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OPTIMAL DESIGNS UNDER A CERTAIN CLASS OF NON-ORTHOGONAL ROW-COLUMN STRUCTURE

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SUMMARY. So far the study of optimality of block designs (eliminating heterogeneity in two directions) has been confined *exclusively* to situations where the row-column incidence structure is *orthogonal* (in the sense that all cells are non-empty). In this article we pose and solve the optimality problem (for inferring on a full set of orthonormal treatment contrasts) in the setting of block designs involving $b \times b$ arrays where *all* the cells along a transversal are empty. For $b \equiv 1 \pmod v$ universally optimal designs are available and for $b \equiv 0 \pmod v$ A-, D-, and E-optimal designs have been characterised and constructed.

1. Introduction

There is a good deal of literature available on the combinatorial, constructional, and analysis aspects of designs eliminating heterogeneity in two or more directions. Usually, in such set ups, it is assumed that the row column structure is orthogonal. That is to say, the incidence pattern of row column is taken as one represented by the matrix $J = ((1))$ of all 1's. Technically, this means that all the cells are assumed to be non-empty. However, in practice, situations may arise when some cells may remain empty because those combinations of levels are infeasible. In such cases, the usual analysis breaks down and appropriate modifications are needed. Aggarwal (1966a) derived the distribution of adjusted row and column sum of squares and the conditions for orthogonality of estimable row, column and treatment contrasts for a general row-column incidence structure, allowing for such empty cells. In subsequent papers (Aggarwal (1966b, 1966c), Aggarwal & Sharma 1976) he presented a series of two-way designs covering the situations where the cells along the principal diagonal are empty. Recently Adhikary and Panda (1983) brought out and explained some concrete physical situations where a sort of peculiarity in the row-column structure *cannot* be overlooked. This is that some combinations of rows and columns may not be feasible when identified with levels of some organic and inorganic manures in the context of agricultural experiments. These are what motivated us to take up a study on the analysis of designs underlying two-way elimination of heterogeneity with a non-orthogonal

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row-column structure, non-orthogonality in the sense of empty cells along a transversal. Without any loss of generality, the transversal can be and will be taken along the diagonal. In this article, we consider an arrangement of v treatments in a square array of size $b \times b$, where the cells along the principal diagonal are supposed to be infeasible. We have characterized the combinatorial aspects of the optimal designs under this non-orthogonal set up assuming usual fixed effects model. We have also considered the construction of some series of such designs. In the process it turns out that construction of such optimal designs is rather involved. This leads us to the study of the relative efficiencies of Aggarwal's (1966b) designs and their generalizations having certain nice simple structure compared to the actual optimal designs characterized by us. It turns out that these designs are indeed highly efficient with respect to the usual optimality criteria.

The usual fixed effects model e.g.,

$$y_{jj'}(h) = \mu+\alpha_j+\beta_{j'}+\tau_h+e_{jj'}, \quad 1 \leqslant h \leqslant v;\ 1 \leqslant j \neq j' \leqslant b$$

where μ, α_j, $\beta_{j'}$, τ_h stand respectively for general effect, j-th row effect, j'-th column effect, h-th treatment effect and $e_{jj'}$'s are i.i.d. $N(0, \sigma^2)$ will be taken for granted under this non-orthogonal set up.

For a specified design

$$L_{(v\times b)} = ((l_{hj})),\ M_{(v\times b)} = ((m_{hj'})),\ N_{(b\times b)} = ((n_{jj'}))$$

stand respectively for 'treatment-row', 'treatment-column' and 'row-column', incidence matrices. In the present set-up, $N = J-I_b$ where I_b is the identity matrix of order b.

As usual, following Kiefer (1958, 1975), Cheng (1978), Bagchi (1982), we are interested in linear inferential problems involving treatment contrasts only and as such we refer to the underyling C-matrix of the design. Let $r = (r_1, r_2, \ldots, r_v)'$ be the vector of treatment replications. Let further

$$n_{.j'} = \sum_{j=1}^{b} n_{jj'}, \quad n_{j.} = \sum_{j'=1}^{b} n_{jj'}.$$

Then, following Aggarwal (1966a),

$$C = B_{22}-X_{12}'A_{11}^{*}X_{12}$$

where

$$B_{22} = \text{diag}(r_1, r_2, \ldots, r_v)-L\,\text{diag}\left(\frac{1}{n_{1.}}, \frac{1}{n_{2.}}, \ldots, \frac{1}{n_{b.}}\right)L'$$

$$X_{12} = -M'+N'\,\text{diag}\left(\frac{1}{n_{1.}}, \ldots, \frac{1}{n_{b.}}\right)L'$$

$$A_{11} = \text{diag}(n_{.1}, n_{.2}, \ldots, n_{.b})-N'\,\text{diag}\left(\frac{1}{n_{1.}}, \ldots, \frac{1}{n_{b.}}\right)N.$$

Here * over a matrix denotes its generalized inverse. We will deduce explicit expression for the C-matrix with $N = J-\varepsilon I$ where $\varepsilon = 0$ or 1. Let k denote the constant row and column sizes i.e. $k = n_{.j'} = n_{j.}$, $1 \leqslant j, j' \leqslant b$. Note that for $\varepsilon = 0$, $k = b$ and for $\varepsilon = 1$, $k = b-1$. Thus we get

$$C_d = D_r - \frac{LL'}{k} - \left\{-M+\frac{LN}{k}\right\}\left\{kI-\frac{N'N}{k}\right\}^*\left\{-M+\frac{N'L'}{k}\right\}$$

$$= D_r - \frac{LL'}{k} - \left\{-M+\frac{LJ-\varepsilon L}{k}\right\}\frac{k}{k^2-\varepsilon^2}\left[I-\frac{b-2\varepsilon}{k^2-\varepsilon^2}J\right]^*$$

$$\times\left\{-M'+\frac{JL'-\varepsilon L'}{k}\right\}.$$

Next note that

$$\left[I-\frac{b-2\varepsilon}{k^2-\varepsilon^2}J\right]^* = [I-J/b]^* = I-J/b \quad \text{for } \varepsilon = 0 \text{ or } 1.$$

Hence $$C_d = D_r - \frac{LL'}{k+\varepsilon} - \frac{MM'}{k+\varepsilon} - \frac{\varepsilon}{k^2-\varepsilon^2}(L+M)(L+M)' + \frac{k+\varepsilon}{(k-\varepsilon)bk}rr'.$$

For $\varepsilon = 0$, we get the usual form of this C-matrix and for $\varepsilon = 1$, we get

$$C_d = D_r - \frac{LL'}{b} - \frac{MM'}{b} - \frac{(L+M)(L+M)'}{b(b-2)} + \frac{rr'}{(b-1)(b-2)}. \quad \ldots \quad (1.1)$$

In our later derivation we will use the above C-matrix in (1.1). In Section 2 we discuss the problems of characterization and construction of optimal designs under the special case of $b = mv+1$. Then, in Sections 3 and 4, we take up the same problem for the case of $b = mv$. In Section 5 we will calculate efficiency of some classes of other designs considered by Aggarwal.

2. Universal optimality results for $b = mv+1$

Let Ω be the relevant class of $b \times b$ array designs. We will see below that whenever $b = mv+1$, universally optimal designs (vide Kiefer, 1975) exist. We state below Proposition 1 in Kiefer (1975) for this purpose.

If there exists a design d^* in Ω such that

(1) d^* is completely symmetric (c.s.) i.e. C_{d^*} is of the form $aI+bJ$.

(2) d^* maximizes the trace of C_d in Ω,

then d^* is universally optimal.

In case of $b = mv+1$, let d^* be a design which assigns each treatment m times in each row and in each column. Then d^* is universally optimal. The proof is extremely simple. First we show that d^* maximizes the trace of C_{d^*}. Referring to (1.1),

$$\text{trace of } C_d = \sum_{i=1}^{v} r_i - \sum_{i=1}^{v}\sum_{j=1}^{mv+1} \frac{l_{ij}^2}{mv+1} - \sum_{i=1}^{v}\sum_{j=1}^{mv+1} \frac{m_{ij}^2}{mv+1}$$

$$- \sum_{i=1}^{v}\sum_{j=1}^{mv+1} \frac{(l_{ij}+m_{ij})^2}{(mv+1)(mv-1)} + \frac{\Sigma r_i^2}{mv(mv-1)}$$

$$= \Sigma r_i - \frac{1}{mv+1} \sum_{i=1}^{v} \left[\sum_{j=1}^{mv+1} \left(l_{ij} - \frac{r_i}{mv+1}\right)^2 \right]$$

$$- \frac{1}{mv+1} \sum_{i=1}^{v} \left[\sum_{j=1}^{mv+1} \left(m_{ij} - \frac{r_i}{mv+1}\right)^2 \right]$$

$$- \frac{1}{(mv-1)(mv+1)} \sum_{i=1}^{v} \left[\sum_{j=1}^{mv+1} \left(l_{ij}+m_{ij} - \frac{2r_i}{mv+1}\right)^2 \right]$$

$$- \frac{1}{mv(mv+1)} \sum_{i=1}^{v} r_i^2.$$

Note that $\sum_{j=1}^{mv+1} l_{ij} = \sum_{j=1}^{mv+1} m_{ij} = r_i,\ i = 1, 2, \ldots, v.$

The sum of squares are all "$\geqslant 0$" and "$=$" 0 for d^* since in d^*, l_{ij}'s and m_{ij}'s are all equal to m. Moreover, $\sum_{i=1}^{v} r_i^2$ is the least for d^* since d^* is equi-replicate. This settles the part on trace maximization. Next, it is also evident that C_{d^*} is completely symmetric. Hence d^* is universally optimal. Such types of designs can be easily constructed. We construct a Latin Square with $b = mv+1$ symbols, say $0, 1, 2, \ldots, mv$ such that along the diagonal the symbol 0 occurs. Then we delete the diagonal and reduce the rest of the symbols mod v.

3. Specific optimality results for $b = mv$

Let Δ be the relevant class of connected designs for $b = mv$. In such a case, the C-matrix of an A-, D-, and E-optimal design is completely symmetric but it does not necessarily produce maximum value of the trace of C_d in Δ.

Take, for example $v = 3$, $m = 2$. The design d_1 has larger trace than that of d^* (which will be shown to be E-optimal) where (0, 1, 2 denoting the treatments)

$$d_1 = \begin{bmatrix} X & 0 & 1 & 0 & 1 & 2 \\ 1 & X & 0 & 1 & 2 & 0 \\ 2 & 1 & X & 1 & 0 & 0 \\ 0 & 2 & 0 & X & 2 & 1 \\ 0 & 1 & 2 & 0 & X & 1 \\ 1 & 0 & 2 & 2 & 0 & X \end{bmatrix} \qquad d^* = \begin{bmatrix} X & 1 & 2 & 0 & 2 & 0 \\ 1 & X & 0 & 2 & 2 & 1 \\ 2 & 0 & X & 0 & 1 & 1 \\ 0 & 0 & 1 & X & 1 & 2 \\ 0 & 2 & 2 & 1 & X & 0 \\ 1 & 2 & 1 & 2 & 0 & X \end{bmatrix}$$

Thus Proposition 1 of Kiefer (1975) is *not* applicable as regards universal optimality. So we look for specific optimality. Let d^* be a equireplicate design for which (i) C_{d^*} is completely symmetric and (ii) the diagonal components are such that

$$C_{d^*hh} = \max_{\{d \epsilon \Delta \,:\, r_h = \bar{r}\}} C_{dhh} \text{ for every } h.$$

(Here $\bar{r}$ stands for the constant replication size of the treatments under d^* i.e. $\bar{r} = m(mv-1)$).

In the following we will establish D-, A-, and E-optimality of d^*. We follow essentially the technique in Kiefer (1975).

Let

$$G = \{0, 1, 2, \ldots, mv(mv-1)\}$$
$$G_1 = \{n : n \,\epsilon\, G, \text{ and } n \text{ is a multiple of } mv\}.$$

$[C, D]$ is defined to be an elementary interval, where C, D are two consecutive integers $\epsilon\ G_1$.

$$\bar{r} = m(mv-1),\ \bar{r} \,\epsilon\, [(m-1)mv, m.mv] = [C_0, D_0] \text{ say.}$$

Suppose it is required to minimize $\phi^*(C_d)$ over all $d \,\epsilon\, \Delta$. Suppose, further, ϕ^* has the representation

$$\phi^*(\lambda_{d1}, \lambda_{d2}, \ldots, \lambda_{d(v-1)}) = \sum_{i=1}^{v-1} f(\lambda_{di}),$$

where f is convex, and assumed to be non-increasing over $[0, \infty]$ and λ_{di}'s $i = 1, 2, \ldots v-1$ are the non-zero characteristic roots of the c_d-matrix. We say that d^* is ψ_f optimal in Δ if $\sum_{i=1}^{v-1} f(\lambda_{d^*i})$ is minimum over all possible designs in Δ. Using steps 4 in Kiefer (1975), one gets

$$\sum_{i=1}^{v-1} f(\lambda_{di}) \geqslant \frac{v-1}{v} \sum_{h=1}^{v} f\left(\frac{v}{v-1}\, g(r_h)\right)$$

where $g(r_h) = \max_{\{d\in\Delta\,:\, h\text{-th replication} = r_h\}} C_{d_{hh}}$.

The problem then reduces to checking that $\Sigma f\left(\frac{v}{v-1}\, g(r_h)\right)$ attains its minimum for $d = d^*$. In effect, we have to establish the validity of

$$\Sigma f\left(\frac{v}{v-1}\, g(r_h)\right) \geqslant vf\left(\frac{v}{v-1}\, g(\bar{r})\right) \qquad \dots \quad (3.1)$$

we define

$$q(r) = -f\left(\frac{v}{v-1}\, g(r)\right)$$

where r is taken to be only non-negative integer. Then (3.1) would follow at once if $q(r)$ were concave i.e. $g(r)$ were concave but this is not the case always. This is what motivates further development (vide Kiefer, 1975).

Let $\bar{q}(r)$ be the concave envelope of $q(r)$ i.e. the function $\geqslant q$ for which second order differences

$$\bar{q}(r+2)+\bar{q}(r)-2\bar{q}(r+1) \text{ are all } \leqslant 0.$$

Then (3.1) will still hold if

$$\bar{q}(\bar{r}) = q(\bar{r}). \qquad \dots \quad (3.2)$$

We will show that a sufficient condition for (3.2) to hold is that

$$q(r+1)-q(r) \not\uparrow \text{ in } r \text{ for } C_0 \leqslant r < D_0-1$$

i.e. q is concave in $C_0 \leqslant r \leqslant D_0-1$ and, moreover,

$$q(\bar{r}+1)-q(\bar{r}) \geqslant q(D_0)-q(D_0-1).$$

$$\text{Set } \quad r_h = mv\left[\frac{r_h}{mv}\right]+t.$$

$$= mv\,u+t \text{ say,}$$

where $\left[\frac{r_h}{mv}\right]$ is the largest integer not greater than $\frac{r_h}{mv}$. Then

$$C_{d_{hh}} = r_h - \frac{1}{mv}\sum_{j=1}^{mv} l_{hj}^2 - \frac{1}{mv}\sum_{j=1}^{mv} m_{hj}^2$$

$$- \frac{1}{mv(mv-2)}\sum_{j=1}^{mv} (l_{hj}+m_{hj})^2 + \frac{r_h^2}{(mv-1)(mv-2)}.$$

We derive below the expression for $g(r_h)$ i.e., max of $C_{d_{hh}}$ subject to the condition $\sum_{j=1}^{mv} l_{hj} = r_h$ and $\sum_{j'=1}^{mv} m_{hj}' = r_h$. It is clear that the minimum of $\sum_{j=1}^{mv} l_{hj}^2$ subject to $\sum_{j=1}^{mv} l_{hj} = r_h$ is attained when t of l_{hj}'s are $u+1$ and $mv-t$ are u. Similarly for $m_{hj'}$'s. Now in order to attain minimum of $\sum_{j=1}^{mv} (l_{hj}+m_{hj})^2$ such that $\sum_{j=1}^{mv} (l_{hj}+m_{hj}) = 2r_h$, first note that
under case (*i*) *e.g.* $t < mv/2$

$$\left[\frac{2r_h}{mv}\right] = 2\left[\frac{r_h}{mv}\right] = 2u$$

and hence the minimum is attained when $2t$ of $(l_{hj}+m_{hj})$ $j = 1, 2, \ldots, mv$ are $2u+1$ and rest are $2u$.

Again, *under case* (*ii*) *e.g. when* $t \geqslant \frac{mv}{2}$,

$$\left[\frac{2r_h}{mv}\right] = 2\left[\frac{r_h}{mv}\right]+1 = 2u+1.$$

Hence, the minimum is attained when $2t-mv$ of $(l_{hj}+m_{hj})$ are $2u+2$ and rest $2(mv-t)$ are $2u+1$.

It can be checked that

under case (*i*) : $t < mv/2$

$$g(r) = \max_{\{d\,:\,r_h=r\}} C_{d_{hh}}$$

$$= r - \frac{2}{mv}\{-mvu^2+(2r-mv)u+r\}$$

$$+\frac{r^2}{(mv-1)(mv-2)} - \frac{-4mvu^2+8ur+2r-2mvu}{mv(mv-2)} \quad [\because t = r-mvu]$$

and *under case* (*ii*) : $t \geqslant mv/2$

$$g(r) = r - \frac{2}{mv}\{-mvu^2+(2r-mv)u+r\} + \frac{r^2}{(mv-1)(mv-2)}$$

$$- \frac{-4mvu^2+8ur+6r-6mvu-2mv}{mv(mv-2)}$$

so that

$$g(r) = \begin{cases} g_1(r) = A(r)+B_1(r) & \text{if } t < \dfrac{mv}{2} \\ & \text{i.e. if } \dfrac{r}{mv} - \left[\dfrac{r}{mv}\right] < \dfrac{1}{2} \\ g_2(r) = A(r)+B_2(r) & \text{if } t \geqslant \dfrac{mv}{2} \\ & \text{i.e. if } \dfrac{r}{mv} - \left[\dfrac{r}{mv}\right] \geqslant \dfrac{1}{2} \end{cases}$$

where

$$A(r) = r - \frac{2}{mv}\left[-mvu^2+(2r-mv)u+r\right] + \frac{r^2}{(mv-1)(mv-2)} + \frac{4mvu^2-8ur}{mv(mv-2)}$$

$$= \frac{1}{(mv-1)(mv-2)}\, r^2 + r\left\{1 - \frac{4u}{mv} - \frac{8u}{mv(mv-2)} - \frac{2}{mv}\right\}$$

$$+2u^2+2u+\frac{4mvu^2}{mv(mv-2)}$$

$$B_1(r) = -\frac{2r-2mvu}{mv(mv-2)}$$

$$B_2(r) = -\frac{6r-6mvu-2mv}{mv(mv-2)}$$

$$g(r) = \begin{cases} g_1(r) & \text{if } umv \leqslant r < umv+mv/2 \\ g_2(r) & \text{if } umv+mv/2 \leqslant r < (u+1)mv \quad u = 0, 1, 2, \ldots, mv-2. \end{cases}$$

Note also that for $umv+mv/2 \leqslant r < (u+1)mv$, $u = 0, 1, 2, \ldots, mv-2$

$$g_1(r)-g_2(r) = \frac{4r-4mvu-2mv}{mv(mv-2)}$$

$$= \frac{4(r-mvu)-2mv}{mv(mv-2)} = \frac{4t-2mv}{mv(mv-2)} > 0 \text{ if } t \geqslant mv/2. \quad \ldots \ (3.3)$$

Since our problem is to find a concave envelope of $q(r) = -f\left(\frac{v}{v-1}\, g(r)\right)$ and $-f$ is increasing, it suffices to find a concave envelope of a larger function. However, for (3.2) to hold, it is necessary that q remains unchanged at $\bar{r}$. So instead of working with the two different expressions for $g(r)$ in the two halves, we will work with $g_1(r)$ alone in all the intervals *except* for the interval $[C_0, D_0]$ containing $\bar{r}$. In other words we will work with

$$g(r) = \begin{cases} g_1(r), & r \leqslant C_0+[mv/2], \quad r \geqslant D_0 \\ g_2(r) & C_0+[mv/2]+1 \leqslant r \leqslant D_0-1. \end{cases}$$

denoting by $[x]$, the greatest integer $\leqslant x$.

Now we study some properties of the function $g(r)$. The first property relates to its nature over the range of values of r. We have

$$\Delta_1(r) = g_1(r+1) - g_1(r)$$

$$= \frac{2mvr + [(mv)^3 - 5(mv)^2 + 7mv - 2] - 4umv(mv-1)}{mv(mv-1)(mv-2)} \quad \ldots \quad (3.4)$$

assuming $$\left[\frac{r}{mv}\right] = \left[\frac{r+1}{mv}\right] = u.$$

For fixed u, as $r \uparrow$, this difference $\uparrow$. Further, in the u-th interval, at the least value of r e.g. mvu, one has

$$\Delta_1(mvu) = \frac{(m^3v^3 - 5m^2v^2 + 7mv - 2) - 2umv(mv-2)}{mv(mv-1)(mv-2)}$$

and this is positive iff

$$\frac{m^3v^3 - 5m^2v^2 + 7mv - 2}{2mv(mv-2)} > u > 0 \quad (\because mv > 2)$$

i.e. iff $$u \leqslant \left[\frac{m^2v^2 - 3mv + 1}{2mv}\right]$$

$$= \text{greatest integer contained in } \frac{m^2v^2 - 3mv + 1}{2mv} \quad \ldots \quad (3.5)$$

Again, when $r = (u+1)mv - 1$ i.e. at the penultimate value of the u-th interval, we get, using $\left[\frac{r}{mv}\right] = u$, $\left[\frac{r+1}{mv}\right] = u+1$,

$$\Delta_1((u+1)mv-1) = \frac{-2u}{(mv-1)} + \frac{m^3v^3 - 5m^2v^2 + 7mv - 2}{mv(mv-1)(mv-2)} \quad \ldots \quad (3.6)$$

and $$\Delta_1((u+1)mv-1) > 0 \text{ iff } u \leqslant \left[\frac{m^2v^2 - 3mv + 1}{2mv}\right] = u_0 \text{ (say)}. \quad \ldots \quad (3.7)$$

Thus (3.5) and (3.7) together imply that $g_1(r) \uparrow$ right from the start (i.e. $r = 0$) to $r = (u_0+1)mv$, covering thereby all the intermediate points in the intervals corresponding to $u = 0$ through $u = u_0$. A similar behaviour of the function $g_2(r)$ can be observed. A study of the functions g_1 and g_2 inside $[C_0, D_0]$ reveal the following facts :

(i) $\frac{mv}{2}$ = integer : $g_1\left(C_0 + \frac{mv}{2}\right) = g_2\left(C_0 + \frac{mv}{2}\right) \leqslant g_2\left(C_0 + \frac{mv}{2} + 1\right)$

$$\leqslant \ldots \leqslant g_2(D_0 - 1) \leqslant g_1(D_0)$$

(ii) $\frac{mv}{2} \neq$ integer : $g_1\left(C_0 + \frac{mv}{2} + 1\right) \leqslant g_2\left(C_0 + \frac{mv+1}{2}\right) \leqslant g_2\left(C_0 + \frac{mv+3}{2}\right)$

$$\leqslant \ldots \leqslant g_2(D_0 - 1) \leqslant g_1(D_0).$$

Again, taking $r = (u+1)mv-2$ in the u-th elementary interval (so that $r+1 = (u+1)mv-1$), we get,

$$\Delta_1((u+1)mv-2) = \frac{-2u}{mv-1} + \frac{m^3v^3-3m^2v^2+3mv-2}{mv(mv-1)(mv-2)}$$

and this is $\quad < 0 \text{ iff } u > \dfrac{m^2v^2-3mv+1}{2mv}+1.$ ... (3.8)

Thus (3.7) and (3.8) together imply that $g_1 \downarrow$ in r for $u \geqslant u_0+2$. In the interval $[(u_0+1)mv, (u_0+2)mv]$ we cannot infer about the behaviour of $g(r)$. To finalise, we see that the function g exhibits the following pattern (noting that u_0 is to the right of D_0).

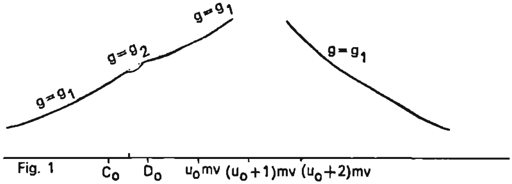

Fig. 1

We now derive a sufficient condition for (3.2) using the properties of g. Suppose $\bar{D} \in G$ is the first integer where g attains its absolute maximum. By the above derivation, $D_0 < \bar{D} \leqslant (u_0+2)mv$. Since $-f$ is monotone, to prove (3.2) it is enough to consider $G' = \{r : r \in G, r \leqslant \bar{D}\}$. Again in any elementary interval $[C, D]$, $C, D \leqslant \bar{D}$ there may exist $r' \in [C, D]$ such that $g(r') \leqslant g(C)$. Then because of the increasing nature of $-f$, we have $q(r') \leqslant q(C)$.

Since we are concerned with a concave envelope of $q(r)$, G' can be replaced by its subset G'' obtained by excluding *such* points r' so that for any two points r_1 and r_2 in G'' we have $q(r_1) \leqslant q(r_2)$ whenever $r_1 \leqslant r_2$. Now a *sufficient condition* for existence of a concave envelope with $\tilde{q}(\bar{r}) = q(\bar{r})$ is local concavity of q at $\bar{r}$. That is,

$$q(r_1+1)-q(r_1) \geqslant q(r_2)-q(r_2-1), \qquad \text{... (3.9)}$$

whatever $r_1+1 \leqslant \bar{r} \leqslant r_2-1$, $r_1, r_2 \in G''$.

We may note that r_2-1 need not belong to G''. As in Kiefer (1975) one can establish (3.9) by proving (i) to (iii) stated below :

$$\left.\begin{array}{ll} \text{(i)} & \min\limits_{0<r_1<\bar{r}=m(mv-1)} [q(r_1+1)-q(r_1)] = q(\bar{r})-q(\bar{r}-1) \\ \text{(ii)} & \max\limits_{r<r_2 \in G''} [q(r_2)-q(r_2-1)] = q(\bar{r}+1)-q(\bar{r}) \\ \text{(iii)} & q(\bar{r})-q(\bar{r}-1) \geqslant q(\bar{r}+1)-q(\bar{r}). \end{array}\right\} \qquad \text{... (3.10)}$$

We will show that for (3.10) to hold, a sufficient condition is

$$\left.\begin{array}{l}\text{concavity of } q \text{ within } C_0 \leqslant r \leqslant D_0-1 \\ \text{and } q(\bar{r}+1)-q(\bar{r}) \geqslant q(D_0)-q(D_0-1)\end{array}\right\} \quad \ldots \quad (3.11)$$

Remark 1 : It has been pointed out later that (3.10)(iii) is not satisfied when $m = 1$. So we take up a proof of sufficiency of (3.11) below for $m \geqslant 2$.

Clearly (3.11) directly implies (3.10)(iii).

To prove (3.10)(i) take r_1 in $[C, D]$ where $[C, D]$ is an elementary interval to the left of $[C_0, D_0]$ so that $D \leqslant C_0$.

By (3.4) $\Delta_1(r_1) \geqslant \Delta_1(C) \;\forall\; r_1 \leqslant C+mv-2$.

Again, for $r_1 = C+mv-1$, by (3.6)

$$\Delta_1(r_1)mv(mv-1)(mv-2) = -2mv(mv-2)u+(mv)^3-5(mv)^2+7mv-2$$

where we write $C = mvu$.

Also, by (3.4),

$$\Delta_1(C)mv(mv-1)(mv-2) = 2(mv)^2u+(mv)^3-5(mv)^2+7mv-2-4umv(mv-1)$$

so that

$$[\Delta_1(r_1)-\Delta_1(C)]mv(mv-1)(mv-2) = 0.$$

Hence, $\quad \Delta_1(r_1) \geqslant \Delta_1(C) \;\forall\; r_1 \in [C, D],\ r_1 < D.$

Let $D_1 = u_1mv$ and $D_2 = u_2mv$, $u_2 > u_1$.

By (3.4),

$$\begin{aligned}\{\Delta_1(D_1)-\Delta_1(D_2)\}&mv(mv-1)(mv-2)\\ &= 2(mv)^2(u_1-u_2)-4mv(mv-1)(u_1-u_2)\\ &= 2mv(u_1-u_2)(2-mv) > 0 \text{ since } u_1 < u_2 \text{ and } mv > 2.\end{aligned}$$

This implies $\Delta_1(D_1) > \Delta_1(D_2)$.

Thus, finally,

$$\Delta_1(r_1) \geqslant \Delta_1(C) > \Delta_1(C_0) > 0$$

where $C_0 = (m-1)mv$. The last inequality follows from (3.5) as

$$m-1 < \frac{m^2v^2-3mv+1}{2mv} \text{ for } v \geqslant 3,\ m \geqslant 2.$$

Set now $g(r_1+1) = y_1$, $g(r_1) = y_2$, $g(C_0+1) = y_3$, $g(C_0) = y_4$.

Then $\Delta_1(r_1) > \Delta_1(C_0) > 0$ means

$$y_2 < y_1 \leqslant y_4 < y_3$$

and

$$y_1-y_2 > y_3-y_4 > 0.$$

Since $-f \uparrow$ and concave, by Mean–Value Theorem,

$$\frac{-f(y_1)+f(y_2)}{y_1-y_2} \geqslant \frac{-f(y_3)+f(y_4)}{y_3-y_4}$$

i.e. $$-f(y_1)+f(y_2) \geqslant \frac{y_1-y_2}{y_3-y_4}\,[-f(y_3)+f(y_4)] \geqslant [-f(y_3)+f(y_4)]$$

$$[\because y_1-y_2 \geqslant y_3-y_4]$$

i.e. $$q(r_1+1)-q(r_1) \geqslant q(C_0+1)-q(C_0)$$

So $$\min_{r_1 \leqslant C_0} [q(r_1+1)-q(r_1)] = q(C_0+1)-q(C_0).$$

This, together with (3.11), i.e. concavity of q within $C_0 \leqslant r \leqslant D_0-1$ implies

$$\min_{0 \leqslant r_1 < \bar{r}} [q(r_1+1)-q(r_1)] = q(\bar{r})-q(\bar{r}-1)$$

so that (3.10)(i) is settled.

We next prove that (3.11) $\Longrightarrow$ (3.10)(ii)

For this take any $r_2 \in [C, D]$ such that $C \geqslant D_0$

By (3.4), for any $r_2 \in \left[\frac{r_2}{mv}\right] = \left[\frac{r_2+1}{mv}\right] = \left[\frac{r_2+2}{mv}\right] = u,$

$$\Delta_1(r_2+1)-\Delta_1(r_2) = \frac{2}{(mv-1)(mv-2)} > 0.$$

Again, when $r_2 = (u+1)mv-2$, $r_2+1 = (u+1)mv-1$, $r_2+2 = (u+1)mv$, by (3.4) and (3.5),

$$\Delta_1((u+1)mv-1)-\Delta_1((u+1)mv-2) = \frac{-2mv}{mv(mv-1)} < 0.$$

Hence $\Delta_1(r_2) \leqslant \Delta_1(D-2)$ for $r_2 \in [C, D]$ with $C = umv$, $D = (u+1)mv$.. Also when $D_1 = u_1mv$, $D_2 = u_2mv$, $u_2 > u_1$, by (3.4),

$$mv(mv-1)(mv-2)\{\Delta_1(D_2-2)-\Delta_1(D_1-2)\} = (u_2-u_1)(-2m^2v^2+4mv) < 0$$

so that $$\Delta_1(D_2-2) < \Delta_1(D_1-2).$$

Thus for $r_2 \in [C, D]$, $C \geqslant D_0$, writing $\Delta(r) = g(r+1)-g(r)$,

$$0 < \Delta(r_2) \leqslant \Delta(D-2) \leqslant \Delta(D_0+mv-2).$$

Again refering to the point D_0, we get

$$\Delta(D_0-2) = g_2(D_0-1)-g_2(D_0-2) = \Delta_1(D_0-2)-\frac{4}{mv(mv-2)}$$

and
$$\Delta(D_0-1) = g_1(D_0)-g_2(D_0-1) = \Delta_1(D_0-1)+\frac{2}{mv}.$$

Further, we verify directly that

$$\Delta(D_0+mv-2) \leqslant \Delta(D_0-2) \leqslant \Delta(D_0-1)$$

Now, an application of the Mean-Value Theorem yields

$$q(D_0)-q(D_0-1) \geqslant q(r_2+1)-q(r_2) \;\forall\; r_2 \geqslant D_0-1.$$

Also, local concavity for $C_0 \leqslant r \leqslant D_0-1$ ensures

$$q(\bar{r}+1)-q(\bar{r}) \geqslant q(\bar{r}+2)-q(\bar{r}+1) \geqslant \dots \geqslant q(D_0-1)-q(D_0-2).$$

However, $\Delta(D_0-2) \leqslant \Delta(D_0-1)$ implies

$$q(D_0-1)-q(D_0-2) \leqslant q(D_0)-q(D_0-1).$$

This explains Remark 1 made earlier as $\bar{r}+1 = D_0$ for $m = 1$. A such, we require an additional condition to be verified. This is that

$$q(\bar{r}+1)-q(\bar{r}) \geqslant q(D_0)-q(D_0-1).$$

This forms the second condition in (3.11).

Thus, whenever condition (3.11) obtains, we get

$$\max_{\bar{r} \leqslant r_2 \in G} q(r_2+1)-q(r_2) = q(\bar{r}+1)-q(\bar{r}) \text{ which is (3.10)(ii).}$$

Thus we find that ψ_f-optimality follows from a verification of (3.11).

We will now adopt specific optimality criteria and interpret (3.11).

D-optimality criterion $\sum_{1}^{v-1} f(\lambda_{di}) = \log\left(\prod_{i=1}^{v-1} \lambda_{di}^{-1}\right)$ so that

$$f(\lambda_{di}) = \log 1/\lambda_{di}$$

and
$$q(r) = -\frac{v}{v-1}\, g(r) = -\log\frac{v}{v-1}\, g(r) = \log\left(\frac{v}{v-1}\, g(r)\right).$$

So (3.11) reads

$$\log g(r+1)-\log g(r) \leqslant \log g(r)-\log g(r-1) \;\forall\; C_0 < r < D_0-1 \quad \dots \quad (3.12a)$$

and
$$\log g(\bar{r}+1)-\log g(\bar{r}) \geqslant \log g(D_0)-\log g(D_0-1) \quad \dots \quad (3.12b)$$

Recalling the expressions for $g(r)$, we rewrite (3.12b) as

$$\frac{g_2(\bar{r}+1)}{g_2(\bar{r})} \geqslant \frac{g_1(D_0)}{g_2(D_0-1)}. \quad \dots \quad (3.12b)$$

Now define $g_i^*(r) = mv(mv-1)(mv-2)g_i(r)$ $i = 1, 2$.

Direct calculations yield

$$g_2^*(\bar{r}+1) = m^5v^3(v-1)-4m^4v^2(v-1)+m^3v(v^2+v-3)$$
$$-m^2v(v-4)-m(v+2)+2$$

$$g_2^*(D_0-1) = m^5v^3(v-1)-m^4v^2(3v-2)-m^3v^2(v-4)+m^2v(v-4)+3mv-2$$

$$g_1^*(\bar{r}) = m^5v^3(v-1)-4m^4v^2(v-1)+3m^3v(v-1)+2m^2v-2m$$

$$g_1^*(D_0) = m^3v^2\{m^2v(v-1)-m(3v-2)+2\}$$

so that

$$g_2^*(\bar{r}+1)g_2^*(D_0-1)-g_2^*(\bar{r})g_1^*(D_0)$$
$$= m^7v^4(v^2-6v+6)-m^6v^3(v^3-4v^2-11v+18)+m^5v^2(2v^3-20v^2+15v+12)$$
$$+m^4v^2(3v^2+11v-33)-2m^3v(4v^2-10v-7)+m^2v(v-22)$$
$$+4m(2v+1)-4 \geqslant 0 \;\forall\; v \geqslant 5 \;\forall\; m \geqslant 2.$$

This verifies (3.12b).

Now to check (3.12a), we have to show

(i) $g_1^2(r) \geqslant g_1(r-1)g_1(r+1) \;\forall\; C_0 < r < C_0+mv/2$

(ii) $g_2^2\left(C_0+\dfrac{mv+1}{2}\right) \geqslant g_1\left(C_0+\dfrac{mv-1}{2}\right)g_2\left(C_0+\dfrac{mv+3}{2}\right)$

in case $mv/2 \neq$ integer. Otherwise a similar inequality has to be verified.

(iii) $g_2^2(r) \geqslant g_2(r+1)g_2(r-1) \;\forall\; C_0+\dfrac{mv}{2}+1 \leqslant r \leqslant D_0-2.$

Recall that

$$g_1^*(r) = r^2mv+r(mv-1)(m^2v^2-4m^2v+2)+2(m-1)(mv-1)(m^3v^2-mv)$$

so that

$$g_1^{*2}(r)-g_1^*(r-1)g_1^*(r+1) = r^2\cdot 2m^2v^2+r\cdot 2mv(mv-1)(m^2v^2-4m^2v+2)-m^2v^2$$
$$+(mv-1)^2(m^2v^2-4m^2v+2)^2-4(m-1)(mv-1)(m^3v^2-mv)mv. \quad \ldots \quad (3.13)$$

Thus, in order to verify (i) above, equivalently one has to achieve non-negativity of (3.13) for $C_0 < r < C_0+\frac{mv}{2}$. Differentiating (3.13) with respect to r, we get,

$$4m^2v^2r+2mv(mv-1)(m^2v^2-4m^2v+2)$$

which is positive for $v > 5$. Therefore, replacing r by $C_0+1 = mv(m-1)+1$ in (3.13), we get

$$g_1^{*2}(r)-g_1^*(r-1)g_1^*(r+1) > 2m^4v^4(m-1)^2+m^2v^2+4m^3v^3(m-1)$$

$$+2m^2v^2(mv-1)(m-1)(m^2v^2-4m^2v+2)+2mv(mv-1)(m^2v^2-4m^2v+2)$$

$$+(mv-1)^2(m^2v^2-4m^2v+2)^2-4(m-1)(mv-1)mv(m^3v^2-mv). \quad \ldots \quad (3.14)$$

In (3.14)

$$2m^2v^2(mv-1)(m-1)(m^2v^2-4m^2v+2)-4(m-1)(mv-1)mv(m^3v^2-mv)$$

$$= 2mv(mv-1)(m-1)(m^3v^2(v-6)+4mv) \geq 0 \ \forall\ v \geq 6,\ \forall\ m \geq 2.$$

The remaining part of (3.14) is always ≥ 0.

So (3.14) is $\geq 0 \ \forall\ v \geq 6,\ \forall\ m \geq 2$.

Again, note that (3.14) can be rewritten as

$$2m^4v^4(m-1)^2+m^2v^2+4m^3v^3(m-1)+(mv-1)^2(m^2v^2-4m^2v+2)^2$$

$$+2mv(mv-1)(m^2v^2-4m^2v+2)+2mv(mv-1)(m-1)(m^3v^3-6m^3v^2+4mv) \quad \ldots \quad (3.15)$$

It can be checked that for $v = 5$, (3.15) is $\geq 0 \ \forall\ m \geq 2$.

This (i) holds for all $v \geq 5$, $m \geq 2$.

Remark 2: It may be noted that in checking (i) we do not make use of the upper bound of r. So it still holds as an adgebraic inequality even for

$$r > C_0+\frac{mv}{2}$$

Next we proceed to check (ii).

$$\text{For } \frac{mv}{2} = \text{integer}, \quad g_2\left(C_0+\frac{mv}{2}\right) = g_1\left(C_0+\frac{mv}{2}\right)$$

and, hence, (ii) follows from the above remark and (3.3)

For $\frac{mv}{2} \neq$ integer using the relation (3.3), we have

$$g_2^2\left(C_0+\frac{mv+1}{2}\right)-g_1\left(C_0+\frac{mv-1}{2}\right)g_2\left(C_0+\frac{mv+3}{2}\right)$$

$$= \left\{g_1^2\left(C_0+\frac{mv+1}{2}\right)-g_1\left(C_0+\frac{mv-1}{2}\right)g_1\left(C_0+\frac{mv+3}{2}\right)\right\}$$

$$+2(mv-1)+g_1\left(C_0+\frac{mv+1}{2}\right)-3\Delta_1\left(C_0+\frac{mv-1}{2}\right). \quad \ldots \quad (3.16)$$

In view of the above remark, writing X for $C_0+\frac{mv+1}{2}$ the quantity inside $\{\ldots\}$ in (3.15) is nonnegative and so (3.15) would be nonnegative whenever $g_1(X)-3\Delta_1(X-1)$ is so.

But

$$g_1(X)-3\Delta_1(X-1) = X^2mv+X\{(mv-1)(m^2v^2-4m^2v+2)-6mv\}$$
$$+3(5m^2v^2-7mv+2)+2(m-1)(mv-1)(m^3v^2+5mv)-3m^2v^3$$

and this is clearly nonnegative for all $v \geqslant 5$, $m \geqslant 2$.

Hence condition (ii) is verified for $v \geqslant 5$, $\forall\, m \geqslant 2$.

It remains to verify condition (iii) which is done below. Using the relations (3.3) and (3.13), we get, on simplification,

$$g_2^{*2}(r)-g_2^*(r+1)g_2^*(r-1)$$
$$= 2m^2v^2r^2+r\cdot 2mv(mv-1)(m^2v^2-4m^2v-2)-m^2v^2$$
$$+(mv-1)^2(m^2v^2-4m^2v+2)^2+4mv(mv-1)(6m^2v+3mv-8m)$$
$$-4m^2v^2(mv-1)(m^3v-m^2v-m+2mv+1). \qquad \ldots \quad (3.16)$$

As in case (i) (3.16) is also an increasing function in r. So putting the least value of r namely, $(m-1)mv+\frac{mv}{2}+1$ one can see that (3.16) is $\geqslant 0\ \forall\, v \geqslant 5$, $\forall\, m \geqslant 2$.

Hence D-optimality holds for $v \geqslant 5$, $\forall\, m \geqslant 2$.

A-optimality For A-optimality (which means $f(\lambda_{di}) = 1/\lambda_{di}$) we have to show

$$g(r)(g(r+1)+g(r-1)) \geqslant 2g(r-1)g(r+1)\ \forall\, C_0 < r < D_0-1 \qquad \ldots \quad (3.17a)$$

and
$$\frac{1}{g(D_0-1)}-\frac{1}{g(D_0)} \geqslant \frac{1}{g(\bar{r})}-\frac{1}{g(\bar{r}+1)}. \qquad \ldots \quad (3.17b)$$

Recalling the expression for $g(r)$ we rewrite (3.17b) as

$$g_2(\bar{r}+1)g_2(\bar{r})\Delta(D_0-1) \leqslant g_1(D_0)g_2(D_0-1)\Delta(\bar{r}). \qquad \ldots \quad (3.17b)$$

Since d^* is completely symmetric, D-optimality of d^* for $v \geqslant 5$ will automatically imply its A-optimality for $v \geqslant 5$. For $v = 4$, both the relations hold; consequently d^* is A-optimal. For $v = 3$, (3.17a) fails to hold. (We omit the details). As such, we cannot infer about A-optimality.

E-optimality. The task of proving E-optimality is so fascinating that by now there have appeared in the literature a considerable number of articles dealing exclusively with E-optimality. In the same spirit, we also provide below a very general result on E-optimality of d^* for *all* $v \geqslant 3$. The proof does not require knowledge of validity of D- and or A-optimality for any combination of m and v.

Referring to Keifer (1975), we note that

$$\text{min eigenvalue of } (PC_dP') \leqslant \frac{v}{v-1} \min_{h \in \{1, 2, \ldots, v\}} C_{d_{hh}}.$$

So it suffices to verify that d^* maximizes $\min_h C_{d_{hh}}$. We do the vertification below :

For any design, there exists a treatment, say h_0, such that $r_{h_0} \leqslant \bar{r}$. Recall that $\bar{r} = (m-1)mv+m(v-1)$ belongs to second half of the elementary interval $[(m-1)mv, m \cdot mv]$. We now distinguish between the following cases :

Case (i).
$$umv \leqslant r_{h_0} \leqslant umv + \frac{mv}{2} \quad u = 0, 1, 2, \ldots, m-1.$$

(r_{h_0} covering all r values in the first halves of the elementary intervals upto and including the one containing $\bar{r}$).

Case (ii). (a)
$$umv + \frac{mv}{2} \leqslant r_{h_0} < (u+1)mv, \; u = 0, 1, 2, \ldots, m-2.$$

(r_{h_0} covering all r values in the second halves of the elementary intervals upto but excluding the one containing $\bar{r}$).

(b)
$$(m-1)mv + \frac{mv}{2} \leqslant r_{h_0} \leqslant \bar{r}.$$

(r_{h_0} covering all the r values in the second half of the elementary intervals $[C_0, D_0]$ containing $\bar{r}$).

Under case (i). It is enough to establish that

$$C_{d_{h_0 h_0}} \leqslant g_1(r_{h_0}) \leqslant g_2(\bar{r}) = C_{d^*_{hh}}.$$

Also under case (ii) (a) and (b) it is enough to verify.

$$C_{d_{h_0h_0}} \leqslant g_2(r_{h_0}) \leqslant g_2(\bar{r}) = C_{d^*_{hh}}.$$

Clearly, these in their turn will establish E-optimality of d^*. We proceed through the following steps.

Step I : For r_{h_0} belonging to the first (second) half of any interval, upto and including $[C_0, D_0]$, we find upper bounds to $g_1(r_{h_0})$ (respectively $g_2(r_{h_0})$) involving g-values at points in the first (respectively second) half of the (basic) elementary interval $[C_0, D_0]$ containing $\bar{r}$.

Step II : We establish that (i) $g_2(r) \uparrow$ in r in the second half of $[C_0, D_0]$ and further that

(ii) $$g_2\left(C_0+\frac{mv+1}{2}\right) \geqslant g_1\left(C_0+\frac{mv-1}{2}\right) \quad \text{in case } mv \text{ is odd.}$$

Step III : Once we are through, with the above two steps (verification given below), we argue as follows :

Case (i). $$g_1(r_{h_0}) \leqslant g_1\left(C_0+\left[\frac{mv}{2}\right]\right) \qquad \text{(by Step I)}$$

$$\leqslant g_2\left(C_0+\left[\frac{mv}{2}\right]+1\right) \qquad \text{(by Step II(ii))}$$

$$\leqslant g_2(\bar{r}) \qquad \text{(by Step II(i))}$$

Case (ii)(a). $$g_2(r_{h_0}) \leqslant g_1(r_{h_0}) \qquad \text{(using 3.3))}$$

$$\leqslant g_1\left(C_0+\left[\frac{mv}{2}\right)\right] \qquad \text{(vide Fig. 1)}$$

$$\leqslant g_2(\bar{r}). \qquad \text{(by steps followed in case (i))}$$

Case (ii)(b). $$g_2(r_{h_0}) \leqslant g_2(\bar{r}) \qquad \text{(by Step II(ii)).}$$

Verifications (Step I) : Figure 1 supports the claim that

$$g_1(r_{h_0}) \leqslant g_1\left(C_0+\left[\frac{mv}{2}\right]\right)$$

whenever r_{h_0} is in the first half of any interval upto and including $[C_0, D_0]$. As regards g_2, the property (3.3) together with the increasing nature of g_1 at least upto $[C_0, D_0]$ justifies the claim

$$g_2(r_{h_0}) \leqslant g_1\left(C_0+\left[\frac{mv}{2}\right]\right)$$

for any r_{h_0} in the second half of any interval upto but excluding $[C_0, D_0]$.

Step II(i) : Referring to definition of $g(r)$ as adopted (just below (3.3)) and (3.3), (3.4), we get,

$$\Delta_2^*(r) = \Delta_1^*(r) - 4(mv-1)$$

which is increasing in r with $\Delta_2^*\left(C_0+\frac{mv}{2}\right) > 0$ for all $v \geqslant 3$, $m \geqslant 2$. Hence the claim.

Step II(ii) : $$g_2^*\left(C_0+\frac{mv+1}{2}\right) - g_1^*\left(C_0+\frac{mv-1}{2}\right)$$

simplifies to

$$m^2v(mv^2-2mv-2v+4)$$

which is nonnegative for $v \geqslant 3$, $m \geqslant 2$.

Thus, finally E-optimality of d^* is settled for any $v \geqslant 3$, and $m \geqslant 2$.

4. Construction of optimal designs

From the c.s. of C_{d^*} it necessarily follows that for d^* to exist $b = mv = \binom{v}{2}\lambda$ for $\lambda \geqslant 1$, integer, i.e. $m = \frac{v-1}{2}\lambda$. For $m = 1$, d^* cannot exist unless $v = 3$. In fact for $v = 3$ also d^* does not exist. We will present here construction of d^* for (i) v even, and (ii) v odd prime or prime power under a special situation. Note that given $v > 3$ it is sufficient to construct d^* for the least possible value of m, since for any other multiple of this m say, denoted by $m^* = km$, the same design for the given m can be inserted as block diagonals k times, off diagonals being filled up suitably by appropriate Latin Squares or F-squares.

For $v = 3$, it is sufficient to consider d^* for $m = 2$, and $m = 3$—the designs for all $m > 3$ following from them easily. These designs are as follows.

$v = 3,\ m = 2$

$$\begin{bmatrix} X & 1 & 2 & 0 & 2 & 0 \\ 1 & X & 0 & 2 & 2 & 1 \\ 2 & 0 & X & 0 & 1 & 1 \\ 0 & 0 & 1 & X & 1 & 2 \\ 0 & 2 & 2 & 1 & X & 0 \\ 1 & 2 & 1 & 2 & 0 & X \end{bmatrix}$$

$v = 3,\ m = 3$

$$\begin{bmatrix} X & 0 & 1 & 2 & 2 & 1 & 0 & 1 & 2 \\ 2 & X & 0 & 1 & 2 & 2 & 0 & 0 & 1 \\ 1 & 2 & X & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & X & 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 & X & 0 & 1 & 2 & 2 \\ 1 & 0 & 0 & 1 & 2 & X & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 & X & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 & 2 & X & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & X \end{bmatrix}$$

Case (i) : v even integer.

For v even, a $v-1$ resolvable BIBD always exists with number of blocks $b^* = \binom{v}{2}$, and block size $k = 2$. The blocks of the BIBD can be split into $v-1$ sets of $v/2$ blocks so that each set contains each of the v treatments exactly once. Let one of the representative sets say t-th set of blocks be $(i_1, j_1), (i_2, j_2), \ldots, (i_{v/2}, j_{v/2})$ where $i_1, i_2, \ldots, i_{v/2}, j_1, j_2, \ldots, j_{v/2}$ comprises the set of all v treatments. A square with missing diagonals can be constructed with the mentioned v symbols as follows :

Consider the square

$$A = \begin{bmatrix} X & v-1 & v-2 & v-3 & \ldots & 1 \\ 0 & X & v-1 & v-2 & \ldots & 2 \\ 1 & 0 & X & v-1 & \ldots & 3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ v-3 & v-4 & \cdot & \cdot & X & v-1 \\ v-2 & v-3 & \cdot & \cdot & 0 & X \end{bmatrix}$$

The pairs missing each twice in the same row and column numbers are precisely $(0, v-1), (1, v-2), \ldots \left(\frac{v}{2}-1, \frac{v}{2}\right)$. Corresponding to the t-th set of the given BIBD construct A_t from A by permuting the symbols $(0, 1, 2, \ldots, v-1)$ to $(i_1, i_2, \ldots, i_{v/2}, j_{v/2}, j_{v/2-1}, \ldots, j_1)$. Thus in A_t, the pairs missing (each twice) in the same row and column numbers are precisely given by the blocks of the t-th set of the BIBD. Each set of blocks thus gives rise to a similar square A_t, $t = 1, 2, \ldots, v-1$. The resultant design with $m = v-1$ (which is its least value) is thus given as

$$d^* = \begin{bmatrix} A_1 & L & L & \ldots & L \\ L & A_2 & L & \ldots & L \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L & L & L & \ldots & A_{v-1} \end{bmatrix}$$

where L's are Latin Squares with v treatments.

Case (ii) : v = odd prime or prime power > 3.

Let $\alpha_0 = 0, \ \alpha_1 = 1, \ \alpha_2, \ldots, \alpha_{\frac{v-1}{2}}, \ \alpha_{\frac{v-1}{2}+1} = -\alpha_1,$

$\alpha_{\frac{v-1}{2}+2} = -\alpha_2, \ldots, \alpha_{v-1} = -\alpha_{\frac{v-1}{2}}$

be the v distinct elements of $GF(v)$.

Let us assume that it is possible to construct a $v \times v$ square A with missing diagonals such that the i-th row (column) contains all the v symbols of $GF(v)$ except α_i (respectively $(\alpha_i+\alpha)$) where $0 \neq \alpha \in GF(v)$. $i = 0, 1, 2, \ldots, v-1$. Let $A_i = \alpha_i A$, or $-\alpha_i A$ $\ i = 1, 2, \ldots, \frac{v-1}{2}$. Then,

$$\begin{bmatrix} A_1 & L & L & \ldots & L \\ L & A_2 & L & \ldots & L \\ L & L & L & \ldots & A_{\frac{v-1}{2}} \end{bmatrix}$$

is the required design with $m = \frac{v-1}{2}$, where L's are any Latin Squares with v symbols of $GF(v)$.

Example : $v = 5$.

$\alpha_0 = 0, \ \alpha_1 = 1, \ \alpha_2 = 2, \ \alpha_3 = -\alpha_1 = 4, \ \alpha_4 = -\alpha_2 = 3.$

are the elements of $GF(5)$.

A can be chosen as

$$A = \begin{bmatrix} X & 1 & 2 & 4 & 3 \\ 0 & X & 4 & 3 & 2 \\ 3 & 4 & X & 1 & 0 \\ 2 & 3 & 0 & X & 1 \\ 4 & 0 & 1 & 2 & X \end{bmatrix} \text{(with } \alpha = 1\text{)}.$$

Below we present the forms of the matrix A for $v = 7$ and $v = 9$ as well.

$v = 7$

$$A = \begin{bmatrix} X & 3 & 4 & 2 & 6 & 5 & 1 \\ 3 & X & 5 & 6 & 2 & 0 & 4 \\ 6 & 5 & X & 0 & 4 & 1 & 3 \\ 4 & 1 & 0 & X & 5 & 2 & 6 \\ 5 & 0 & 1 & 3 & X & 4 & 2 \\ 2 & 4 & 6 & 1 & 3 & X & 0 \\ 0 & 6 & 2 & 5 & 1 & 3 & X \end{bmatrix} \text{(with } \alpha = 1\text{)}$$

$v = 9$

$$A = \begin{bmatrix} X & x+1 & x+2 & 1 & 2 & x & 2x & 2x+2 & 2x+1 \\ x+2 & X & 2 & x+1 & 2x & 2x+2 & 0 & 2x+1 & x \\ 2x+1 & 2x+2 & X & 2 & 1 & x+1 & x+2 & 0 & 2x \\ 0 & 2x & 2x+1 & X & 2x+2 & 2 & 1 & x & x+2 \\ 2x+2 & 0 & 1 & 2x+1 & X & 2x & x & x+1 & 2 \\ 2x & 2x+1 & 2x+2 & x & x+2 & X & x+1 & 1 & 0 \\ x & 1 & 0 & 2x+2 & 2x+1 & x+2 & X & 2 & x+1 \\ x+1 & x+2 & x & 2x & 0 & 2x+1 & 2 & X & 1 \\ 2 & x & 2x & 0 & x+1 & 1 & 2x+2 & x+2 & X \end{bmatrix}$$

(with $\alpha = 1$)

5. Efficiency of Aggarwal's designs

Aggarwal (1966b) presented a series of two-way Latin Square designs with all distinct elements missing along the diagonal. These designs can be generalized to the case of $mv \times mv$ arrays (for $m > 1$) by placing such $v \times v$ designs along the diagonal block matrices and ordinary Latin Squares along the off-diagonal blocks. Clearly such designs are not optimal. However, they possess a high degree of efficiency as is demonstrated below tith respect to the A-optimality criteron.

Note that for the optimal design (d^*) as well as for the above-mentioned Aggarwal's design (d_0 say), the C-matrices are completely symmetric. Hence, one gets for the efficiency of d_0 the expression

$$E_{d_0} = \frac{\Sigma\,\Sigma\, V(\hat{\tau}_i-\hat{\tau}_j) \text{ using } d^*}{\Sigma\,\Sigma\, V(\hat{\tau}_i-\hat{\tau}_j) \text{ using } d_0} = \Sigma \frac{1}{\lambda_{d_0^{*i}}} \quad \Sigma \Big/ \frac{1}{\lambda_{d_0^i}}$$

$$= \frac{a_0}{a^*} \text{ using the representations } C_{d_0} = a_0(I-J/v) \text{ and } C_{d^*} = a^*\,(I-J/v).$$

Recalling now the expressions for $C_{d_{hh}}$, $A(\bar{r})$ and $g_2(\bar{r})$ (vide Section 3), we get

$$\left(1-\frac{1}{v}\right) a^* = A(\bar{r}) - \frac{2(2v-3)}{v(mv-2)}$$

$$\left(1-\frac{1}{v}\right) a_0 = \bar{r} - \frac{2}{mv}\,\{-mv(m-1)^2+(2\bar{r}-mv)(m-1)+\bar{r}\} + \frac{\bar{r}^2}{(mv-1)\,mv-2)}$$

$$- \frac{4m^2\cdot m(v-1)+4m(m-1)^2}{mv(mv-2)}$$

$$= A(\bar{r}) - \frac{4(v-1)}{v(mv-2)}$$

$$= E(\bar{r}) \quad \text{(say)}.$$

E_{d_0} now simplifies to $\dfrac{E(\bar{r})}{E(\bar{r}) + \dfrac{2}{v(mv-2)}}$.

Calculations indicate that even for moderate values of v and m, E_{d_0} is close to unity.

6. Concluding remarks

Our investigation on non-orthogonal row-column incidence structure admits of the following generalizations. First, a general m-way heterogeneity non-orthogonal set-up could be studied. A search for optimal nested two-way designs under orthogonal row-column structure has been recently undertaken (Sinha, Mukhopadhyaya and Bagchi, 1984). We intend to carry out similar investigations under the present frame-work. A study of similar problems under mixed effects model will be undertaken shortly.

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