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DE BRANGES SPACES CONTAINED IN SOME BANACH SPACES OF ANALYTIC FUNCTIONS

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1. Introduction

L. de Branges has proved in Theorem 15 of [2] an invariant subspace theorem which generalizes not only Beurling's famous theorem [1] but also its generalizations due to Lax [7] and Halmos [4]. The scalar version of the theorem says:

THEOREM A. *Let M be a Hilbert space contractively contained in the Hardy space H^2 of the unit D such that $S(M) \subset M$ (where S is the operator of multiplication by the coordinate function z) and S acts as an isometry on M. Then there exists a unique b in the unit ball of H^∞ such that*

$$M = b(z)H^2.$$

Further,

$$\|bf\|_M = \|f\|_{H^2}.$$

In this note we characterize those Hilbert spaces M which are algebraically contained in various Banach spaces of analytic functions on the unit disc D. We drop the contractivity requirement on M (no continuity assumptions are made on the inclusion relation). Thus even in the particular case of $M \subset H^2$, we obtain an extension of de Branges Theorem by having characterized the class of all Hilbert spaces which are vector subspaces of H^2 and on which S acts as an isometry. See Corollaries 5.1 and 4.1.

2. Preliminary notations, definitions and results

Let D be the unit disc in the complex plane and H^p $(0 < p \leq \infty)$ the well known Hardy spaces on D. Let L^p $(0 < p \leq \infty)$ be the familiar Lebesgue spaces on the unit circle T. It is well known that H^p can be viewed as a space of functions on T for each p. The Dirichlet space A^2 consists of all analytic functions $f(z)$ such that

$$\int_D |f'(z)|^2 \, dx\, dy < \infty.$$

The Bergman space B^2 consists of all analytic functions $f(z)$ on D such that

$$\int_D |f(z)|^2 \, dx\, dy < \infty.$$

Let BMO be the class of all L^1 functions f such that

$$\|f\|_* = \operatorname{Sup} \frac{1}{|I|} \int_I \left| f - \frac{1}{|I|} \int_I f \right| < \infty$$

where the supremum is taken over all subarcs I of T and $|I|$ denotes the normalized Lebesgue measure of I.

BMO is a Banach space under the norm

$$\|f\| = \|f\|_* + |f(0)|.$$

VMO is the closure of the continuous functions in BMO.

$BMOA = BMO \cap H^1$ and $VMOA = VMO \cap H^1$.

It is well known that $BMOA \subset H^p$ $(p < \infty)$.

A positive Borel measure μ on D is said to be a Carleson measure if

$$\mu(S(I)) = O(|I|)$$

for every subarc I of T where

$$S(I) = \left\{ z : \frac{z}{|z|} \in I, 1 - |I| \le |z| \le 1 \right\}.$$

Excellent references for all that has been said above are [3], [5] and [11].

We shall also use the following result:

LEMMA 2.1. *Let H be a Hilbert space and let A be an isometry on H such that $\bigcap_{n=0}^{\infty} A^n(H) = \{0\}$. Then*

$$H = N \oplus A(N) \oplus A^2(N) \oplus \cdots$$

where $N = H \ominus A(H)$.

Proof. See page 2, Section 1.3 of [8].

3. The main result

PROPOSITION. *Let M be a Hilbert space such that M is a vector subspace of the vector space of all analytic functions on D. Further, suppose $S(M) \subset M$*

and S acts as an isometry (S denotes multiplication by the coordinate function z). Then

$$M = N \oplus S(N) \oplus S^2(N) \oplus \cdots$$

where $N = M \ominus S(M)$.

Proof. In view of Lemma 2.1, all that is required is to show that $\bigcap_{n=0}^{\infty} S^n(M) = \{0\}$. But this is a simple consequence of the fact that any $f(z)$ in M has a power series expansion

$$f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots$$

because it is analytic in D.

On the other hand, if f is in $\bigcap_{n=0}^{\infty} S^n(M)$ then $f(z) = z^n g_n(z)$ for each positive n. Hence $\alpha_n = 0$ for each n and thus $f = 0$.

4. Consequences of the proposition: the case when M is contained in B^2

Note. Throughout, M is assumed to satisfy the hypothesis of the proposition in Section 3.

COROLLARY 4.1. *Let M be contained in the Bergman space B^2. Then there is a collection of unit vectors $\{b_i\}$ in M such that:*

(i) $M = \oplus \sum_i b_i H^2$;

(ii) $|b_i(z)|^2\, dx\, dy$ *is a Carleson measure for each i;*

(iii) $\|b_i f\|_M = \|f\|_{H^2}$ *for each i and for each f in* H^2.

Proof. From the proposition we conclude that

$$M = N \oplus S(N) \oplus S^2(N) \oplus \cdots$$

where $N = M \ominus S(M)$.

Let b be any element of unit norm in N and let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be any element of H^2. Let $f_n(z) = \sum_{k=0}^{n} \alpha_k z^k$ so that $f_n \to f$ in H^2.

Now by the above decomposition, bf_n is in M for each n and

$$\begin{aligned}
\|bf_n\|_M^2 &= \left\| \sum_{k=0}^{n} b\alpha_k z^k \right\|_M^2 \\
&= \sum_{k=0}^{n} \|b\alpha_k z^k\|_M^2 = \sum_{k=0}^{n} |\alpha_k|^2 \|bz^k\|_M^2 \\
&= \sum_{k=0}^{n} |\alpha_k|^2 \|S^k b\|_M^2 \\
&= \sum_{k=0}^{n} |\alpha_k|^2 \quad (\text{as } S \text{ is an isometry and } \|b\|_M = 1) \\
&= \|f_n\|_{H^2}^2.
\end{aligned}$$

This means that bf_n is a Cauchy sequence in M and so there is a g in M such that $bf_n \to g$. Now for any positive integer k, it is easy to see that

$$bf_n = \alpha_0 + \alpha_1 zb + \cdots + \alpha_k z^k b + z^{k+1} bh_n$$

where $h_n = \alpha_{k+1} + \alpha_{k+2} z + \cdots + \alpha_n z^{n-k-1}$. So bh_n is a Cauchy sequence in M by the same argument and hence bh_n converges to some h in M. Thus

$$\alpha_0 + \alpha_1 zb + \cdots + \alpha_k z^k b + z^{k+1} h = g.$$

Hence, using the fact that every element above is in B^2 and so has a Taylor series expansion, we conclude that the kth Taylor coefficient of g is the kth Taylor coefficient of $\alpha_0 + \alpha_1 zb + \cdots + \alpha_k z^k b$ which is the same as the kth Taylor coefficient of the formal product of the Taylor series of b and f. Thus we see that $g = bf$ and since f is an arbitrary element of H^2, we conclude that $bH^2 \subset B^2$. In other words, b multiplies H^2 into B^2. It now follows by Theorems 1.1 and 1.2 of [9] that

$$|b(z)|^2 \, dx \, dy$$

is a Carleson measure. Further, since $\|bf_n\|_M = \|f_n\|_{H^2}$, it follows that $\|bf\|_M = \|f\|_{H^2}$ (Since $bf_n \to bf$ in M).

The rest of the corollary now follows by fixing an orthonormal basis $\{b_i\}$ in N.

Remark 4.2. We observe that the index set for $\{i\}$ may contain more than one element, for one can construct a space $M = bH^2 \oplus gH^2$ contained in B^2 where b, g satisfy the Carleson measure condition and

$$\|bf + gh\|_M^2 = \|f\|_{H^2}^2 + \|h\|_{H^2}^2.$$

All that is required is to choose b, g in such a way that $bH^2 \cap gH^2 = \{0\}$. One way of doing this is as follows:

By the remarks following Theorem 1.7 in [9], each element of the Bergman space B^4 satisfies the Carleson measure condition since it is trivially (by virtue of Schwarz's Inequality) a multiplier of H^2 into B^2. From the same remarks, $H^2 \subset B^4$. Hence H^2 functions also satisfy the Carleson measure condition. Now choose a B^4 function b whose zeros $\{z_n\}$ do not satisfy the Blaschke condition (see [6, Theorem 4.6]) $\Sigma_n(1 - |z_n|) < \infty$. Hence

$$bH^2 \cap H^2 = \{0\}$$

because the zeros of any H^2 function satisfy the Blaschke condition. Let g be any H^∞ function so that gH^2 is contained in H^2 and hence in B^2. Clearly

$$bH^2 \cap gH^2 = \{0\}.$$

5. The case when M is contained in H^p

COROLLARY 5.1. *Let $M \subset H^p$ $(1 \le p \le \infty)$. Then*

$$M = bH^2$$

for a unique b:

(i) *If $1 \le p \le 2$, $b \in H^{2p/2-p}$.*

(ii) *If $p < 2$, $b = 0$.*

Further, $\|bf\|_M = \|f\|_{H^2}$ for all f in H^2 $(1 \le p \le 2)$.

Proof. *Case* 1. $1 \le p \le 2$. By the proposition,

$$M = N \oplus S(N) \oplus S^2(N) \oplus \cdots$$

where $N = M \ominus S(M)$. Further, by arguments identical to the proof of Corollary 4.1, we conclude that each b in N multiplies H^2 into H^p. Thus using the fact that on the circle $L^2 = H^2 \oplus zH^2$, we conclude that b multiplies L^2 into L^p.

Let $g \in L^q$, for some q, be such that g multiplies L^2 into L^p. Then,

$$\int |fg|^p < \infty \quad \text{for all } f \in L^2.$$

That is,

$$\int |f|^p |g|^p < \infty \quad \text{for all } |f|^p \in L^{2/p}.$$

Hence,

$$\int |g|^p h < \infty \quad \text{for all } h \in L^{2/p} \text{ and } h \ge 0.$$

As every $h \in L^{2/p} = (h_1 - h_2) + i(h_3 - h_4)$ where $h_i \in L^{2/p}$ and $h_i \ge 0$, we have

$$|g|^p h \in L^1 \quad \text{for all } h \in L^{2/p}.$$

Thus by the converse to Hölder's Inequality (see, [10, page 136]), $|g|^p$ is in the dual of $L^{2/p}$; that is,

$$|g|^p \in L^{2/2-p}$$

Hence,

$$g \in L^{2p/2-p}.$$

So the set of multipliers of L^2 into L^p $(1 < p < 2)$ is the space $L^{2p/2-p}$. Thus $b \in H^{2p/2-p}$.

Note that $2p/2 - p \geq 2$ as $2 \geq p \geq 1$. Hence $b \in H^2$.

Next we show that N is one dimensional. Suppose b and d are two mutually orthogonal elements in N. Then it is not difficult to see that $bH^2 \perp dH^2$. Further, $bd = db$ lies in bH^2 as well as dH^2. This means that $bd = 0$. As b and d are analytic functions, one of them is zero. Hence $M = bH^2$. Again using the same arguments as in the proof of Corollary 4.1, we can show that

$$\|bf\|_M = \|f\|_{H^2}.$$

Case 2. $2 < p$. In the decomposition of M, we shall show that $N = \{0\}$. This shall establish that $M = \{0\}$. So let b be any element in N. Proceeding as in the previous case we conclude that b multiplies L^2 into L^p $(\subseteq L^2)$ and hence b is in $L^\infty \cap H^p = H^\infty$. Choose a suitable $\varepsilon > 0$ such that $E = \{\vartheta: |b(\vartheta)| > \varepsilon\}$ has a positive measure. Let g be a function such that g vanishes on the complement of E and g is in L^2 but not in L^p. But bg is in L^p and so g will lie in L^p since b is invertible on E. This contradiction stems from the assumption that $b \neq 0$. Hence every b in N is zero and thus $N = \{0\}$.

Hence $M = \{0\}$.

6. The theorem of de Branges

COROLLARY 6.1 (THEOREM A). *Let M be contractively contained in H^2. Then there is a unique b in the unit ball of H^∞ such that $M = bH^2$ and $\|bf\|_M = \|f\|_{H^2}$.*

Proof. In view of Corollary 5.1, case 1, $p = 2$, all that is required is to show that $\|b\|_\infty \leq 1$. Now

$$\begin{aligned}\|bf\|_{H^2} &\leq \|bf\|_M \text{ (as } M \text{ is contractively contained in } H^2\text{)}\\ &= \|f\|_{H^2}\end{aligned}$$

So $\text{Sup}\{\|bf\|_{H^2}: \|f\|_{H^2} \leq 1\} \leq 1$; that is $\|b\|_\infty \leq 1$.

7. The case when M is contained in *BMOA* (*VMOA*)

COROLLARY 7.1. *Let M be contained in BMOA (VMOA). Then $M = \{0\}$.*

Proof. Note that *BMOA* (*VMOA*) is contained in $\cap H^p$ and hence in H^p for $p \geqslant 2$. The corollary is now obvious by applying Corollary 5.1, case 2.

8. The case when M is contained in the Dirichlet space A^2

COROLLARY 8.1. *Let M be contained in A^2. Then $M = \{0\}$.*

Proof. Proceeding as in Corollary 4.1, we conclude that for any non-zero b in N, bH^2 is contained in A^2 and $\|bf\|_M = \|f\|_{H^2}$. Further by the closed graph theorem, multiplication by b is a bounded linear operator from H^2 into A^2. Thus there exists a constant k such that

$$\|bf\|_{A^2} \le k\|f\|_{H^2} \quad \text{for all } f \text{ in } H^2.$$

Let $f(z) = z^n$; then as $n \to \infty$, $\|bz^n\|_{A^2} \to \infty$. On the other hand $\|z^n\|_{H^2} = 1$ for all n. This contradiction implies that b must be zero. Hence $N = \{0\}$, so $M = \{0\}$.

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