the left lower corner of a jaw, and one on molar at upper right

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jaw. The reduction in the thickness of plaque for subjects are usually recorded as belonging to four different categories, viz., no reduction, slight reduction, moderate reduction and vast reduction. One of the objects of the study was to evaluate effectiveness of brushing. In such cases natural question can be: Is it possible to reduce the number of records per individual per day? If there is some sort of dependence, it may be possible to reduce the dimension of the data. Das and Chattopadhyay (2004) developed a latent mixture regression model to study this categorical multivariate data.

Canonical correlation factor analysis are the tools used for non-longitudinal measurable data. To deal with reliability data Barlow and Proschan (1975) defined various concepts of bivariate and multivariate dependence and studied their relationships. For sake of completeness we give definitions of dependence concepts needed in sequence. Given random variables S and T following are some of the concepts of bivariate dependence.

Definition 1.1 (*Right-tail increasing*). A random variable T is RTI in a random variable S if $P[T>t \mid S>s]$ is increasing in s for all t.

Definition 1.2 (*Stochastically increasing*). A random variable T is SI in a random variable S if $P[T>t \mid S=s]$ is increasing in s for all t.

Definition 1.3 (*Multivariate stochastically increasing*). A random variable T is stochastically increasing in random variables $S_1, S_2, \dots, S_k$ if $P[T>t \mid S_1 = s_1, S_2 = s_2, \dots, S_k = s_k]$ is increasing in $s_1, s_2, \dots, s_k$.

Definition 1.4 (*Conditionally increasing in sequence*). Random variables $T_1, T_2, \dots, T_n$ are conditionally increasing in sequence if $P[T_i > t_i \mid T_{i-1} = t_{i-1}, \dots, T_1 = t_1]$ is increasing in $t_1, t_2, \dots, t_{i-1}$ for $i = 1, 2, \dots, n$, that is, T_i is stochastically increasing in $T_1, T_2, \dots, T_{i-1}$.

Definition 1.5 (*Associated*). Random variables $T_1, T_2, \dots, T_n$ are associated if $\mathrm{Cov}(\Gamma(\underline{T}), \Delta(\underline{T})) \geqslant 0$ for all pairs of co-ordinatewise increasing functions Γ and Δ.

Remark 1.6. An infinite sequence of random variables $\{T_n, n \geqslant 1\}$ is said to be associated if it is associated for every finite n.

Barlow and Proschan (1975) showed that $\mathrm{SI}(T \mid S)$ implies $\mathrm{RTI}(T \mid S)$ and if $T_1, T_2, \dots, T_n$ are conditionally increasing in sequence then they are associated. Associated random variables arise in reliability, statistical mechanics, percolation theory, etc. For a detailed review see Roussas (1999) and Prakasa Rao and Dewan (2001). The concept of association in time was defined by Hjort et al. (1985).

Definition 1.7 (*Associated in time*). The stochastic process $\underline{X}$ is said to be associated in time iff, for any integer m and $\{t_1, \dots, t_m\}$, the random variables in the above array are associated.

Hjort et al. (1985) and Kuber and Dharamadikari (1996) discuss sufficient conditions under which association in time for Markov and semi-Markov processes holds.

We model a vector valued stochastic process, recognize its multivariate structure for a specific time, and longitudinal aspects over the period of time and identify sufficient conditions for such a process to be associated in time. In Section 2 we discuss the discrete case with special reference to multivariate Bernoulli random vectors. In Section 3 we discuss the continuous case with special reference to multivariate normal random vectors.

2. The discrete case

Consider the stochastic process $\{\{X_i(m), m \geqslant 1\}, 1 \leqslant i \leqslant k\}$. For $i, j \in E$, let

$$P[X_1(1) = i] = \pi_i, \quad i = 1, 2, \dots, n,$$

$$P[X_a(b) = j \mid X_{a-1}(b) = i] = P_{i,j}, \quad \forall 2 \leqslant a \leqslant k,$$

$$P[X_1(b) = j \mid X_k(b-1) = i] = L_{i,j}, \quad b \geqslant 1. \tag{2.1}$$

Note that π_i is the initial probability, $P_{i,j}$ are the usual one step transition probabilities, $L_{i,j}$ link two vectors $\underline{X}(b)$ and $\underline{X}(b-1)$ in terms of the first entry of the bth column and the last entry of the $(b-1)$th column.

Hence we call them linkage probabilities. Assume that $\forall 2 \leqslant a \leqslant k,\ b \geqslant 1$,

$$\begin{aligned} \mathrm{A}_1: &P[X_a(b) = i_{b,a} \mid X_{a-1}(b) = i_{b,a-1}, \ldots, X_1(b) = i_{b1}] \\ &= P[X_a(b) = i_{b,a} \mid X_{a-1}(b) = i_{b,a-1}] = P_{i_{b,a-1},i_{b,a}}(b). \end{aligned} \tag{2.2}$$

Further, suppose that

$$\mathrm{A}_2: P[X_1(b) = i_{b1} \mid \underline{X}_{b-1} = \underline{i}_{b-1}] = P[X_1(b) = i_{b,1} \mid X_k(b-1) = i_{(b-1),k}]. \tag{2.3}$$

Note that A_1 is a Markov-like assumption for a finite collection of chronologically ordered random variables and A_2 is a Markov-like assumption for the probabilities which link a component of a vector with the last component of the previous vector.

For $b = 1$, the joint distribution of $\{X_1(b), \ldots, X_k(b)\}$ will be determined by $\{\underline{\pi_i}, P_{i,j}, i,j = 1, 2, \ldots, n\}$. For $s = 1, 2, \ldots, m$, let $\underline{i}_s = (i_{s,1}, i_{s,2}, \ldots, i_{s,k})$. Then from assumption $(\mathrm{A}_1, \mathrm{A}_2)$ it follows that

$$\begin{aligned} &P[X_k(1) = i_{1,k}, X_{k-1}(1) = i_{1,k-1}, \ldots, X_1(1) = i_{1,1}] \\ &XP[X_k(1) = i_{1,k} \mid X_{k-1}(1) = i_{1,k-1}, \ldots, X_1(1) = i_{1,1}] \\ &\quad = P[X_{k-1}(1) = i_{1,k-1} \mid X_{k-2}(1) = i_{1,k-2}, \ldots, X_1(1) = i_{1,1}] \ldots P[X_1(1) = i_{1,1}] \\ &\quad = \prod_{j=2}^{k} P[X_j(1) = i_{1,j} \mid X_{j-1}(1) = i_{1,j-1}, \ldots, X_1(1) = i_{1,1}] P[X_j(1) = i_{1,1}] \\ &\quad = \prod_{j=2}^{k} P[X_j(1) = i_{1,j} \mid X_{j-1}(1) = i_{1,j-1}] P[X_j(1) = i_{1,1}]. \end{aligned} \tag{2.4}$$

Further,

$$\begin{aligned} &P[\underline{X}(2) = \underline{i_2}, \underline{X}(1) = \underline{i_1}] \\ &\quad = P[X_k(2) = i_{2k} \mid X_{k-1}(2) = i_{2,k-1}, \ldots, X_1(2) = i_{2,1}, \underline{X}(1) = \underline{i_1}] \\ &\qquad \ldots P[X_2(1) = i_{21} \mid X_k(1) = i_{1k}, X_{k-1}(1) = i_{1,k-1}, \ldots, X_1(1) = i_{1,1}] \\ &\quad = \prod_{j=2}^{k} P[X_j(2) = i_{2,j} \mid X_{(j-1)}(2) = i_{2,j-1}] P[X_1(2) = i_{21} \mid X_k(1) = i_{1k}] \\ &\qquad \times \prod_{j=2}^{k} P[X_j(1) = i_{1,j} \mid X_{j-1}(1) = i_{1,j-1}] P[X_1(1) = i_{1,1}]. \end{aligned} \tag{2.5}$$

In general, let $(a-1)k < \ell \leqslant ak,\ a \geqslant 1$. Then $\ell = (a-1)k + d$ for some $d \in \{1, 2, \ldots, k\}$. Consider the joint distribution of $\{\underline{X}(1), \underline{X}(2), \ldots, \underline{X}(b-1), X_1(b) = i_{b,1}, \ldots, X_d(b) = i_{b,d}\}$. Then, using A_1 and A_2, we get

$$\begin{aligned} &P[\underline{X}(1) = \underline{i}_1, \underline{X}(2) = \underline{i}_2, \ldots, \underline{X}(b-1) = \underline{i}_{(b-1)}, X_1(b) = i_{b,1}, \ldots, X_d(b) \geqslant i_{b,d}] \\ &\quad = P[X_d(b) \geqslant i_{b,d}, X_{d-1}(b) \geqslant i_{b,d-1}, \ldots, X_1(1) - i_{1,1}] \\ &\quad = P[X_d(b) \geqslant i_{b,d} \mid X_d(b) \geqslant i_{b,d}] \prod_{s=1}^{b} \prod_{a=2}^{k} P[X_a(s) = i_{s,a} \mid X_{a-1}(s) = i_{s,a-1}] \\ &\qquad \times \prod_{s=2}^{b} P[X_1(s) = i_{s,1} \mid X_k(s-1) = i_{s-1,k}] P[X_1(1) = i_{1,1}]. \end{aligned} \tag{2.6}$$

This expression involves $(b-1)$ linkage probabilities and $(b-1)(k-1) + d$ one step conditional probabilities and one initial probability. Hence

$$\begin{aligned} &P[X_d(b) \geqslant i_{b,d} \mid X_{d-1}(b) \geqslant i_{b,d-1}, \ldots, X_1(1) = i_{1,1}] \\ &\quad = \begin{cases} P[X_1(b) \geqslant i_{b,1} \mid X_k(b-1) = i_{b-1,k}] & \text{if } d = (b-1)k + 1, \\ P[X_d(b) \geqslant i_{b,d} \mid X_{d-1}(b) \geqslant i_{b,d-1}] & \text{if } (b-1)k + 1 < d \leqslant bk. \end{cases} \end{aligned}$$

Thus, to write all finite dimensional distributions of such a vector valued process one would require information about $\{\underline{\pi}_i, P_{i,j}, L_{i,j}, i,j \in E\}$. Here $\sum_i \pi_i = 1$, $\sum_j P_{ij} = 1\ \forall i$, $\sum_j L_{ij} = 1\ \forall i$.

For the given process $\underline{X}$, let $\{Z_n, n \geqslant 1\}$ be a process, where

$$Z_{(i-1)k+j} \stackrel{\text{st}}{=} X_i(j), \quad j \geqslant 1,\ 1 \leqslant k. \tag{2.7}$$

Hence all finite dimensional distributions of $\underline{X}$ and $\{Z_n, n \geqslant 1\}$ coincide. Now we consider a set of sufficient conditions for association in time for the process $\{Z_n, n \geqslant 1\}$.

Theorem 2.1. *If $\{Z_n, n \geqslant 1\}$ is conditionally stochastically increasing then it is associated in time.*

Proof follows from the fact that conditionally stochastically increasing random variables are associated (see Barlow and Proschan, 1975).

Theorem 2.2. *Suppose conditions* A_1 *and* A_2 *hold for the stochastic process* $\underline{X}$. *Further suppose that*

$$P[X_a(b) \geqslant i_{b,a} \mid X_{a-1}(b) = i_{b,a-1}] \text{ is increasing in } i_{b,a-1} \quad \forall b \geqslant 1,\ a \geqslant 2, \tag{2.8}$$

$$P[X_1(b) \geqslant i_{b,1} \mid X_k(b-1) = i_{b-1,k}] \text{ is increasing in } i_{b-1,k} \quad \forall b \geqslant 1,\ a \geqslant 2. \tag{2.9}$$

Then it is associated in time.

Proof follows immediately from (2.6).

Thus sufficient conditions for $\underline{X}$ to be associated in time are that all one-step conditional and linkage survival probabilities are stochastically increasing. Since right-tail increasing implies stochastically increasing, it is sufficient that these conditional probabilities are right-tail increasing.

Lemma 2.3. *Suppose X, Y are discrete random variables on the same finite sample space E. Further suppose that*

$$\begin{aligned} P[X = x \mid Y = y] \quad & \text{is increasing in } y \text{ for each } x \geqslant y, \\ & \text{is decreasing in } y \text{ for each } x < y. \end{aligned} \tag{2.10}$$

Then

$$P[X \geqslant x \mid Y = y] \text{ is increasing in } y \text{ for each } x. \tag{2.11}$$

Proof. First note that $P[X \geqslant x \mid Y = y] = \sum_{z=x}^{m} P[X = z \mid Y = y]$.

The proof is trivial when $y \leqslant x \leqslant m$. When $x < y \leqslant m$, we have

$$\begin{aligned} P[X \geqslant x \mid Y = y] &= \sum_{z=x}^{y-1} P[X = z \mid Y = y] + \sum_{z=y}^{m} P[X = z \mid Y = y] \\ &= P[X < y \mid Y = y] - P[X < x \mid Y = y] + P[X \geqslant y \mid Y = y] \\ &= 1 - P[X < x \mid Y = y]. \end{aligned}$$

The proof follows from the fact that $P[X = x \mid Y = y]$ is decreasing in y for each $x < y$. $\square$

The lemma leads to the following theorem.

Theorem 2.4. *Suppose conditions* A_1 *and* A_2 *hold for the stochastic process* $\underline{X}$. *Further suppose that*

$$\begin{aligned} P[X_a(b) = i_{b,a} \mid X_{a-1}(b) = i_{b,a-1}] \quad & \text{is increasing in } i_{b,a-1} \quad \forall i_{b,a} \geqslant i_{b,a-1} \quad b \geqslant 1,\ a \geqslant 2, \\ & \text{is decreasing in } i_{b,a-1} \quad \forall i_{b,a} < i_{b,a-1} \quad b \geqslant 1,\ a \geqslant 2, \end{aligned} \tag{2.12}$$

$$\begin{aligned} P[X_1(b) = i_{b,1} \mid X_k(b-1) = i_{b-1,k}] \quad & \text{is increasing in } i_{b-1,k} \quad \forall i_{b,1} \geqslant i_{b-1,k} \quad \forall b \geqslant 1,\ a \geqslant 2, \\ & \text{is decreasing in } i_{b-1,k} \quad \forall i_{b,1} < i_{b-1,k} \quad \forall b \geqslant 1,\ a \geqslant 2. \end{aligned} \tag{2.13}$$

Then it is associated in time.

Theorems 2.1, 2.2 and 2.4 give sufficient conditions for $\underline{X}$ to be associated in time. Theorem 2.1 does not require the Markovian assumption $(\mathrm{A}_1, \mathrm{A}_2)$. Theorem 2.2 requires one-step conditional and linkage survival probabilities to be stochastically increasing. Since right-tail increasing implies stochastically increasing, it is sufficient that these conditional probabilities are right-tail increasing.

However, the conditions in Theorem 2.4 are in terms of conditional mass function, that is, in terms of the kernel of $\underline{X}$. Hence these are easily verifiable. Theorems 2.2 and 2.4 give weaker conditions for verifying associated in time provided the underlying process is Markovian in the sense defined by $(\mathrm{A}_1, \mathrm{A}_2)$.

2.1. The Bernoulli case

Marshall and Olkin (1985) considered a bivariate Bernoulli distribution whose marginals are Bernoulli random variables. They have used this bivariate Bernoulli distribution to generate bivariate binomial, Poisson and hypergeometric distributions.

Now we consider two models, a multiplicative and an additive, which arise from independent Bernoulli random variables and can be considered as an extension of Bernoulli random variables to k dimensional dependent variables. They will be used to study association in time for the processes like those discussed above. However, they are of independent interest as well.

2.1.1. The multiplicative model

Let $Y_1, Y_2, \ldots, Y_{k-1}, Y_k$ be independent $B(1, p_i)$, $i = 1, 2, \ldots, k$ random variables. Define a new random vector $\underline{W}$ as follows:

$$W_i = Y_i * Y_k, \quad i = 1, 2, \ldots, k-1, \ \ W_k = Y_k. \tag{2.14}$$

Note that each W_i is increasing in its arguments. Since independent random variables are associated and increasing functions of associated random variables are associated (Esary et al., 1967), we have $\underline{W} = (W_1, W_2, \ldots, W_k)$ associated random variables. The joint distribution of $(W_1, W_2, \ldots, W_k)$ is given by

$$P[W_1 = W_2 = \cdots, W_k = 0] = 1 - p_k,$$

$$P[W_1 = W_2 = \cdots, W_k = 1] = \prod_{i=1}^{k} p_i,$$

$$P[W_1 = w_1, W_2 = w_2, \ldots, W_{k-1} = w_{k-1}, W_k = 0] = 0, \quad \text{if } w_j = 1 \text{ for any } 1 \leqslant j < k,$$

$$P[W_1 = w_1, W_2 = w_2, \ldots, W_{k-1} = w_{k-1}, W_k = 1] = p_k \left[\prod_{j=1}^{k-1} p_j^{w_j} (1 - p_j)^{1 - w_j} \right] \quad \text{otherwise.} \tag{2.15}$$

For completeness note that for $i = 1, 2, \ldots, k-1$,

$$P[W_i = 1] = p_i p_k, \quad P[W_i = 0] = 1 - p_i p_k,$$

$$\mathrm{Cov}(W_i, W_j) = p_i p_j p_k (1 - p_k), \quad i \neq j. \tag{2.16}$$

$$\mathrm{Cov}(W_i, W_k) = p_i p_k (1 - p_k) \quad \text{for } i \neq k. \tag{2.17}$$

However, note that

$$P[W_3 = 1 \mid W_2 = 0, W_1 = 0] = \frac{(1 - p_1)(1 - p_2) p_3 p_k}{1 - p_1 p_k},$$

$$P[W_3 = 1 \mid W_2 = 0] = \frac{(1 - p_2) p_3 p_k}{1 - p_2 p_k}. \tag{2.18}$$

Clearly, the Markovian property defined in (A_1) does not hold. The following result is true.

Theorem 2.5. *Let $\{X_a(b), a = 1, 2, \ldots, k\}$ be independent $B(1, p_a)$ random variables for all $b \geqslant 1$. Define*

$$Z_\ell = X_a(1) * X_k(1), \quad 1 \leqslant \ell \leqslant k - 1,$$

$$Z_{jk} = \prod_{a=1}^{j} X_k(a), \quad j \geqslant 1,$$

$$Z_\ell = X_a(b) * Z_{bk}, \quad \ell = (b-1)k + a, \ \ 1 \leqslant a < k. \tag{2.19}$$

Then $\{Z_n, n \geqslant 1\}$ *are associated in time.*

2.1.2. The additive model

We also consider another additive model which describes the dental data mentioned earlier. Now, let $Y_1, Y_2, \ldots, Y_k$ be independent $B(1, p_i), i = 1, 2, \ldots, k$ random variables and U be $B(1, p)$ random variable independent of $Y_i, i = 1, 2, \ldots, k$. Define a new random vector $\underline{W}^*$ as follows:

$$W_i^* = Y_i + U, \quad i = 1, 2, \ldots, k. \tag{2.20}$$

Note that $(W_1, W_2, \ldots, W_k)$ are associated random variables. Further each W_i^* takes values $\{0, 1, 2\}$. We have

$$P[W_i^* = 0] = (1 - p_i)(1 - p), \quad P[W_i^* = 1] = (1 - p_i)p + (1 - p)p_i, \quad P[W_i^* = 2] = p_i * p. \tag{2.21}$$

Note that

$$E(W_i^*) = p + p_i, \quad \mathrm{Var}(W_i^*) = p(1 - p) + p_i(1 - p_i) \quad \forall i,$$

$$\mathrm{Cov}(W_i^*, W_j^*) = p(1 - p) \quad \forall i \neq j. \tag{2.22}$$

This idea can be extended such that W_i^* is a sum of two or more independent Bernoulli random variables and a common effect. In this case also

$$P[W_3^* = 1 \mid W_2^* = 1, W_1^* = 0] = p_3,$$

$$P[W_3^* = 1 \mid W_2^* = 1] = \frac{p(1 - p_2)(1 - p_3) + (1 - p)p_2 p_3}{p(1 - p_2) + p_2(1 - p)}. \tag{2.23}$$

Hence the Markovian property does not hold. However, the process $\{Z_n, n \geqslant 1\}$, defined below, is associated in time.

Theorem 2.6. *Let* $\{X_a(b), a = 1, 2, \ldots, k, b \geqslant 1\}$ *be independent* $B(1, p_a)$ *random variables for all* $b \geqslant 1$. *Let* $U_j, j = 1, 2, \ldots$ *be independent* $B(1, p_j^*)$ *random variables, independent of* $\{X_a(b)\}$. *Define*

$$Z_\ell = X_a(1) + U_1, \quad 1 \leqslant \ell \leqslant k,$$

$$Z_\ell = X_a(b) + \prod_{j=1}^{b} U_j, \quad \ell = (b-1)k + a, \ \ 1 \leqslant a \leqslant k. \tag{2.24}$$

Then $\{Z_n, n \geqslant 1\}$ *are associated in time.*

Hence for both the models considered above the stochastic processes of interest are not Markovian. Hence Theorems 2.2 and 2.4 cannot be used. However, both the processes are associated in time.

3. The continuous case

In Section 2, the state space of the process $\underline{X}$ was considered to be discrete. When the records are on actual measurements on an individual (device) at a given point, the random variables $X_a(b)$ take values in an interval. For example, in case of dental data amount of stain may be measurable. In such cases one has to study the vector valued process $\{\underline{X}\}$, discrete in time and continuous in state space. In what follows we provide sufficient conditions for association in time for such a process. To begin with, as in Section 2, we provide a result based on conditionally stochastically increasing sequence and then following Pitt (1982) obtain the sufficient condition for association when the finite dimensional distribution follows multivariate normal distribution.

As before $\underline{X} = \{\underline{X}(b), b \in N\}$. $\{\underline{X}(b), b = 1, 2, \ldots, m\}$ is a collection of km random variables. One would know the behaviour of these km random variables completely if one knows the corresponding km dimensional multivariate distribution completely. We note that there are two co-ordinates of this family of random variables. For a fixed $b \in N$, there are finitely many, i.e., k random variables, say $\{X_1(b), X_2(b), \dots, X_k(b)\}$, each taking value in R^+. This k dimensional multivariate distribution will be known completely if we know one marginal and successive conditionals, say $P[X_1(b) \geqslant x_1], P[X_2(b) \geqslant x_2 \mid X_1(b) \geqslant x_1], \dots, P[X_k(b) \geqslant x_k \mid X_{k-1}(b) \geqslant x_{k-1}, \dots, X_1(b) \geqslant x_1]$. Assume that

$$\begin{aligned} \mathrm{B}_1: & P[X_j(b) \geqslant x_j \mid X_1(b) \geqslant x_1, \dots, X_{j-1}(b) \geqslant x_{j-1}] \\ & = P[X_j(b) \geqslant x_j \mid X_{j-1}(b) \geqslant x_{j-1}] = \bar{F}_{j-1,j}(x_j \mid x_{j-1}) \quad \text{(say)}, \end{aligned}$$

$$\mathrm{B}_2: \bar{F}_{j-1,j}(x \mid y) = \bar{F}_{1,2}(x \mid y). \tag{3.1}$$

Then, in order to know the k dimensional distributions one would require to know $\{\bar{F}_1(x), \bar{F}_{j-1,j}(x_j \mid x_{j-1})\ \forall x_j \in R, j = 1, 2, \dots, k\}$, where $\bar{F}_1(x) = P[X_1(b) \geqslant x_1]$. Further, if the conditional survival functions satisfy the stationarity property B_2 and the marginals are identical, then the kernel of this k dimensional vector would be $\{\bar{F}(x), \bar{F}_{1,2}(x \mid y)\}$. Above is the multivariate expect of the family of random variables described.

Now to consider the "process" aspect of it, $\forall 0 < s < t, s, t \in N$ $\{\underline{X}(s), \underline{X}(t)\}$ are jointly distributed. One can say that $\{\underline{X}(t), t \geqslant 0\}$ is a vector valued Markov process if

$$\mathrm{B}_3: P[\underline{X}(t) \geqslant \underline{x}_t \mid \underline{X}(s) = \underline{x}_s,\ \forall s \leqslant t] = P[\underline{X}(t) \geqslant \underline{x}_t \mid \underline{X}(s) = \underline{x}_s] = \bar{F}_{s,t}(x_t \mid x_s). \tag{3.2}$$

Further we assume stationarity, that is,

$$\bar{F}_{s,t}(x_t \mid x_s) = \bar{F}_{t-s}(x_t \mid x_s), \quad \forall\, \underline{x_t}, \underline{x_s} \in R^+,\ \forall 0 < s < t. \tag{3.3}$$

In light of (3.1) and (3.3), in order to write the joint distribution of $\{\underline{X}(s), \underline{X}(t)\}$, one needs the conditional distribution function, say from one of the $X_i(s)$ to one of the $X_j(t)'s$. The linkage probabilities are given by

$$P[X_1(t) \geqslant x_{1t} \mid X_k(s) = x_{ks}] = \bar{G}^{k,1}_{s,t}(x_{1t} \mid x_{ks}) \quad \text{(say)}. \tag{3.4}$$

Then, using $\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3$, the joint distribution of $\{\underline{X}(1), \underline{X}(2), \dots, \underline{X}(m)\}$ can be determined by $\{\bar{F}(s), \bar{F}_{1,2}(t \mid s), \bar{G}^{(k,1)}_{s,t}(t \mid s)\}$. Further for $0 < s < t$,

$$P[\underline{X}(t) \geqslant \underline{x}_t, \underline{X}(s) \geqslant \underline{x}_s] = \bar{F}(x_{s1}) \prod_{s=1}^{t} \prod_{j=2}^{k} \bar{F}_{1,2}(x_{s_j} \mid x_{s_{j-1}}) \prod_{s=2}^{t} \bar{G}^{k,1}_{s,t}(x_{1t} \mid x_{ks}). \tag{3.5}$$

In a similar way finite distribution of any order can be written.

As before, we consider the process $\{Z_n, n \geqslant 1\}$ given in (2.9) and study sufficient conditions for the process to be associated in time. Theorem 2.1 holds even in this case when the state space is continuous.

Theorem 3.1. *Suppose that for the stochastic process $\underline{X}$ conditions B_1–B_3 hold. Further suppose that*

$$P[X_j(s) \geqslant x_{s_j} \mid X_{j-1}(s) = x_{s_{j-1}}] \textit{ is increasing in } x_{s_{j-1}} \quad \forall s \geqslant 1,\ j \geqslant 2, \tag{3.6}$$

$$P[X_1(s) \geqslant x_{s_1} \mid X_k(s-1) = x_{s-1_k}] \textit{ is increasing in } x_{s-1_k} \quad \forall s \geqslant 1,\ j \geqslant 2. \tag{3.7}$$

Then it is associated in time.

Pitt (1982) showed that positively correlated normal random variables are associated. Hence we have the following two results.

Theorem 3.2. *If $\{Z_n, 1 \leqslant n \leqslant m\}$ have $N_m(\mu, \Sigma)$, with $\sigma_{i,j} \geqslant 0$, then $\{Z_n, n \geqslant 1\}$ are associated in time.*

Theorem 3.3. *If* $\{\underline{X}(t_i)1 \leqslant i \leqslant m\}$ *have* $N_k(\mu_k, \Sigma^k)$, *with* $\sigma^k_{i,j} \geqslant 0$, *and*

$$P[X_1(s) \geqslant x_{s_1} \mid X_k(s-1) \geqslant x_{(s-1),k}] \text{ is increasing in } x_{(s-1),k} \ \forall s, \tag{3.8}$$

then the process $\{\underline{X}\}$ *is associated in time.*

Note that we do not need the process $\{\underline{X}\}$ to be Markovian for Theorems 3.2 and 3.3 to be true.

4. Applications to dental data

Tables 1 and 2 give a part of dental data analysed by Das and Chattopadhyay (2004). It gives stain on the same tooth at all the four positions before and after brushing, respectively. Numbers under (P_1, P_2, P_3, P_4) indicate the amount of stain at each of the four positions on the selected tooth of an individual.

It is easy to verify that data in Table 1 are conditionally increasing in its co-ordinates. However, for data in Table 2 all probability inequalities are in the desired direction except that $P[P_4 \geqslant 3 \mid P_1 = 1, P_2 = 0, P_3 = 1] = \frac{1}{2}$, while $P[P_4 \geqslant 3 \mid P_1 = 0, P_2 = 0, P_3 = 0] = \frac{11}{12}$. Note that the first probability is based on only two observations and the departure can be attributed to sampling/measuring errors. With such an understanding, both the data sets can be considered to be associated in time. Hence measurement only at one of the four positions, say at P_4, would suffice for statistical analysis. To the best of our knowledge there are no statistical tests for testing if a sequence of random variables is conditionally increasing.

In general, the philosophy in this paper is analogous to the philosophy behind generators, fractions, alias structure and the resolution of a fraction in factorial experiments. These concepts in factorial experiments help user to minimize the experimental work. On the same lines, if the user has some technical or statistical evidence that the data are going to have a dependence structure of specific type, he/she can plan economic data acquisition methods. We have provided various situations wherein collecting fraction of data may be sufficient to take decisions.

Table 1
Dental data: stain before brushing

Individual	P_1	P_2	P_3	P_4
1	1	1	1	2
2	1	1	2	2
3	1	1	2	2
4	1	1	2	2
5	1	1	2	2
6	1	2	2	2
7	1	2	2	2
8	1	2	2	2
9	1	2	2	2
10	1	2	2	2
11	1	2	2	2
12	1	2	2	2
13	1	2	2	3
14	2	1	2	2
15	2	2	2	2
16	2	2	2	2
17	2	2	2	2
18	2	2	2	2
19	2	2	2	2
20	2	2	2	2
21	2	2	2	2
22	2	2	2	2
23	2	2	2	2
24	2	2	2	3
25	2	2	2	3

Table 2
Dental data: stain after brushing

Individual	P_1	P_2	P_3	P_4
1	0	0	0	0
2	0	0	0	1
3	0	0	0	1
4	0	0	0	1
5	0	0	0	1
6	0	0	0	1
7	0	0	0	1
8	0	0	0	1
9	0	0	0	1
10	0	0	0	1
11	0	0	0	1
12	0	0	0	2
13	0	0	0	2
14	0	0	1	1
15	0	0	1	1
16	0	0	1	1
17	0	0	1	1
18	0	0	1	1
19	0	0	1	1
20	0	0	1	1
21	0	1	1	1
22	0	1	1	1
23	0	1	1	1
24	0	1	1	1
25	1	1	1	2

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