Classroom

In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

B Bagchi, Indian Statistical Institute, Banglaore 560 059.

! Bachet's Problem

A grocery shopkeeper keeps five stones of different weights. He is able to use a common balance and weigh out quantities ranging from 1 to 100 kg, in steps of 1 kg. What are the weights of these five stones?

The above is the problem *100 kg with five stones* posed by R Yusufzai in the *Think it Over* column of the July 1996 issue of *Resonance*. A much better problem will result if the figure 100 is replaced by 121. This is because the question *what are the weights of these five stones*? seems to suggest that there are uniquely determined weights to be found! However, as may easily be verified, the weights in kg of the stones might be 1, 3, 9, 27 and m, where m is any integer in the range $60 \leq m \leq 81$. In fact, there are many other solutions to the problem as posed. If, however, it was given that the grocer can weigh any object of weight between 1 kg and 121 kg (in steps of 1 kg) using his five stones, then the weights (in kg) of the stones must have been 1, 3, 9, 27 and 81. This is the case $k = 5$ of the result stated and proved below.

The problem is a well-known variation of an old problem due to Bachet (see Suggested Reading). In the original *binary* version, the grocer cannot subtract, so he must put the stones in one pan and the object in the other. Mr Yusufzai's problem is an instance

of the *ternary* version where this restriction is removed. The general problem (in its *ternary* version) may be stated as follows: Given a positive integer k, find the largest integer N_k such that any object whose weight is an integer between 1 and N_k (ends included) can be weighed using k stones of suitable integral weights. In this notation, the *Think it Over* problem is to show that $N_5 \geq 100$.
In fact, we have —

Theorem : $N_k = \frac{3^k - 1}{2}$. If k stones are such that all integral weights between 1 and N_k can be measured using them, then the weights of these stones must be 3^j, $0 \leq j \leq k-1$.

This is, essentially, Theorem 141 in the book by Hardy and Wright (see Suggested Reading).

In order to prove this, we must convert it into a precise mathematical statement. To this end, let $a_0, \cdots, a_{k-1}$ be the (positive integral) weights of k stones. In order to weigh an object of integral weight m, the grocer places the object together with some of the stones on the right pan (say) and puts some other stones on the left pan. For $0 \leq j \leq k-1$, put $\varepsilon_j = 1$ if the stone of weight a_j is placed on the left pan, $\varepsilon_j = -1$ if it is on the right pan, $\varepsilon_j = 0$ if it is not used. Since the two pans must balance, we get

$$m = \sum_{j=0}^{k-1} \varepsilon_j \, a_j \quad \text{where } \varepsilon_j \in \{0, 1, -1\} \text{ for } 0 \leq j \leq k-1 \qquad (1)$$

This leads us to:

Definition : If $A = \{a_0, \cdots, a_{k-1}\}$ is a finite set of positive integers then the *capacity* $C(A)$ of A is the largest integer M such that for every integer m in the range $1 \leq m \leq M$, the equation (1) has a solution.

Informally, the capacity $C(A)$ is the largest M such that all weights between 1 and M can be measured using k stones whose weights are in A. In terms of this definition, the above theorem may be restated as follows.

Theorem : If A is of size k then $C(A) \le \frac{3^k-1}{2}$. Equality holds here if and only if $A = \{3^j : 0 \le j \le k-1\}$.

To prove the theorem, note that if m can be written as in (1) then so can $-m$ (just change the signs of all ε_j); also, trivially, $m = 0$ can be written thus (take $\varepsilon_j = 0$ for all j). Therefore, if $C(A) = M$, then all the $2M+1$ integers m in the range $-M \le m \le M$ can be expressed as in (1). But there are 3 choices for ε_j for each j, hence only 3^k choices for the right hand side of (1). Hence $2M+1 \le 3^k$, or $C(A) \le \frac{3^k-1}{2}$. Now, if we take $A = \{3^j : 0 \le j \le k-1\}$, then for $1 \le m \le \frac{3^k-1}{2}$ write $\frac{3^k-1}{2} - m$ in base 3: $\frac{3^k-1}{2} - m = \sum_{j=0}^{k-1} \delta_j 3^j$, where $\delta_j \in \{0,1,2\}$. Put $\varepsilon_j = 1 - \delta_j$. Then (1) holds. Thus $C(A) \ge \frac{3^k-1}{2}$ for this set. Together with the previous inequality, we get $C(A) = \frac{3^k-1}{2}$.

Only the uniqueness part of the theorem remains to be proved. In fact, this is the only non-trivial and interesting part. To prove this, let $A = \{a_0, \cdots, a_{k-1}\}$ have capacity N_k. Since, now, equality holds in the inequality $C(A) \le \frac{3^k-1}{2}$ which appears in the statement of the theorem, the proof of the inequality shows that every integer m in the range $-\frac{3^k-1}{2} \le m \le \frac{3^k-1}{2}$ has a *unique* representation (1); conversely any m of the form (1) belongs to this range. Therefore, letting X be an indeterminate, we get

$$\prod_{j=0}^{k-1} (X^{-a_j} + 1 + X^{a_j}) = \sum_{|m| \le \frac{3^k-1}{2}} X^m \tag{2}$$

as may be verified by multiplying out the left hand. Since, in particular, the largest integer (viz. $\sum_{j=0}^{k-1} a_j$) of the form (1) must be the largest integer in the range $[-\frac{3^k-1}{2}, \frac{3^k-1}{2}]$, we also have:

$$\sum_{j=0}^{k-1} a_j = \frac{3^k - 1}{2}. \tag{3}$$

Using (3) and a little algebra, (2) simplifies to

$$\prod_{j=0}^{k-1} \frac{X^{3a_j} - 1}{X^{a_j} - 1} = \frac{X^{3^k} - 1}{X - 1}. \tag{4}$$

Now fix j, $0 \le j \le k-1$. Let w be a primitive $3a_j$ - *th* root of unity. That is, w is a complex number such that $w^l = 1$ if and only if l is an integral multiple of $3a_j$. (For instance, we may take $w = \exp(2\pi\sqrt{-1}/3a_j))$. Then w is a zero of the left hand side, and hence also of the right hand, of (4). Thus $w^{3^k} = 1$. So $3a_j$ divides 3^k. That is, $a_j \in \{3^i, 0 \le i \le k-1\}$. Since this holds for all j, we have $A \subseteq \{3^i : 0 \le i \le k-1\}$. Since both sets have size k, we must have $A = \{3^i : 0 \le i \le k-1\}$. This proves the uniqueness of the set of given size and maximum capacity.

The reader may like to look up the proof in the book by Hardy and Wright, which is very different from the proof given here. It is a clever use of mathematical induction.

Tail-piece : Bachet is better remembered by mathematicians for another reason. It was on Bachet's edition of Diophantus' Arithmetic that Fermat scribbled his famous marginal notes. Bachet was also the first man to state, (without proof) what is now known as Lagrange's four square theorem: every natural number is the sum of at most four perfect squares.

Suggested Reading

- **F Schuh. *The Master Book of Mathematical Recreations*.Dover. New York. pp115-118, 1968.**
- **G H Hardy and E M Wright. *An Introduction to the Theory of Numbers*. Oxford Univ. Press.London.pp115-117,1971.**