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ON SOME ENUMERATION PROBLEMS OF IMMERSIONS IN CRITICAL DIMENSION

GOUTAM MUKHERJEE

Stat-Math. Division, Indian Statistical Institute
203, B. T. Road, Calcutta 700 035

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We present the enumeration of regular homotopy classes of immersions of the real Grassmann manifold $G_k(\mathbb{R}^{n+k})$ into $\mathbb{R}^{2nk}$.

1. INTRODUCTION

In this paper we present the enumeration of regular homotopy classes of immersions of the nk-dimensional real Grassmann manifold $G_k(\mathbb{R}^{n+k})$ into $\mathbb{R}^{2nk}$. The dimension $2nk$ is critical because above this dimension any two immersions become regularly homotopic.

If X is a smooth m-manifold, let $[X \subseteq \mathbb{R}^{2m}]$ denote the set of regular homotopy classes of immersions of X into $\mathbb{R}^{2m}$. Then we prove the following

Theorem 1.1 — If nk is even then $[G_k(\mathbb{R}^{n+k}) \subseteq \mathbb{R}^{2nk}] = Z$ and if nk is odd then $[G_k(\mathbb{R}^{n+k}) \subseteq \mathbb{R}^{2nk}] = Z_2$.

In particular, when $k = 1$, the theorem implies that $[P^n \subseteq \mathbb{R}^{2n}] = Z$ if n is even, and $[P^n \subseteq \mathbb{R}^{2n}] = Z_2$ if n is odd and $n \geq 1$. This result for the real projective space P^n gives a mild improvement of a result of Larmore and Rigdon[3]. It may be noted that the case $n = 1$ is the famous Whitney-Graustein Theorem[7] which says that $[S^1 \subseteq \mathbb{R}^2] = Z$.

The proof of Theorem 1.1 is obtained by revitalizing an old technique, the primary classification theorem for sections of a fibration[6], in the equivariant setting. Møller[4] has introduced for G-complexes, where G is a finite group, Bredon cohomology groups with local coefficients, and shown that obstructions to equivariant sections of δ G-fibrations lie in these groups. Using these groups Møller has built up an equivariant obstruction theory for equivariant sections of a G-fibration. If the local coefficients system is simple, then the Møller groups reduce to Bredon cohomology with a generic coefficients system[1]. The principal feature of our proof is to reduce the problem to this special case, and then identify using the equivariant classification theorem the set of regular homotopy classes of immersions with a suitable Bredon cohomology group. The method of proof also applied to the Lens space L_p^{2n-1}, and we have

Theorem 1.2 — $[L_p^{2n-1} \subseteq \mathbb{R}^{4n-2}] = Z_2$.

It is interesting to note that unlike the non-equivariant theory our computations avoid twisted coefficients.

2. Equivariant Obstruction Theory

Let us look at the obstruction theory of Møller[4], and obtain the primary classification theorem in the equivariant context. Throughout this section, a space will always be compactly generated, and G will be a finite group. The abbreviation LCS will stand for local coefficients system or systems.

Let O_G be the orbit category, whose objects are G-maps $\hat{g}: G/H \to G/K$ arising from subconjugacy relations $g^{-1}Hg \subseteq K, g \in G$. On the other hand, let τ be the category of pairs (X, L) where X is a space and L is an ordinary LCS on it; a morphism $(\phi_1, \phi_2): (X, L) \to (Y, M)$ consists of a map $\phi_1 : X \to Y$ and a homomorphism $\phi_2 : L \to \phi_1^* M$ between LCS on X. Then an equivariant LCS on a G-space X is a contravariant functor $\mathcal{L} : O_G \to \tau$ such that (i) $\mathcal{L}(G/H) = (X^H, \mathcal{L}_H)$, where $\mathcal{L}_H$ is an ordinary LCS on the fixed point set X^H, and (ii) if $\hat{g}: G/H \to G/K$ is a morphism in O_G then $\mathcal{L}(\hat{g}) = (g, \mathrm{L}(\hat{g}))$, where $g: X^K \to X^H$ is the left translation by g, and $\mathrm{L}(\hat{g}) : \mathcal{L}_K \to g^* \mathcal{L}_H$ is a morphism of LCS on X^K.

It is easy to conceive of examples of equivariant LCS in a manner parallel to the non-equivariant case. Suppose that $p: E \to B$ is a G-fibration so that, for each $H \subseteq G$, $p^H : E^H \to B^H$ has n-simple fibre. Then, if $\pi_n(\mathcal{F}^H)$ is the ordinary LCS on B^H induced by the fibration p^H, we have a LCS $\pi_n(\mathcal{F})$ on B defined as follows. Set $\pi_n(\mathcal{F})\ (G/H) = (B^H, \pi_n(\mathcal{F}^H))$, and, for $\hat{g}: G/H \to G/K$, $g^{-1}Hg \subseteq K$, set $\pi_n(\mathcal{F})\ (\hat{g}) = (\hat{g}, \pi_n(g))$ where $g: (p^K)^{-1}(x) \to (p^H)^{-1}(gx)$, $x \in B^K$, is the left translation by g, and $\pi_n(g)$ is the induced homomorphism between the n-th homotopy groups.

Now suppose that X is a G-CW complex with an equivariant LCS $\mathcal{L}$ on it. Then, as G is finite, each X^H is an ordinary CW-complex. Let $C^n(X^H; \mathcal{L}_H)$ be the nth cellular cochain group of X^H with values in the ordinary LCS $\mathcal{L}_H$, and δ_n^H be the coboundary. The elements of this cochain group are functions c_H defined on n-cells $\sigma : D^n \to X^H$ such that $c_H(\sigma) \in \mathcal{L}(\sigma(e_0))$, where e_0 is the base point of D^n. Then the groups

$$C^n(X; \mathcal{L}) = \oplus_{H \subseteq G} C^n(X^H; \mathcal{L}_H)$$

with coboundaries $\delta^n = \oplus_{H \subseteq G} \delta_H^n$ form a cochain complex $C(X; \mathcal{L})$.

Let $\Gamma^n(X;\mathcal{L})$ be the subgroup of $C^n(X;\mathcal{L})$ consisting of all $c = \{c_H\}_{H\subseteq G}$ such that for any n-cell $\sigma: D^n \to X$ the equation $c_H(g\sigma) = L(\hat{g})(\sigma(e_0))(c_K(\sigma))$ is satisfied whenever $K \subseteq G$ fixes σ and $g^{-1}Hg \subseteq K$. As shown in Møller[4], $\Gamma(X;\mathcal{L})$ is a cochain subcomplex of $C(X;\mathcal{L})$. Then the Bredon cohomology of X with equivariant LCS $\mathcal{L}$ is defined by $H^n(X;\mathcal{L}) = H^n(\Gamma(X;\mathcal{L}))$.

If A is a sub-G-CW-complex of X with inclusion $i: A \subset X$, then the restriction maps $(i^H)\#: C^n(X^H;\mathcal{L}_H) \to C^n(A^H;(i^H)^*\mathcal{L}_H)$ induce a cochain map $i^\#: C(X;\mathcal{L}) \to C(A; i^*\mathcal{L})$ so that $i^\#$ maps $\Gamma(X;\mathcal{L})$ into $\Gamma(A; i^*\mathcal{L})$. Then the groups $\Gamma^n(X,A;\mathcal{L}) = \mathrm{Ker}\,(i^\# \mid \Gamma^n(X;\mathcal{L})$, form a cochain subcomplex $\Gamma^n(X, A;\mathcal{L})$, and it's cohomology is defined to be the Bredon cohomology of (X, A) with values in $\mathcal{L}$, denoted by $H^n(X, A;\mathcal{L})$.

Let $p: E \to B$ be a G-fibration so that, for each subgroup H of G, the fibration $p^H: E^H \to B^H$ has path connected base and fibre, and (X, A) be a G-CW-complex pair so that (X^H, A^H) is connected for each subgroup H of G. Let $\phi: X \to B$ and $f: A \to E$ be G-maps so that $p \circ f = \phi/A$, that is f is an equivariant partial lifting of ϕ. Then the lifting problem is to find a G-map $\psi: X \to E$ so that $p \circ \psi = \phi$ and $\psi/A = f$.

Let X_n be the n-skeleton of (X, A), and suppose now that $\psi: X_n \to E$ is an equivariant partial lifting of ψ for $n \geq 1$. Then define the obstruction to extending ψ as

$$c^{n+1}(\psi) = \{c_H^{n+1}(\psi^H)\}_{H\subseteq G} \in C^{n+1}(X,A;\phi^*\pi_n(\mathcal{F}))$$

where $c_H^{n+1}(\psi^H)$ is the non-equivariant obstruction to extending $\psi^H: X_n^H \to E^H$, and $\pi_n(\mathcal{F})$ is the equivariant LCS on B as defined earlier. It can be verified that $c^{n+1}(\psi) \in \Gamma^{n+1}(X,A;\phi^*\pi_n(\mathcal{F}))$.

Next, let $\psi_0, \psi_1: X_n \to E$ be equivariant partial liftings of ϕ, and $\lambda: I \times X_{n-1} \to E$ be a vertical G-homotopy rel A between ψ_0/X_{n-1} and ψ_1/X_{n-1}. These maps fit together to give an equivariant partial lifting $\mu: I \times X_n \bigcup I \times X_{n-1} \to E$ of $\phi \circ \pi$ where $\pi: I \times X \to X$ is the projection. Then define the equivariant difference cochain of ψ_0 and ψ_1 with respect to λ as

$$d^n(\psi_0, \psi_1, \lambda) = \{d_H^n(\psi_0^H, \psi_1^H, \lambda^H)\}_{H\subseteq G} \in C^n(X,A;\phi^*\pi_n(\mathcal{F}))$$

where $d_H^n(\psi_0^H, \psi_1^H, \lambda^H)$ is the non-equivariant difference cochain of ψ_0^H and ψ_1^H with respect to λ^H. Again it can be verified that $d^n(\psi_0, \psi_1, \lambda) \in \Gamma^n(X,A;\phi^*\pi_n(\mathcal{F}))$. Similarly, the primary difference $\delta^n(\psi_0, \psi_1) \in H^n(X,A;\phi^*\pi_n(\mathcal{F}))$ can be defined for the liftings $\psi_0, \psi_1: X \to E$ of ϕ.

The motivation of the above definition is the observation that there is a 1-1 correspondence between equivariant $(n+1)$-cells $G/H \times D^{n+1} \to X$ and non-equivariant $(n+1)$-cells $D^{n+1} \to X^H$.

All the properties of the non-equivariant obstruction and difference cochains transform to the equivariant case in a natural way leading us to the following classification theorem.

Theorem 2.1 — With the same notations as above, suppose that

1. the fibre of each $p^H : E^H \to B^H$ is q-simple for $n+1 \leq q < \dim(X,A)$,
2. $H^q(X,A\,;\phi^* \pi_q(\mathcal{F})) = 0$ for $n+1 \leq q < 1 \dim(X,A)$,
3. $H^{q+1}(X,A\,;\phi^* \pi_q(\mathcal{F})) = 0$ for $n+1 \leq q < \dim(X,A)$,

and that $\psi_0 : X \to E$ is an equivariant lifting of $\phi : X \to B$. Then the correspondence $\psi \to \delta^n(\psi_0, \psi)$ is a bijection between the set of vertical G-homotopy classes (rel A) of equivariant liftings of ϕ which agree with ψ_0 on A and the group $H^n(X,A;\phi^* \pi_n(\mathcal{F}))$.

3. Proof of Theorem 1.1

If X is a smooth m-manifold, let $E(X) \to X$ denote the bundle associated to the tangent bundle of X with fibre $V_m(\mathbb{R}^{2m})$ which is the Steifel manifold of m-frames in $\mathbb{R}^{2m}$. Then according to Hirsch[2], the set of regular homotopy classes of immersions $[X \subseteq \mathbb{R}^{2m}]$ corresponds bijectively with the set of vertical homotopy classes of sections of $E(X) \to X$.

Let $\tilde{G}_k(\mathbb{R}^{n+k})$ be the Grassmann manifold of oriented k-planes through the origin of $\mathbb{R}^{n+k}$. Since Z_2 acts freely on $\tilde{G}_k(\mathbb{R}^{n+k})$ by orientation reversing diffeomorphism $x \to -x$, the bundle $E(\tilde{G}_k(\mathbb{R}^{n+k})) \to \tilde{G}_k(\mathbb{R}^{n+k})$ becomes a Z_2-fibration, where the action of Z_2 on $E(\tilde{G}_k(\mathbb{R}^{n+k}))$ is given by

$$(-1)(x; v_1, \ldots, v_{nk}) = (-x; -v_1, \ldots, -v_{nk})$$

where $(x; v_1, \ldots, v_{nk})$ is an orthonormal nk-frame in $\mathbb{R}^{2nk}$ associated to x. This bundle produces the bundle $E(G_k(\mathbb{R}^{n+k})) \to G_k(\mathbb{R}^{n+k})$ by passing to the quotient. Moreover note that $\tilde{G}_k(\mathbb{R}^{n+k})$ is the simply connected covering of $G_k(\mathbb{R}^{n+k})$. Therefore the vertical homotopy classes of sections of $E(G_k(\mathbb{R}^{n+k})) \to G_k(\mathbb{R}^{n+k})$ are in $1-1$ correspondence with the vertical Z_2-homotopy classes of Z_2-equivariant sections of $E(\tilde{G}_k(\mathbb{R}^{n+k})) \to \tilde{G}_k(\mathbb{R}^{n+k})$.

Since the Z_2 action is free on $\tilde{G}_k(\mathbb{R}^{n+k})$, we have only one fixed point set, namely $\tilde{G}_k(\mathbb{R}^{n+k})$ itself, corresponding to the trivial subgroup $H = \{e\}$ of Z_2. Let $\pi_{nk}(\mathcal{F})$ be the equivariant LCS on $\tilde{G}_k(\mathbb{R}^{n+k})$ induced by $E(\tilde{G}_k(\mathbb{R}^{n+k})) \to G_k(\mathbb{R}^{n+k})$. Since $\tilde{G}_k(\mathbb{R}^{n+k})$ is simply connected and $V_{nk}(\mathbb{R}^{2nk})$ is $(nk-1)$-connected, the induced LCS on $\tilde{G}_k(\mathbb{R}^{n+k})$ is simple and assigns to each point the group $\pi_{nk}(V_{nk}(\mathbb{R}^{2nk}))$. Thus, in view of the homotopy group $\pi_{nk}(V_{nk}(\mathbb{R}^{2nk}))$, we have on $\tilde{G}_k(\mathbb{R}^{n+k})$ the constant coefficient Z_2-module Z when nk is even, and the constant coefficient the trivial Z_2-module Z_2 when nk is odd.

Let us now compute the equivariant cohomology $H^{nk}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F}))$. First note that the Pontryagin's construction of a CW-complex structure on $\widetilde{G}_k(\mathbb{R}^{n+k})$, as given in Pontryagin[5], provides a Z_2-CW-complex structure on $\widetilde{G}_k(\mathbb{R}^{n+k})$. Recall that each Schubert symbol σ of order k

$$\sigma = (0 \le \sigma_1 \le \sigma_2 \le \ldots \le \sigma_k \le n)$$

(and one has to take $\sigma_0 = 0$ and $\sigma_{k+1} = n$) determines two cells $e(\sigma)_+$ and $e(\sigma)_-$ of $G_k(\mathbb{R}^{n+k})$, each of dimension $\sum_{i=1}^{i=k} \sigma_i$, which satisfy the following conditions :

1. $e(\sigma)_+ \bigcap e(\sigma)_- = \phi$

2. $e(\sigma)_+ \bigcup e(\sigma)_-$ consists of k-planes P such that $\dim (P \bigcap \mathbb{R}^{\sigma_i + i}) = i$, $i = 1, 2, \ldots, k$,

3. $A(e(\sigma)_+) = e(\sigma)_-$, $A(e(\sigma)_-) = e(\sigma)_+$, where A is the orientation reversing diffeomorphism $P \to -P$.

The boundary $\overline{e(\sigma)}_+ - e(\sigma)_+ = \overline{e(\sigma)}_- - e(\sigma)_-$ is the union of cells $e(\tau)_+ \bigcup e(\tau)_-$ where τ runs over the symbols obtained from σ by replacing one σ_i by $\sigma_i - 1$, $1 \le i \le k$, provided the function τ is non-negative and non-decreasing.

Pontryagin determined explicit formulas of the boundary operator for the Schubert chains $e(\sigma)_+$ and $e(\sigma)_-$ (with suitable orientations). Dualizing these formulas we get the following coboundary relations for the Schubert cochains $e\{\sigma\}_+$ and $e\{\sigma\}_-$, which are dual to $e(\sigma)_+$ and $e(\sigma)_-$. For a Schubert symbol $\sigma = (\sigma_1, \ldots, \sigma_k)$ define

$$s(\sigma, i) = \sigma_i + i + k + 1 \text{ and } t(\sigma, i) = \sum_{j=1}^{k} w_j$$

where $$w_j = \begin{cases} \sigma_j & \text{for } j < 1 \\ \sigma_{i+1} & \text{for } j \ge i \end{cases} \quad 1 \le j \le k.$$

Then

$$\delta e\{\sigma\}_+ = \sum_{i=1}^{k} (-1)^{t(\sigma,i)} [\, e\{\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \ldots, \sigma_k\} + (-1)^{s(\sigma,i)} e\{\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \ldots, \sigma_k\}_- \,]$$

$$\delta e\{\sigma\}_- = \sum_{i=1}^{k} (-1)^{t(\sigma,i)} [\, e\{\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \ldots, \sigma_k\}_- + (-1)^{s(\sigma,i)} e\{\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \ldots, \sigma_k\}_+ \,]$$

where only the meaningful symbols appear on the right hand side.

Let us denote the generators $e\{n-1, n, \ldots, n\}_\pm$ of $C^{nk-1}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F}))$ by $e_\pm^{nk-1}$, and the generators $e\{n, n, \ldots, n\}_\pm$ of $C^{nk}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F}))$ by $e_\pm^{nk}$. Now if a is the nontrivial element of Z_2, then, for the relation $a^{-1}Ha = H$,

$$\pi_{nk}(\hat{a})(x) : \pi_{nk}(\mathcal{F}^H)(x) \to \pi_{nk}(\mathcal{F}^H)(ax)$$

is multiplication by -1, and for $e^{-1}He = H$, $\pi_{nk}(\hat{e})(x)$ is the identity, where $x \in \widetilde{G}_k(\mathbb{R}^{n+k})$ and $H = \{e\}$. Then we find that $\Gamma^{nk-1}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F}))$ is generated by $e_+^{nk-1} - e_-^{nk-1}$, and $\Gamma^{nk}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F}))$ is generated by $e_+^{nk} - e_-^{nk}$. It follows from the coboundary relations that

$$\delta(e_+^{nk-1} - e_-^{nk-1}) = (-1)^{nk}\{1 + (-1)^{n+k}\}\ (e_+^{nk} - e_-^{nk}).$$

Therefore if nk is even then

$$\Gamma^{nk-1}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F})) = \Gamma^{nk}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F})) = Z$$

and the map

$$\delta : \Gamma^{nk-1}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F})) \to \Gamma^{nk}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F}))$$

is the zero homomorphism. If, on the other hand, nk is odd then

$$\Gamma^{nk-1}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F})) = \Gamma^{nk}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F})) = Z_2$$

and the map δ is again zero. It follows then

$$H^{nk}(\widetilde{G}_k(\mathbb{R}^{n+k}); \pi_{nk}(\mathcal{F})) = \begin{cases} Z & \text{if } nk \text{ is even} \\ Z_2 & \text{if } nk \text{ is odd}. \end{cases}$$

Thus all the conditions of Theorem 2.1 are satisfied, and we obtain Theorem 1.1.

4. PROOF OF THEOREM 1.2

This proof may also be treated similarly. Recall that the Lens space L_p^{2n-1} is the space of orbits S^{2n-1}/Z_p of the cyclic group $Z_p = Z/pZ$ (p an odd prime) acting freely on $S^{2n-1} \subset C^n$ by the action $k.(z_0, ..., z_{n-1}) = (w^k z_0, ..., w^k z_{n-1})$ with $\omega = \exp(2\pi i/p)$. The CW decomposition of S^{2n-1} compatible with this action is given by the cells

$$e_r^{2k} = \{z \in S^{2n-1} : z_j = 0 \text{ for } j > k,\ \arg(z_k) = 2\pi r/p\}$$

$$e_r^{2k+1} = \{z \in S^{2n-1} : z_j = 0 \text{ for } j > k, 2\pi r/p < \arg(z_k) < 2\pi(r+1)/p\}$$

where $z = (z_0, ..., z_{n-1})$, $0 \le r < p$, $0 \le k < n$, and with suitable orientations of cells the boundaries are given by

$$\partial(e_r^{2k}) = \sum_{j=0}^{p-1} e_j^{2k-1} \cdot \partial(e_r^{2k+1}) = e_r^{2k} - e_{r+1}^{2k}\ \ (r \bmod p), \text{ where } e_p^{2k} = e_0^{2k}.$$

This provides a $Z_p - CW$ structure on S^{2n-1} with one equivariant cell in each dimension $q = 0, 1, ..., 2n - 1$, the action being $w \cdot e_r^q = e_{r+1}^q$.

The situation here is similar to that of the Grassmannian. Here also we have one fixed point set, and the equivariant LCS $\pi_{2n-1}(\mathcal{F})$ induced on S^{2n-1} by the Z_p-fibration $E(S^{2n-1}) \to S^{2n-1}$ is simple and assigns to each point the group Z_2. Therefore the cochain complex $\Gamma^k(S^{2n-1}; \pi_{2n-1}(\mathcal{F}))$ becomes

$$0 \to Z_2 \to Z_2 \to .... \to Z_2 \to 0$$

where the coboundary $\delta^q = 0$ if q is even, and $\delta^q = id$ if q is odd. Consequently,

$$H^q(S^{2n-1}; \pi_{2n-1}(\mathcal{F})) = \begin{cases} Z_2 & \text{if } q \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}$$

Now proceeding as before, we may complete the proof.

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