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Denting and strongly extreme points in the unit ball of spaces of operators

T S S R K RAO

Indian Statistical Institute, R V College Post, Bangalore 560 059, India
E-mail: tss@isibang.ac.in

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Abstract. For $1 \leq p \leq \infty$ we show that there are no denting points in the unit ball of $\mathcal{L}(\ell^p)$. This extends a result recently proved by Grząślewicz and Scherwentke when $p = 2$ [GS1]. We also show that for any Banach space X and for any measure space $(\Omega, \mathcal{A}, \mu)$, the unit ball of $\mathcal{L}(L^1(\mu), X)$ has denting points iff $L^1(\mu)$ is finite dimensional and the unit ball of X has a denting point. We also exhibit other classes of Banach spaces X and Y for which the unit ball of $\mathcal{L}(X, Y)$ has no denting points. When X^* has the extreme point intersection property, we show that all 'nice' operators in the unit ball of $\mathcal{L}(X, Y)$ are strongly extreme points.

Keywords. Denting point; strongly extreme point; M-ideal.

1. Introduction

Let X be a Banach space and let X_1 denote its closed unit ball. In this paper we consider several aspects of the extremal structure of the unit ball of the space of operators $\mathcal{L}(X, Y)$. A point $x_0 \in X_1$ is said to be a denting point if for all $\epsilon > 0$, $x_0 \notin \overline{CO}(X_1 \setminus B(x_0, \epsilon))$ ($B(x_0, \epsilon)$ denotes the open ball and $\overline{CO}$, the closed convex hull) (see [DU]). Any denting point is an extreme point. For an infinite dimensional Hilbert space H, Grząślewicz and Scherwentke have showed recently that there are no denting points in the unit ball of $\mathcal{L}(H)$, the space of bounded linear operators (see [GS1]). Their proof makes use of the description of the extreme points of $\mathcal{L}(H)_1$ as isometries and co-isometries and shows that they are not denting points. However for the case of $1 < p < \infty, p \neq 2$, there is no complete description of extreme points of $\mathcal{L}(\ell^p)_1$ known (see [G] for more information). In this paper we take the equivalent definition of a denting point given by the result of [LLT] as an extreme point and a point of weak-norm continuity for the identity mapping on the unit ball. Most of our arguments involve ideas from M-structure theory for which we refer to [HWW].

In the first section we show that there are no denting points in the unit ball of $\mathcal{L}(\ell^p, Y)$ whenever there is a non-compact operator and the space of compact operators $\mathcal{K}(\ell^p, Y)$ is a M-ideal in $\mathcal{L}(\ell^p, Y)$. Since this is the case when $Y = \ell^p$ (see [HWW]) we have an extension of the result from [GS1]. For measure spaces $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ (μ_1 and μ_2 are positive measures) the same authors have proved in [GS2] that there are no denting points in $\mathcal{L}(L^1(\mu_1), L^1(\mu_2))_1$ when $L^1(\mu_1)$ is infinite dimensional. We generalize this by showing that $\mathcal{L}(L^1(\mu), X)_1$ has a point of weak-norm continuity iff $L^1(\mu)$ is finite dimensional and X_1 has a point of weak-norm continuity. Our ideas also work for other operator ideals in $\mathcal{L}(L^1(\mu), X)$. By taking advantage of the description of operators defined on $C(K)$ spaces (see [DU] Chap. VI), in some special cases we could describe points of weak-norm continuity in $\mathcal{AS}(C(K),X)_1$ (the ideal of absolutely summing operators with the absolutely summing norm).

For continuous function spaces, we show that for any infinite, totally disconnected, compact Hausdorff space K and for any Banach space X there are no points of weak-norm continuity in $\mathcal{L}(X,C(K))_1$. Since ℓ^∞ can be identified as $C(\beta(N))$, this covers the case $p=\infty$. We also show that if Y is a Banach space with infinite dimensional centralizer then no 'nice' operator (defined in § 1) can be a point of weak-norm continuity for the identity map on $\mathcal{L}(X,Y)_1$.

We next consider a weaker extremal form, namely, strongly extreme point. A point $x_0 \in X_1$ is said to be a strongly extreme point, if for every pair of sequences $\{x_n\}$, $\{y_n\} \subset X_1$ such that $(x_n+y_n)/2 \to x_0$, implies $\|x_n - y_n\| \to 0$. Any denting point is clearly a strongly extreme point. It is well known that for any compact set K, and for any measure space $(\Omega, \mathcal{A}, \mu)$, every extreme point in $C(K)_1$ and $L^1(\mu)_1$ is a strongly extreme point. Extending this aspect to operator spaces the authors of [GS2] show that all the extreme points in $\mathcal{L}(L^1(\mu_1), L^1(\mu_2))_1$ are strongly extreme. Isolating a property called the E. P. I .P that is common to both $C(K)$ and $L^1(\mu)$ we show that if X^* has the E. P. I. P then any 'nice' operator in $\mathcal{L}(X,Y)_1$ is a strongly extreme point, thus extending Theorem 1 and Corollary 4 of [GS2].

In the concluding part of the paper we briefly consider the stability aspect of another geometric property shared by the $C(K)$ and $L^1(\mu)$ spaces, namely every extreme point of the unit ball is also an extreme point of the unit ball of the bidual. These were called weak* extreme points in [KR] and it is known that any denting or strongly extreme point of the unit ball is a weak* extreme point. Having noted the non-existence of denting points and the availability of strongly extreme points, the permanence of extreme points is a natural question to consider. Our result considered in the more general setting of vector-valued continuous functions, states that, if K is a dispersed space then $\partial_e C(K,X)_1 \subset \partial_e C(K,X)_1^{**}$ whenever X has the same property.

All the Banach spaces considered here are infinite dimensional.

2. Denting points

In this section, using ideas from M-structure theory and the structure of basic sequences in ℓ^p spaces, we first show that there are no denting points in $\mathcal{L}(\ell^p, Y)_1$ for $1<p<\infty$ in the non-trivial situation. For $p=1$ we get a better result by showing that there are no points of weak-norm continuity in the unit ball of $\mathcal{L}(\ell^1, Y)$ for any Y.

We refer the reader to [Ans] Proposition 2.5 for a characterization of Banach spaces Y for which $\mathcal{K}(\ell^p,Y)=\mathcal{L}(\ell^p,Y)$ (see Proposition 2.c.3 in [LT] when $Y=\ell^r$). In what follows we assume that Y is such that there is a non-compact operator from ℓ^p and $\mathcal{K}(\ell^p,Y) \subset \mathcal{L}(\ell^p,Y)$ is a M-ideal. Examples of such Y include, ℓ^q for $p \leq q$, $L^p([0,1])$, and the Schatten class c_p for $2 \leq p$. We refer the reader to [KW] for more examples and characterizations of such Y. In particular we recall from Corollary 6.4 of [KW] that this property is hereditary for Y.

Theorem 2.1. *Let $1<p<\infty$. Suppose Y is such that $\mathcal{K}(\ell^p,Y)$ is a proper M-ideal in $\mathcal{L}(\ell^p,Y)$. There are no denting points in the unit ball of $\mathcal{L}(\ell^p,Y)$.*

Proof. Let $T \in \mathcal{L}(\ell^p,Y)$, $\|T\|=1$ be a non-compact operator. We shall show that T is not a point of weak-norm continuity for the identity mapping on the unit ball. Once this is established it would follow that any denting point must be a compact operator. However since $\mathcal{K}(\ell^p, Y)$ is a proper M-ideal in $\mathcal{L}(\ell^p, Y)$, applying Proposition 4.2 and Theorem 4.4 of Chapter 2 in [HWW] we see that there are no denting points in $\mathcal{K}(\ell^p, Y)_1$. This completes the proof.

Now let $T \in \mathcal{L}(\ell^p, Y)_1$ be a non-compact operator.

Case i. Let $\{e_n\}$ be the canonical basis of ℓ^p. Suppose $T(e_n) \not\to 0$ in the norm. It is easy to see that $T \circ (I - (e_n \otimes e_n)) \to T$ weakly. Also

$$\begin{aligned} \|I - (e_n \otimes e_n)\| &\leq 1 \text{ and} \\ \|T \circ (e_n \otimes e_n)\| &= \sup_{\|x\| \leq 1} \|e_n(x) T(e_n)\| \\ &= \|T(e_n)\|. \end{aligned}$$

We thus get the required contradiction.

Case ii. The general case follows from a similar argument. Since T is non-compact on a reflexive domain, assume without loss of generality that there exists a sequence $\{x_n\}$ such that $\|x_n\| = 1, x_n \xrightarrow{w} 0$ but $T(x_n) \not\to 0$ in the norm topology. Applying Proposition 1.a.12 in [LT] we may assume that $\{x_n\}$ is equivalent to a block basis of the canonical basis $\{e_n\}$. Also by Proposition 2.a.1 in [LT], for any normalized block basis, its closed span is isometric to ℓ^p. Hence the conclusion follows from arguments similar to the ones given during the proof in Case i).

Remark 2.1. We note that in the situation $p = 2$, $Y = \ell^2$ where it is well-known that there are no extreme points in the unit ball of $\mathcal{K}(\ell^2)$, the second half of the proof gives the result, and the M-ideal argument is not needed. In the general case, even when $Y = \ell^p$, this does not immediately lead to a contradiction since there are plenty of compact extremal operators(see [H]).

Remark 2.2. It is apparent from the arguments given above that an isometry or co-isometry is not a point of weak-norm continuity in the unit ball. In the case of a complex Hilbert space H, since any operator is an average of an isometry and co-isometry, it is easy to see that there are no points of weak-norm continuity in $\mathcal{L}(H)_1$. Thus there are no points of weak*-norm continuity either. Since $\mathcal{L}(H)$ is the bidual of $\mathcal{K}(H)$, one can conclude that there are no points of weak-norm continuity in $\mathcal{K}(H)_1$ (see [HL]). For other finite p, we do not know if there can be points of weak-norm continuity in the unit ball of the space of operators.

Remark 2.3. If Y has the R. N. P and $\mathcal{K}(\ell^p, Y) = \mathcal{L}(\ell^p, Y)$, it follows from a Corollary in [DM] that $\mathcal{L}(\ell^p, Y)$ has the R. N. P and hence has plenty of denting points in the unit ball (see [DU]). We do not know an example of a space Y for which $\mathcal{K}(\ell^p, Y) = \mathcal{L}(\ell^p, Y)$ fails to have denting points in the unit ball.

In the following proposition we exhibit another class of Banach spaces where there are denting points in the unit ball of $\mathcal{L}(\ell^p, X)$. If X has the Schur property (i.e., weak and norm sequential convergences coincide) then clearly, $\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X)$. If further X is infinite dimensional, then it contains an isomorphic copy of ℓ^1. In the following proposition we assume that X contains 'better' copies of ℓ^1.

PROPOSITION 2.1

Let X be a Banach space having the Schur property and an isometric copy of ℓ^1. Suppose there exists a projection $P: X^* \to X^*$ of norm one such that $\mathrm{Ker}(P) = \ell^{1\perp}$ *and* $P(X^*)_1$ *is weak* dense in* X_1^*. *Then there are denting points in* $\mathcal{L}(\ell^p, X)_1$.

Proof. Consider $\mathcal{L}(\ell^p, \ell^1) \subset \mathcal{L}(\ell^p, X)$. Since the latter space being a separable dual space has the R. N. P., it has denting points in the unit ball. We shall show that any denting point of this space is also a denting point of $\mathcal{L}(\ell^p, X)_1$. This is achieved by exhibiting a projection, $Q: \mathcal{L}(\ell^p, X)^* \to \mathcal{L}(\ell^p, X)^*$ of norm one such that $\mathrm{Ker}(Q) = \mathcal{L}(\ell^p, \ell^1)^\perp$ and $Q(\mathcal{L}(\ell^p, X)^*)_1$ is weak* dense in $\mathcal{L}(\ell^p, X)_1^*$. We then appeal to Proposition 2 and its proof in [R3] to conclude that denting points get preserved.

Since $\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X)$, we identify $\mathcal{L}(\ell^p, X)^*$ with the space of integral operators from ℓ^q to X^* (see [DU] p. 232). Now Q is defined by composing such an operator with P. Using the properties of P it is fairly routine to verify that Q has the desired properties. Hence there are denting points in $\mathcal{L}(\ell^p, X)_1$.

Using arguments similar to the ones given above and results from [DM] , the following corollary is easy to prove.

COROLLARY 2.1

Let X be a Banach space having the R. N. P. Suppose $\mathcal{L}(\ell^p, X^{**}) = \mathcal{K}(\ell^p, X^{**})$. *Then there are denting points in the unit ball of* $\mathcal{L}(\ell^p, X^{**})$.

Our next result deals with the question of points of weak-norm continuity in the unit ball of ℓ^∞-sums of Banach spaces.

PROPOSITION 2.2

If $\{X_i\}_{i \in I}$ *is any infinite family of non-trivial Banach spaces then there is no point of weak-norm continuity in the unit ball of the space* $X = \oplus_\infty X_i$.

Proof. Write $X = Y \oplus_\infty Z$ where Y and Z consist of ℓ^∞-sum of infinitely many X_i's. Suppose $x \in X_1$ is a point of weak-norm continuity. Let $x = y + z$, $y \in Y$, $z \in Z$.

If $\|y\| < 1$ then since Y is infinite dimensional, we can get a net $\{y_\alpha\}$ with $\|y_\alpha\| = 1$ and $y_\alpha \to y$ in the weak topology. Now $\|y_\alpha + z\| = \max\{\|y_\alpha\|, \|z\|\} = 1$ and $y_\alpha + z \to x$ weakly but not in the norm. Therefore $\|y\| = 1$. Similarly $\|z\| = 1$. It is also easy to see that each of y and z are points of weak norm continuity in Y_1 and Z_1 respectively.

Thus there is no loss of generality in assuming that I is countable. The same argument also shows that x can have at most finitely many components zero and for any infinite set $A \subset N$, $\sup_{a \in A} \|x(a)\| = 1$. Thus $\|x(n)\| \not\to 0$.

We may assume without loss of generality that there exists $0 < \delta < 1$ such that $\|x(n)\| > \delta \; \forall n$. Now for $e_n \in \ell^\infty$, let $e_n x \in \oplus_{c_0} X_n$ (the c_0-direct sum) denote the vector with x in the n-th place and zeros elsewhere. Clearly $e_n x \to 0$ weakly. Note that $\|x - e_n x\| = 1$ and $x - e_n x \to x$ weakly but not in the norm. Therefore there are no points of weak-norm continuity in the unit ball.

Remark 2.4. The corresponding question for ℓ^p-direct sums for $1 \leq p < \infty$ has positive answer (see [HL]).

We recall that for any discrete set I, the space $\mathcal{L}(\ell^1(I), X)$ can be identified with $\oplus_\infty X_i$ where $X_i = X$ for all $i \in I$.

Theorem 2.2. *Let X be a Banach space and $(\Omega, \mathcal{A}, \mu)$ (μ-positive) be a measure space. $\mathcal{L}(L^1(\mu), X)_1$ has a point of weak-norm continuity if and only if $L^1(\mu)$ is finite dimensional and X_1 has a point of weak-norm continuity.*

Proof. Suppose $\mathcal{L}(L^1(\mu), X)_1$ has a point of weak-norm continuity.

For any $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$, the projection $P : L^1(\mu) \to L^1(\mu)$ defined by $Pf = f\chi_A$ has the property that $\|f\| = \|P(f)\| + \|f - P(f)\|$ for all $f \in L^1(\mu)$. For such a projection P (a so called L-projection) Å. Lima observed in [L1] that $Q : \mathcal{L}(L^1(\mu), X) \to \mathcal{L}(L^1(\mu), X)$ defined by $Q(T) = T \circ P$ is a projection and satisfies $\|T\| = \max\{\|Q(T)\|, \|T - Q(T)\|\}$ for all $T \in \mathcal{L}(L^1(\mu), X)$.

Thus if $L^1(\mu)$ is infinite dimensional we can choose a sequence $\{A_n\}$ of pairwise disjoint sets with $0 < \mu(A_n) < \infty$ such that

$$\mathcal{L}(L^1(\mu), X) = \bigoplus_\infty \mathcal{L}(L^1(\mu_n), X) \bigoplus_\infty M$$

where $\mu_n = \mu | A_n$ and M is a closed (possibly trivial) subspace of $\mathcal{L}(L^1(\mu), X)$. In view of the above proposition we obtain a contradiction.

Clearly if $L^1(\mu)$ is of dimension n, then $\mathcal{L}(L^1(\mu), X) = \oplus_\infty^n X$ (n-many copies of X) and the conclusion follows from the arguments given during the proof of the proposition.

From the definition of a denting point we chose, and from arguments similar to the ones indicated above the following corollary is immediate.

COROLLARY 2.2

$\mathcal{L}(L^1(\mu), X)_1$ has a denting point if and only if $L^1(\mu)$ is finite dimensional and X_1 has a denting point.

Remark 2.5. Note that the same argument shows that for any closed subspace $\mathcal{H} \subset \mathcal{L}(L^1(\mu), X)$ that is closed under composition by operators from $\mathcal{L}(L^1(\mu))$, there is no point of weak-norm continuity in the unit ball of $\mathcal{H}$. Examples of such $\mathcal{H}$ include the spaces of compact operators, weakly compact operators.

Similar idea is again used in the following proposition which generalizes also Theorem 3 of [GS2]. Recall that a compact set is totally disconnected, if it has a base consisting of clopen sets.

PROPOSITION 2.3

Let K be any infinite, compact, totally disconnected space. For any Banach space X, there are no points of weak-norm continuity in $\mathcal{L}(X, C(K))_1$.

Proof. For any clopen set $A \subset K$, the projection $R : C(K) \to C(K)$ defined by $R(f) = f\chi_A$ has the property

$$\|f\| = \max\{\|R(f)\|, \|f - R(f)\|\}.$$

For such projections R it again follows from [L1], that $S : \mathcal{L}(X, C(K)) \to \mathcal{L}(X, C(K))$ defined by $S(T) = R \circ T$ is a projection and satisfies

$$\|T\| = \max\{\|S(T)\|, \|T - S(T)\|\}.$$

Thus since K is infinite we can find an infinite maximal family $\{A_i\}_{i\in I}$ of pairwise disjoint clopen sets. Hence $\mathcal{L}(X, C(K))$ is a ℓ^∞-sum of infinitely many non-trivial spaces. Thus there is no point of weak-norm continuity in $\mathcal{L}(X, C(K))_1$.

The main difficulty in dealing with the question of points of weak-norm continuity is that in general one does not have a description of the weak topology of $\mathcal{L}(X)$. We do not know if the identity operator can be a denting point of $\mathcal{L}(X)_1$. In the above arguments we took advantage of weak convergence of sequences in $\mathcal{K}(X)$. Thus for a general X a more reasonable space to consider is span$\{\mathcal{K}(X), I\}$.

In contrast with the situation for ℓ^∞-sums, for ℓ^1-sums of Banach spaces, we have a positive result. It is clearly enough to consider sum of two spaces. The following Lemma is easy to prove.

Lemma. Let X be a Banach space. Suppose M and N are two closed subspaces such that $X = M \oplus N$, *is an* ℓ^1 *direct sum.* x_0 *is a denting point of* X_1 *if and only if* $x_0 \in M$ *or N and is a denting point of the corresponding unit ball.*

Turning back to the question of denting points of span$\{\mathcal{K}(X), I\}_1$, suppose $\|T + I\| = \|T\| + 1$ for all $T \in \mathcal{K}(X)$. Then it follows from the above Lemma that I is a denting point of span$\{\mathcal{K}(X), I\}_1$. That this hypothesis is satisfied when $X = C[0, 1]$ is a well known result of Daugavet (see [A] for more information). This also shows that the preceding technique of working with finite rank or compact operators does not work here. In spite of this, this author has recently proved that for any infinite compact space Ω and for any Banach space X, there are no denting points in $\mathcal{L}(X, C(\Omega))_1$ (see [R2]). In the case of a non-atomic measure it was shown in [R3] that there are no points of weak-norm continuity in $L^1(\mu, X)_1$.

Before exhibiting another class of Banach spaces for which there are no denting points in $\mathcal{L}(X, Y)_1$, we need some notation and terminology.

An operator $T \in \mathcal{L}(X, Y)_1$ is said to be a 'nice' operator if $T^*(\partial_e Y_1^*) \subset \partial_e X_1^*$. Any such operator is clearly an extreme point of $\mathcal{L}(X, Y)_1$ (see [S]).

For any X, its centralizer $Z(X)$ is the set of all $T \in \mathcal{L}(X)$ for which there is a bounded function α and a $S \in \mathcal{L}(X)$ such that

$$\left.\begin{aligned} T^*(p) &= \alpha(p)p \\ S^*(p) &= \bar{\alpha}(p)p \end{aligned}\right\} \quad \text{for all } p \in \partial_e X_1^*.$$

In our concluding result of this section we again consider extra conditions on the range space to conclude that certain operators cannot occur as points of weak-norm continuity. Since $Z(C(K))$ is isometric to $C(K)$ (see Ch. 3 in [B]), the next theorem generalizes Theorem 2 of [GS2].

Theorem 2.3. *Let Y be a Banach space such that* $Z(Y)$ *is infinite dimensional. Then any* $T \in \mathcal{L}(X, Y)_1$ *such that* T^* *maps extreme points of* Y_1^* *to unit vectors is not a point of weak-norm continuity. In particular I is not a denting point of* $\mathcal{L}(Y)_1$.

Proof. It is easy to see that if K is any infinite compact Hausdorff space, there exists a sequence $\{f_n\} \in C(K)_1, 0 \le f_n \le 1, f_n \xrightarrow{w} 0$ and $\|f_n\| = 1, \|1 - f_n\| \le 1$ for all n.

Using Theorem 4.14 of [B], we represent Y as a maximal function module over a compact space K_Y. By our hypothesis on $Z(Y)$, K_Y is an infinite set. Therefore using the isometric correspondence between $C(K_Y)$ and $Z(Y)$, we may choose a sequence $T_n \in Z(Y)$

that corresponds to the f_n's mentioned above such that $\|T_n\| = 1$ and $T_n \xrightarrow{w} 0$ and $\|I - T_n\| \leq 1$ for all n.

Now let $T \in \mathcal{L}(X,Y)_1$ be such that T^* maps extreme points of Y_1^* to unit vectors.

Clearly $(I - T_n) \circ T \to T$ weakly and $\|(I - T_n) \circ T\| \leq 1$. Also

$$\|T_n \circ T\| = \|T^* \circ T_n^*\| = \sup_{p \in \partial_e Y_1^*} \|T^*(T_n^*(p)\|$$
$$= \sup_{p \in \partial_e Y_1^*} \|\alpha_n(p) T^*(p)\|$$

(since $T_n \in Z(Y)$ we have that $T_n^*(p) = \alpha_n(p)p$)

$$= \sup_{p \in \partial_e Y_1^*} |\alpha_n(p)|$$
$$= \|\alpha_n\| = \|T_n\| = 1.$$

Therefore T is not a point of weak-norm continuity.

COROLLARY 2.3

Let X and Y be such that extreme points of $\mathcal{L}(X,Y)_1$ are 'nice' operators. Suppose that $Z(Y)$ is infinite dimensional. Then there are no denting points in $\mathcal{L}(X,Y)_1$.

Remark 2.6. It is worth noting that in the case of ℓ^p, $Z(\ell^p)$ is trivial (see [B] Corollary 4.23).

Remark 2.7. We do not know an answer to the denting point (point of weak-norm continuity) question for the space $\mathcal{L}(\ell^\infty, X)_1$ for a general X. When X is an infinite dimensional space with the Schur property, then since X has no copy of ℓ^∞, it follows from Corollary 3 on p. 149 (see also Theorem 15 on p. 159) of [DU] that every operator here is weakly compact and hence compact. When $X = \ell^1$ we first note that $\mathcal{K}(c_0, X) = \ell^1 \otimes_\epsilon \ell^1 = \mathcal{L}(c_0, X)$, being a separable dual space has the R. N. P. Also $\mathcal{K}(\ell^\infty, X) = \ell^{\infty *} \otimes_\epsilon \ell^1 = \mathcal{L}(\ell^\infty, X)$. Using the canonical embedding of ℓ^1 in its bidual and arguments similar to the ones given during the proof of Proposition 1 we see that $\mathcal{L}(\ell^\infty, \ell^1)_1$ has points of weak-norm continuity. Also for any infinite compact set K and for any X with the R. N. P, it follows from the results in Chap. VI of [DU] that $\mathcal{AS}(C(K), X)$ can be identified as a subspace of rcabv(X) (space of X-valued countably additive regular Borel measures of finite variation). A complete description of points of weak-norm continuity of $\mathcal{AS}(C(K), X)_1$ can be deduced from Theorem 3 of [R3].

3. Strongly extreme points

In this section we consider the existence of strongly extreme points in the unit ball of the space of operators $\mathcal{L}(X,Y)$ and the permanence of extreme points.

It is known that for any compact set K, every extreme point of $C(K)_1$ is a strongly extreme point and a similar result is true of $L^1(\mu)_1$. The corresponding operator version ie., all extreme points in $\mathcal{L}(L^1(\mu_1), L^1(\mu_2))_1$ are strongly extreme has been recently proved in [GS2]. The authors of [GS2] also exhibit certain class of operators as strongly extreme points in $\mathcal{L}(C(K_1), C(K_2))_1$ for compact sets K_1, K_2.

We first isolate a property that is common to $C(K)$ and $L^1(\mu)$ spaces and use it to obtain a general version of the results in [GS2].

DEFINITION [L1]

A Banach space X is said to have the extreme point intersection property (E. P. I. P. for short) if for all $x \in \partial_e X_1$ and for all $x^* \in \partial_e X_1^*$, $|x^*(x)| = 1$. It is easy to see that both $C(K)$ and $L^1(\mu)$ have this property. Any Banach space whose dual is isometric to $L^1(\mu)$ also has this property.

Suppose $x_0 \in X_1$ is such that $|x^*(x_0)| = 1$ for all $x^* \in \partial_e X_1^*$ then we note that x_0 is a strongly extreme point of X_1. To see this, if $(x_n + y_n)/2 \to x_0$ for two sequences $\{x_n\}$, $\{y_n\} \subset X_1$. Then for $\epsilon > 0 \, \exists N$ such that $\forall n \geq N$, $\forall x^* \in \partial_e X_1^*$, $|(x^*(x_n) + x^*(y_n))/2 - x^*(x_0)| < \epsilon$. Since $|x^*(x_0)| = 1$, $|x^*(x_n)| \leq 1$ and $|x^*(y_n)| \leq 1$ we get that $\|x_n - y_n\| = \sup_{x^* \in \partial_e X_1^*} |x^*(x_n) - y^*(x_n)| \leq 2\epsilon$ for all $n \geq N$. Hence the claim.

Theorem 3.1. *Suppose X^* has the E.P.I.P. For any Banach space Y, any 'nice' operator $T \in \mathcal{L}(X, Y)_1$ is a strongly extreme point.*

Proof. Let $T \in \mathcal{L}(X, Y)_1$ be a 'nice' operator. In view of our observation above, we shall show that $|\wedge(T)| = 1$ for all $\wedge \in \partial_e \mathcal{L}(X, Y)_1^*$. Since $T^*(\partial_e Y_1^*) \subset \partial_e X_1^*$ and since X^* has the E. P. I. P., the conclusion now follows from the arguments given during the proof of Theorem 1 in [R1].

Remark 3.1. Since every extreme point of $\mathcal{L}(L^1(\mu_1), L^1(\mu_2))_1$ is a nice operator (see [S] Theorem 2.2) we get that every extreme point is strongly extreme (see [GS2] Corollary 4).

Another extremal property enjoyed by both $C(K)$ and $L^1(\mu)$ spaces is that any extreme point of the unit ball is also an extreme point of the unit ball of the second dual of the space (under the canonical embedding). We do not know if this property holds for spaces of operators as well. We however have the following stability results.

PROPOSITION 3.1

Let $\{X_i\}_{i \in I}$ be a family of Banach spaces. If for all i, every extreme point of $(X_i)_1$ is an extreme point of the unit ball of the bidual then the same is true of their ℓ^1-direct sum $X = \oplus_1 X_i$.

Proof. Let $x_0 \in X_1$ be an extreme point. Clearly there exists a $i_0 \in I$ such that $x_0(i) = 0$ for all $i \neq i_0$.

It is easy to see using the hypothesis that x_0 is an extreme point of $\oplus_1 X_i^{**}$. It is well-known that $X^{**} = \oplus_1 X_i^{**} \oplus_1 (\oplus_{c_0} X_i^*)^\perp$. Therefore x_0 is an extreme point of X_1^{**}.

We next consider this property for the space of vector-valued continuous functions $C(K, X)$ equipped with the supremum norm. We deal with the cases K dispersed (or scattered) and K containing a perfect set separately. The well-known identification of the space of compact operators $\mathcal{K}(X, C(K))$ with $C(K, X^*)$ thus gives corresponding result for the space of compact operators.

The situation when K is dispersed is very similar to Proposition 1.

PROPOSITION 3.2

Let K be a dispersed compact set. Let X be such that $\partial_e X_1 \subset \partial_e X_1^{**}$. Then $\partial_e C(K, X)_1 \subset \partial_e C(K, X)_1^{**}$.

Proof. Let I denote the set of isolated points of K. It is well-known that $C(K,X)^*$ can be identified with $\oplus_1 X_i^*$ where $X_i^* = X^*$ for all i. Thus the bidual has the identification $\oplus_\infty X_i^{**}$.

If $f \in \partial_e C(K,X)_1$ then for any $i \in I$, since i is an isolated point, $f(i) \in \partial_e X_1 \subset \partial_e X_1^{**}$. Therefore $f \in \partial_e C(K,X)_1^{**}$.

We need stronger assumptions on X and $C(K,X)$ to deal with the case when K contains a perfect set. We have proved in [R1] that if X has the E. P. I. P. and for every $f \in \partial_e C(K,X)_1$, $f(K) \subset \partial_e X_1$ then $C(K,X)$ has the E. P. I. P. and thus from our observation above every element of $\partial_e C(K,X)_1$ is a weak* extreme point.

Our concluding remark deals with the question of permanence of extreme points in the unit ball of projective tensor products.

Remark 3.2. Let μ be a finite measure and X be such that $\partial_e X_1 \subset \partial_e X_1^{**}$. Since any extreme point of $L^1(\mu,X)_1$ is of the form $\chi_A x$ where $x \in \partial_e X_1$ and $A \in \mathcal{A}$ is a μ-atom (see [Su]), it is easy to see that $L^1(\mu,X) = X \oplus_1 M$ for a closed subspace $M \subset L^1(\mu,X)$ (X is identified as functions that are constant on the atom A). Therefore $\partial_e L^1(\mu,X)_1 \subset \partial_e L^1(\mu,X)_1^{**}$.

In the case of general projective tensor product $X \otimes_\pi Y$, it is known that any denting point of $(X \otimes_\pi Y)_1$ is of the form $x \otimes y$ where $x \in X_1$ and $y \in Y_1$ are denting points (see [RS]). Since $(X \otimes_\pi Y)^* = \mathcal{L}(X, Y^*)$ (see [DU] p. 230), it follows from Theorem 3.7 of [L2] that if $x \in X_1$ is a denting point and $y \in Y$ is a weak* extreme point then $x \otimes y$ is a weak* extreme point of $(X \otimes_\pi Y)_1$.

It is known (see [La]) that on the surface of the unit ball of $K(\ell^2)^* = \ell^2 \otimes_\pi \ell^2$ the weak and norm topologies coincide and thus any unit vector is a point of weak-norm continuity, whereas denting points in the unit ball are of the form $x \otimes y$ for $\|x\| = \|y\| = 1$. In view of these remarks it is natural to ask the following question.

Question. If $x \in X_1$ is a denting point and $y \in Y_1$ is a point of weak-norm continuity then will $x \otimes y$ always be a point of weak-norm continuity in $(X \otimes_\pi Y)_1$?

This is indeed the case when of X or Y is a $L^1(\mu)$ space (see [R3]). More generally the following proposition gives another instance when the above question has affirmative answer. It can be proved using arguments similar to the ones given during the proof of Corollary 3 in [R3].

PROPOSITION 3.3

Suppose X is such that the answer to the above question is affirmative for $X \otimes_\pi X$. For any compact set K, let $F \in M(K,X)_1$ be a denting point and $x \in X_1$ be a point of weak-norm continuity. Then $F \otimes x$ is a point of weak-norm continuity of $(M(K,X) \otimes_\pi X)_1$. If either K is dispersed or X also has the R. N. P., then the same conclusion holds, when F is a point of weak norm continuity and x is a denting point.

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References

[A] Abramovich Y, New classes of spaces on which compact operators satisfy the Daugavet equation, *J. Operator Theory* **25** (1991) 331–345

[Ans] Ansari S I, On Banach spaces Y for which $\mathcal{B}(C(\Omega), Y) = \mathcal{K}(C(\Omega), Y)$, *Pac. J. Math.* **169** (1995) 201–218

[BE] Behrends E, *M-Structure and the Banach–Stone Theorem*, Springer LNM No. **736** (1979)

[DU] Diestel J and Uhl J J, *Vector measures, Math. Surveys*, No.15 (Providence, Rhode Island) (1977)

[DM] Diestel J and Morrison T J, The Radon–Nikodym property for the space of operators, *Math. Nach.* **92** (1979) 7–12

[GR] Grząślewicz R, A note on extreme contractions on ℓ^p spaces, *Port. Math.* **40** (1981) 413–419

[GS1] Grząślewicz R and Scherwentke P, On strongly extreme and denting points in $\mathcal{L}(H)$, *Math. Jpn.* **41** (1995) 283–284

[GS1] Grząślewicz R and Scherwentke P, On strongly extreme and denting contractions in $\mathcal{L}(C(X), C(Y))$, *Bull. Acad. Sinica* **25** (1997) 155–160

[HWW] Harmand P, Werner D and Werner W, *M-ideals in Banach spaces and Banach algebras*, (Springer LNM No 1547, Berlin) (1993)

[HL] Hu Z and Lin B L, RNP and CPCP in Lebesgue–Bochner function spaces, *Ill. J. Math.* **37** (1993) 329–347

[H] Hennefeld J, Compact extremal operators, *Il. J. Math.* **21** (1977) 61–65

[KW] Kalton N J and Werner D, Property (*M*), *M*-ideals, and almost isometric structure of Banach spaces, *J. Reine Angew. Math.* **461** (1995) 137–178

[KR] Kunen K and Rosenthal H P, Martingale proofs of some geometrical results in Banach space theory, *Pac. J. Math.* **100** (1982) 153–175

[La] Lau A T M and Mah P F, Quasinormal structures for certain spaces of operators on a Hilbert space, *Pac. J. Math.* **121** (1986) 109–118

[L1] Lima Å, Intersection properties of balls in spaces of compact operators, *Ann. Inst. Fourier*, **28** (1978) 35–65

[L2] Lima Å, Oja E, Rao T S S R K and Werner D, Geometry of operator spaces, *Mich. Math. J.* **41** (1994) 473–490

[LLT] Bor-Luh Lin, Pei-Kee Lin and Troyanski S L, Characterizations of denting points, *Proc. Amer. Math. Soc.* **102** (1988) 526–528

[LT] Lindenstrauss J and Tzafriri L, *Classical Banach spaces I*, (Springer, Berlin) (1977)

[RS] Ruess W M and Stegall C P, Weak*-denting points in duals of operator spaces, in "Banach spaces", ed.: Kalton N and Saab E, *Lecture Notes in Mathematics*, No. 1166, Springer (1984) 158–168

[R1] Rao T S S R K, On the extreme point intersection property, in "Function Spaces, the second conference", *Lecture Notes in pure and applied mathematics* Ed.: K. Jarosz, (No. 172, Marcel Dekker) (1995) 339–346

[R2] Rao T S S R K, There are no denting points in the unit ball of $WC(K, X)$, *Proc. Am. Math. Soc.* (to appear)

[R3] Rao T S S R K, Points of weak*-norm continuity in the unit ball of the space $WC(K, X)^*$, *Can. Math. Bull.* (to appear)

[SM] Sharir M, Extremal structures in operator spaces, *Trans. Am. Math. Soc.* **186** (1973) 91–111

[SuK] Sundaresan K, Extreme points of the unit cell in Lebesgue–Bochner function spaces, *Colloq. Math.* **22** (1970) 111–119

Note added in proof

We now use the ideas contained in § 1 of this paper to completely answer the question on points of weak-norm continuity in $\mathcal{L}(X, C(K))_1$. In what follows we shall use the identification of $\mathcal{L}(X, C(K))$ as $W^*C(K, X^*)$, the space of functions on K that are continuous when X^* has the weak* topology, equipped with the supremum norm.

Theorem. *Let K be an infinite compact Hausdorff space and let X be any Banach space. Let* $f \in W^*C(K, X^*)$ *be a unit vector. There exists a sequence* $\{f_n\}_{n \geq 1} \subset W^*C(K, X^*)_1$ *such that* $f_n \to f$ *weakly but not in the norm. Hence there are no points of weak-norm continuity in* $\mathcal{L}(X, C(K))_1$.

Proof. Let $f \in W^*C(K, X^*)$, $\|f\| = 1$. It follows from Theorem 3 of [DHS] (see also [R2]) that the result is true if f is continuous w.r.t the norm topology on X^*. Note that if $f(K)$ is a norm compact subset of X^*, then since weak* and norm topologies coincide on $f(K)$, we get that f is continuous w.r.t the norm topology. Thus we assume w.l.o.g that $f(K)$ is not norm compact. Therefore there exists a sequence $\{t_n\}_{n \geq 1} \subset K$ of distinct terms such that $\{f(t_n)\}_{n \geq 1}$ has no convergent subsequence ($f(K)$ being weak* compact is norm closed).

We choose as before sequences of pairwise disjoint open sets $\{U_n\}$ and $\{g_n\} \subset C(K)_1^+$ such that $g_n(t_n) = 1$ and $g_n = 0$ on $K - U_n$ for all n. Using the 'dominated convergence' and the 'Riesz representation' theorems it is easy to see that $(1 - g_n) \to 0$ in the weak topology of $C(K)$. Since the map $g \to gf$ is a bounded linear contraction from $C(K)$ into $W^*C(K, X^*)$, it preserves weak convergence. Thus it follows that $f_n = (1 - g_n)f \to f$ in the weak topology. Since $\text{Sup}\|f(t_n)\| \leq \text{Sup}\|g_n f\|$ we get that $f_n \not\to f$ in the norm.

[DHS] Dowling P N, Hu Z and Smith M A, Extremal structure of the unit ball of $C(K, X)$, *Contemp. Math.*, **144** (1993) 81–85