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Fixed Width Confidence Interval of $P(X < Y)$ in Partial Sequential Sampling Scheme

Uttam Bandyopadhyay,[1,*] Radhakanta Das,[2] and Atanu Biswas[3,*]

[1]Department of Statistics, University of Calcutta, Kolkata, India
[2]Department of Statistics, Bankura Sammilani College, Bankura, West Bengal, India
[3]Applied Statistics Unit, Indian Statistical Institute, Kolkata, India

ABSTRACT

The article is related to nonparametric fixed-width confidence interval estimation of the parameter $\theta = \int F(y)\,dG(y)$, where F and G are two unknown univariate continuous distribution functions, by adopting a partial sequential sampling scheme. Different asymptotic results associated with the proposed procedures are formulated and examined.

*Correspondence: Uttam Bandyopadhyay, Department of Statistics, University of Calcutta, 35 B. C. Road, Kolkata 700 019, India; E-mail: ubandyopadhyay@yahoo.com. Atanu Biswas, Applied Statistics Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700 108, India; E-mail: atanu@isical.ac.in.

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1. INTRODUCTION

Let X and Y be two real-valued random variables having unknown continuous distribution functions (d.f.'s) F and G respectively. In this article we are concerned with fixed-width confidence interval estimation of the parameter

$$\theta = P(X < Y) = \int F(y)\, dG(y) \tag{1.1}$$

by using a partially sequential sampling scheme in which there is a random sample of fixed size from one of the populations and, using this information, the observations from the other population are drawn one-by-one sequentially until a prefixed requirement is attained.

There are many practical situations in which the observations from one of these distributions are easy and relatively inexpensive to collect, while the sample observations corresponding to the other population are costly and difficult to obtain. For example, the easily collected sample might be on X, a standard treatment observations and the sample that is more difficult to obtain corresponds to Y, a new treatment data. In such situations we would like to gather data (may be large) on X and collect only enough observations necessary in order to reach a decision regarding the problem under consideration. Quite often, in several industrial and biomedical applications, one may wish to compare a new technique or therapy with some existing one. Since the existing technique or therapy is in use for quite some time some observations on it are readily available, and one may wish to collect some observations on the new technique or therapy as per requirement. To achieve these goals we consider collecting Y-observations in a sequential manner and the sampling is terminated in the sense that it will never require more than a preset number of treatment observations.

In connection with the problem of testing H: $F = G$ against $F > G$, several partial sequential designs were proposed and studied by Wolfe,[1] Orban and Wolfe[2,3] and Chatterjee and Bandyopadhyay.[4] For details one can also see the book by Randles and Wolfe.[5] This article, motivated by Hjort and Fenstad[6] (see also Khan,[7] provides a fixed-width confidence interval of θ using the following partial sequential sampling procedure. Let $\mathbf{X_m} = (X_1, \ldots, X_m)'$ be a random sample of fixed size m (≥ 1) on X and $\{Y_n, n \geq 1\}$ be a sequence of observations on Y. Let $\hat{\theta}_n$ be a sequence of estimators based on $\mathbf{X_m}$ and $(Y_1, \ldots, Y_n)$, $n \geq 1$. Suppose for each m, there is a positive integer $r = r(m)$ such that, as $m \to \infty$, we have $r \to \infty$. We also assume that as $m \to \infty$,

$$\sup_{n \geq r} |\hat{\theta}_n - \theta| \to 0, \tag{1.2}$$

in probability. Then, for given $d(>0)$, we introduce the random variable

$$N(d) = \sup\{n \geq 1 : |\hat{\theta}_n - \theta| \geq d\}, \tag{1.3}$$

which is related to a partially sequential fixed-width confidence interval of θ in the sense that there exists a positive integer $\nu = \nu(m)$ such that asymptotically, for given $\alpha \in (0,1)$,

$$P(|\hat{\theta}_n - \theta| < d \text{ for all } n \geq \nu) = 1 - \alpha. \tag{1.4}$$

The random variable of the type $N(d)$ is well-studied by Hjort and Fenstad[8] in connection with sequential fixed-width confidence interval estimation. A terminal version corresponding to Eq. (1.4) is that there exists a positive integer $\nu^* = \nu^*(m)$ (different from ν) such that asymptotically

$$P(|\hat{\theta}_n - \theta| < d) \geq 1 - \alpha \quad \text{for all} \quad n \geq \nu^*. \tag{1.5}$$

Under sequential set-up, Eq. (1.5) is well-studied by many researchers. For details one can go through the book by Ghosh et al.[9] But, unlike the existing procedure, we, as in Eq. (1.3), can also attach here a random variable $N^*(d)$ such that, for each $n \geq 2$, $N^*(d) \geq$ or $< n$ is completely determined by $|\theta_n - \theta| \geq d$ or $< d$. It is worth to mention that, under Eq. (1.2), the variables $N(d)$ and $N^*(d)$ are both finite with probability one. Also the asymptotic results related to $N^*(d)$ can easily be found from that of $N(d)$ as a terminal case.

We organize the rest of the article in the following way. Section 2 presents, along with some asymptotic results, the limiting distributions of $\sup\{\sqrt{r}|\hat{\theta}_n - \theta|, n \geq r\}$ and $d^2N(d)$. Section 3 discusses the different partially sequential fixed-width confidence intervals of θ. The convergence of $E(d^2N(d))$ is also studied in Sec. 4. Section 5 contains some numerical computations for comparing the different procedures. Some concluding remarks appear in Sec. 6.

2. SOME ASYMPTOTIC RESULTS

Let $F_m(x)$ be the empirical d.f. based on $\mathbf{X_m}$. Then, for each observed Y, we set

$$\hat{\theta}_n = \frac{1}{n}\sum_{j=1}^{n} F_m(Y_j) = \frac{1}{nm}\sum_{j=1}^{n}\sum_{i=1}^{m} u(X_i, Y_j), \tag{2.1}$$

where $u(a, b) = 1$ or 0 according as $a <$ or $> b$. We now provide the following result that is useful in the subsequent derivations. For this we assume that, for each m, there is a positive integer $r = r(m)$ such that, as $m \to \infty$,

$$r \to \infty \quad \text{but} \quad \frac{r}{m} \to \lambda \in [0, \infty). \tag{2.2}$$

Throughout in this article, whenever $m \to \infty$, we mean that Eq. (2.2) holds. In this context, we like to mention that explicit sequential methodologies based on generalized U-statistics and associated asymptotics were studied by Williams and Sen.[10,11] Their methods are relevant for studying various asymptotics related to the variable $N^*(d)$ but not to $N(d)$. Hence we need some further development in the direction appropriate for the variable $N(d)$. As a first step towards this development, we consider the following lemma.

Lemma 2.1. *As* $m \to \infty$,

$$\sup_{n \geq r}\left|\sqrt{r}(\hat{\theta}_n - \theta) - \frac{\sqrt{r}}{m}\sum_{i=1}^{m}(\bar{G}(X_i) - \theta) - \frac{\sqrt{r}}{n}\sum_{j=1}^{n}(F(Y_j) - \theta)\right| \xrightarrow{P} 0,$$

where $\bar{G}(x) = 1 - G(x)$.

Proof. We write

$$V_{mj} = F_m(Y_j) - F(Y_j) - \frac{1}{m}\sum_{i=1}^{m}(\bar{G}(X_i) - \theta)$$

so that

$$\frac{1}{n}\sum_{j=1}^{n} V_{mj} = (\hat{\theta}_n - \theta) - \frac{1}{m}\sum_{i=1}^{m}(\bar{G}(X_i) - \theta) - \frac{1}{n}\sum_{j=1}^{n}(F(Y_j) - \theta).$$

Observe that, given $\mathbf{X_m}$, V_{mj}'s are independently and identically distributed (i.i.d.) random variables with "zero" expectation and

$$\mathrm{Var}(V_{mj}/\mathbf{X_m}) = \int [F_m(y) - F(y)]^2\, dG(y) - \left(\frac{1}{m}\sum_{i=1}^{m} \bar{G}(X_i) - \theta\right)^2$$
$$= \sigma^2(\mathbf{X_m}) \quad \text{(say)}.$$

By Glivenko-Cantelli's lemma, it is not difficult to show that, as $m \to \infty$,

$$\sigma^2(\mathbf{X_m}) \xrightarrow{P} 0,$$

and hence, since $\sigma^2(\mathbf{X_m})$ is uniformly bounded, we get

$$E(\sigma^2(\mathbf{X_m})) \to 0. \tag{2.3}$$

Now, for every $\epsilon > 0$,

$$P\left(\sup_{n\geq r}\left|\frac{1}{n}\sum_{j=1}^{n} V_{mj}\right| > \frac{\epsilon}{\sqrt{r}}\right) = EP\left(\sup_{n\geq r}\left|\frac{1}{n}\sum_{j=1}^{n} V_{mj}\right| > \frac{\epsilon}{\sqrt{r}}\middle|\mathbf{X_m}\right),$$

which, by Hajek-Renyi inequality (see Rao,[12] pp. 143), is

$$\leq E\left[\frac{r}{\epsilon^2}\left(\frac{1}{r}\sum_{i=1}^{r}\sigma^2(\mathbf{X_m}) + \sum_{i=r+1}^{\infty}\sigma^2(\mathbf{X_m})/i^2\right)\right] < \frac{2}{\epsilon^2}E(\sigma^2(\mathbf{X_m})),$$

and hence, by Eq. (2.3), the required result follows. □

Note. From the above lemma, it is easy to check that, as $m \to \infty$,

$$\sup_{n\geq r} |\hat{\theta}_n - \theta| \xrightarrow{P} 0.$$

Remark. While proving the above lemma, we could incorporate earlier methodologies (see e.g., Williams and Sen[11]).

Limiting Distribution of $sup\{\sqrt{r}|\hat{\theta}_n - \theta|,\ n \geq r\}$: Here we introduce the following random variables:

$$\bar{U}_n = \frac{1}{n}\sum_{j=1}^{n}\left(F_m(Y_j) - \frac{1}{m}\sum_{i=1}^{m}\bar{G}(X_i)\right), \tag{2.4}$$

$$\bar{U}_{1m} = \frac{1}{m}\sum_{i=1}^{m}(\bar{G}(X_i) - \theta)/\sigma_1, \tag{2.5}$$

and

$$\bar{U}_{2n} = \frac{1}{n}\sum_{j=1}^{n}(F(Y_j) - \theta)/\sigma_2, \tag{2.6}$$

where

$$\sigma_1^2 = \int [\bar{G}(x) - \theta]^2 dF(x), \qquad \sigma_2^2 = \int [F(y) - \theta]^2 dG(y). \tag{2.7}$$

Clearly

$$\hat{\theta}_n - \theta = \bar{U}_n + \sigma_1 \bar{U}_{1m}, \tag{2.8}$$

and, by Lemma 2.1, we have

$$\sqrt{r}\bar{U}_n = \sigma_2\sqrt{r}\bar{U}_{2n} + \epsilon_n, \tag{2.9}$$

where, as $m \to \infty$,

$$\sup_{n\geq r} |\epsilon_n| \xrightarrow{P} 0. \tag{2.10}$$

Let $\{n(t) = n_m(t),\ 0 < t \leq 1\}$ be a sequence of non-increasing, right-continuous and positive integers such that

$$n(1) = r \quad \text{and} \quad n(t) \to \infty \quad \text{as} \quad t \to 0. \tag{2.11}$$

In particular, for every m ($\geq$1), if we take

$$n(t) = \min\left\{n : \frac{r}{n} \leq t\right\}, \quad 0 < t \leq 1, \tag{2.12}$$

then Eq. (2.11) is satisfied. Let us then introduce the following stochastic processes:

$$\begin{aligned} W_m(t) &= \sqrt{r}\bar{U}_{n(t)}, \quad 0 < t \leq 1, \\ &= 0, \qquad\qquad\ \ t = 0. \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} Z_m(t) &= \sqrt{r}\bar{U}_{2n(t)}, \quad 0 < t \leq 1, \\ &= 0, \qquad\qquad\ \ t = 0. \end{aligned} \tag{2.14}$$

Let us now write $\lim_{t\downarrow 0} Z_m(t) = Z_m(0+0)$ and $\lim_{t\downarrow 0} W_m(t) = W_m(0+0)$. Then from Lemma 2.1 of Sen[13] it is easy to verify that $Z_m(0+0)$ is equal to "zero" with probability one, and hence it is not difficult to extend the definition of Eq. (2.14) at $t = 0$. Thus the stochastic process $Z_m = \{Z_m(t), 0 \leq t \leq 1\}$ belongs to $D[0, 1]$ equipped with Skorkhod J_1-topology. But, with a view to extending the definition (2.13) at $t = 0$, we consider the following lemma.

Lemma 2.2. *For every m (≥ 1), $W_m(0+0) = 0$ with probability one.*

Proof. Here we show that, for every m (≥ 1),

$$P\left(\lim_{t \downarrow 0} W_m(t) = 0\right) = 1. \tag{2.15}$$

For this it is enough to show that, for every $\epsilon > 0$,

$$\lim_{k \to \infty} P\left(\max_{n \geq r \times 2^k} |\bar{U}_n| > \frac{\epsilon}{\sqrt{r}}\right) = 0. \tag{2.16}$$

Now, using the fact that, given $\mathbf{X_m}$, $\bar{U}_n$ is the mean of i.i.d. random variables having "zero" expectation and variance

$$S_m^2 = E_{Y_1}\left(F_m(Y_1) - \frac{1}{m}\sum_{i=1}^{m} \bar{G}(X_i)\right)^2 \leq E_{Y_1} F_m^2(Y_1) \leq 1,$$

we have, on applying Hajek-Renyi inequality, the expression under limit of the left hand side of Eq. (2.16) as

$$\leq \left(\frac{2}{\epsilon}\right)^2 2^{-k} E(S_m^2) \leq \epsilon^{-2} 2^{2-k} \to 0 \quad \text{as} \quad k \to \infty.$$

Hence Eq. (2.15) is true. □

Thus, using the above lemma, the stochastic process $W_m = \{W_m(t), 0 \leq t \leq 1\}$ belongs to $D[0, 1]$ equipped with Skorkhod J_1-topology. Let $W = \{W(t), 0 \leq t \leq 1\}$ be a Wiener process on $D[0, 1]$ (with $C(0, 1]$ as support). Then we prove the following theorems.

Theorem 2.1. *As $m \to \infty$,*

$$Z_m \xrightarrow{d} W, \text{ in the } J_1\text{-topology on } D[0, 1].$$

Proof. From Sen,[13] as $\bar{U}_{2n}$ is the mean of i.i.d. random variables with "zero" expectation and unit variance, it can be easily established that the process Z_m converges in distribution to the process W. □

Theorem 2.2. *For any fixed* x (≥ 0),

$$\lim_{m\to\infty} P\left(\sqrt{r}\sup_{n\geq r}|\hat{\theta}_n - \theta| \leq x\right) = P\left(\sup_{0\leq t\leq 1}|\sigma_2 W(t) + \sqrt{\lambda}\sigma_1 Z| \leq x\right),$$

where Z *is an* $N(0, 1)$ *random variable distributed independently of* W.

Proof. Using Eqs. (2.8) and (2.13), we get

$$\sqrt{r}\sup_{n\geq r}|\hat{\theta}_n - \theta| = \sup_{0\leq t\leq 1}\left|W_m(t) + \sigma_1\sqrt{r}\bar{U}_{1m}\right|. \tag{2.17}$$

By CLT, as $m \to \infty$,

$$\sqrt{r}\bar{U}_{1m} \xrightarrow{d} N(0, \lambda). \tag{2.18}$$

Hence, as $\bar{U}_{1m}$ is distributed independently of $\{\bar{U}_{2n}\}$, we get, using Eq. (2.14) and Theorem 2.1, as $m \to \infty$,

$$\sup_{0\leq t\leq 1}\left|\sigma_2 Z_m(t) + \sigma_1\sqrt{r}\bar{U}_{1m}\right| \xrightarrow{d} \sup_{0\leq t\leq 1}\left|\sigma_2 W(t) + \sqrt{\lambda}\sigma_1 Z\right|. \tag{2.19}$$

But, in view of Eqs. (2.9) and (2.10), we get, as $m \to \infty$,

$$\left|\sup_{0\leq t\leq 1}\left|W_m(t) + \sigma_1\sqrt{r}\bar{U}_{1m}\right| - \sup_{0\leq t\leq 1}\left|\sigma_2 Z_m(t) + \sigma_1\sqrt{r}\bar{U}_{1m}\right|\right|$$
$$\leq \sup_{0\leq t\leq 1}|W_m(t) - \sigma_2 Z_m(t)| \xrightarrow{P} 0,$$

and hence, Eq. (2.19) implies the required result. □

Note. From the above theorem it is easy to find

$$\lim_{m\to\infty} P\left(\sqrt{r}|\hat{\theta}_r - \theta| \leq x\right) = P\left(|\sigma_2 W(1) + \sqrt{\lambda}\sigma_1 Z| \leq x\right)$$
$$= 2\Phi\left(\frac{x}{\sqrt{\sigma_2^2 + \lambda\sigma_1^2}}\right) - 1, \tag{2.20}$$

where $\Phi(\cdot)$ is the d.f. of an $N(0, \mathbf{1})$ random variable.

Limiting Distribution of $d^2N(d)$: Here we assume that, for each m (≥ 1), there is $d = d(m)$ (>0) such that, as $m \to \infty$,

$$d \to 0 \quad \text{but} \quad d\sqrt{m} \to b(>0). \tag{2.21}$$

In particular, if we take $d = b/\sqrt{m}$, $b > 0$. Equation (2.21) is satisfied. Now, given b, $v > 0$ and $m \geq 1$, we can find r to be the smallest integer $\geq vm/b^2$ and v_0 ($= rd^2$) satisfying: $v_0 - b^2/m < v \leq v_0$. Let V be a random variable such that, for a random variable U having the d.f. given by Result 1 of the Appendix,

$$\frac{V}{\sigma_2^2 + (\sigma_1^2/b^2)V} \sim U^2. \tag{2.22}$$

Then we get the following theorem.

Theorem 2.3. *As* $m \to \infty$,

$$d^2N(d) \xrightarrow{d} V.$$

Proof. For every v (>0), we have

$$\begin{aligned} P(d^2N(d) \geq v) &= P(N(d) \geq r) \\ &= P\left(\sqrt{r}\sup_{n \geq r} |\hat{\theta}_n - \theta| \geq \sqrt{v_0}\right), \end{aligned}$$

which, by Theorem 2.2, tends to

$$P\left(\sup_{0 \leq t \leq 1}\left|\sigma_2 W(t) + \frac{\sqrt{v}}{b}\sigma_1 Z\right| \geq \sqrt{v}\right),$$

and hence, from Result 1 of the Appendix, we get

$$\lim_{m \to \infty} P(d^2N(d) < v) = \psi\left(\frac{\sqrt{v}}{\sqrt{\sigma_2^2 + (\sigma_1^2/b^2)v}}\right).$$

This implies our required result. □

Note. From Eq. (2.20) it is easy to see that, for every v (>0),

$$\lim_{m \to \infty} P(d^2N^*(d) < v) = 2\Phi\left(\frac{\sqrt{v}}{\sqrt{\sigma_2^2 + (\sigma_1^2/b^2)v}}\right) - 1, \tag{2.23}$$

which means that

$$d^2N^*(d) \xrightarrow{d} V^*, \tag{2.24}$$

where $\sqrt{V^*}\Big/\sqrt{\sigma_2^2 + (\sigma_1^2/b^2)V^*} \sim$ Chi-square one degree of freedom.

3. FIXED-WIDTH CONFIDENCE INTERVALS

In this section, we discuss the construction of different fixed-width confidence intervals using the limiting distributions obtained in the previous section. These are obtained as follows.

First we consider the fixed-width confidence interval using the limiting distribution of $d^2N(d)$. Let, for given $\alpha \in (0,1)$, u_α be such that

$$\psi(u_\alpha) = 1 - \alpha.$$

Then, for given $(m, b, \sigma_1^2, \sigma_2^2)$, we have ν as the smallest integer n ($\geq$1) satisfying

$$\psi\left(\frac{b\sqrt{(n/m)}}{\sqrt{\sigma_2^2 + (n/m)\sigma_1^2}}\right) \geq 1 - \alpha \Longleftrightarrow n \geq \frac{m\sigma_2^2 u_\alpha^2}{b^2 - \sigma_1^2 u_\alpha^2},$$

$$\text{provided } b^2 - \sigma_1^2 u_\alpha^2 > 0. \tag{3.1}$$

One sufficient condition for satisfying the second inequality of Eq. (3.1) is that $b > u_\alpha$. For ν satisfying Eq. (3.1), we get

$$\lim_{m\to\infty} P(N(d) < \nu) = 1 - \alpha, \tag{3.2}$$

and hence a sequence of fixed-width confidence interval for θ of length $2d$ with confidence coefficient $1-\alpha$ is given by

$$(\hat{\theta}_n - d, \hat{\theta}_n + d), \quad n \geq \nu. \tag{3.3}$$

In practice σ_1 and σ_2 are unknown and hence Eq. (3.3) cannot be used. We then modify the above procedure as follows. For each n ($\geq$2), we provide the estimators:

$$\hat{\sigma}_{1n}^2 = \left[m\binom{n}{2}\right]^{-1} \sum_{i=1}^{m} \sum_{1\leq j< j'\leq n} u(X_i, Y_j)u(X_i, Y_{j'}) - \hat{\theta}_n^2,$$

$$\hat{\sigma}_{2n}^2 = \left[n\binom{m}{2}\right]^{-1} \sum_{1\leq i<i'\leq m} \sum_{j=1}^{n} u(X_i, Y_j)u(X_{i'}, Y_j) - \hat{\theta}_n^2.$$

Then, as $m \to \infty$,

$$\sup_{n \geq r} \left| \frac{\hat{\sigma}_{kn}}{\sigma_k} - 1 \right| \xrightarrow{P} 0, \quad k = 1, 2,$$

for every r satisfying Eq. (2.2). Further, if we write $\hat{\nu}$ to be the smallest integer n (≥ 2) satisfying

$$n \geq \frac{m\hat{\sigma}_{2n}^2 u_\alpha^2}{b^2 - \hat{\sigma}_{1n}^2 u_\alpha^2}, \quad \text{provided} \quad b^2 - \hat{\sigma}_{1n}^2 u_\alpha^2 > 0, \tag{3.4}$$

then we get the following theorem.

Theorem 3.1. *As* $m \to \infty$,

(i) $\hat{\nu} \sim \nu$,
(ii) $P(N(d) < \hat{\nu}) \to 1 - \alpha$,

where "$\hat{\nu} \sim \nu$" *means that* $\hat{\nu}/\nu \to 1$, *in probability.*

Proof. For any $\epsilon > 0$,

$$\left| \frac{\hat{\nu}}{m} - \frac{\nu}{m} \right| \Longleftrightarrow \hat{\nu} < \nu_1 \quad \text{or} \quad \hat{\nu} > \nu_2,$$

where $\nu_1 = \nu - m\epsilon$ and $\nu_2 = \nu + m\epsilon$. Now

$$\hat{\nu} < \nu_1 \Rightarrow \sup_{k < \nu_1} \frac{m\hat{\sigma}_{2k}^2 u_\alpha^2}{b^2 - \hat{\sigma}_{1k}^2 u_\alpha^2} < \nu_1,$$

and

$$\hat{\nu} > \nu_2 \Rightarrow \frac{m\hat{\sigma}_{2[\nu_2]}^2 u_\alpha^2}{b^2 - \hat{\sigma}_{1[\nu_2]}^2 u_\alpha^2} > \nu_2.$$

As $m \to \infty$, both the left hand members of the above inequalities divided by m converge in probability to $\lambda = (\sigma_2^2 u_\alpha^2)/(b^2 - \sigma_1^2 u_\alpha^2)$, but $(\nu_k/m) \to \lambda + (-1)^{k+1}\epsilon$, $k = 1, 2$. Hence we get

$$\lim_{m \to \infty} P(\hat{\nu} < \nu_1) = \lim_{m \to \infty} P(\hat{\nu} > \nu_2) = 0.$$

This implies (i).

Again

$$P(N(d) \geq \hat{\nu}) = P\left(\frac{\nu}{\hat{\nu}} d^2 N(d) \geq \nu d^2\right) \to P(V \geq b^2\lambda).$$

Thus we get

$$\lim_{m\to\infty} P(N(d) \geq \hat{\nu}) = \lim_{m\to\infty} P(N(d) \geq \nu),$$

which implies (ii). □

Hence, instead of Eq. (3.3), we get the following random sequence of fixed-width confidence intervals of length $2d$:

$$(\hat{\theta} - d, \hat{\theta} + d), \quad n \geq \hat{\nu}, \tag{3.5}$$

because, by Theorem 3.1 (ii), we get, as $m \to \infty$,

$$P\left(\sup_{n\geq\hat{\nu}} |\hat{\theta}_n - \theta| < d\right) = P(N(d) < \hat{\nu}) \to 1 - \alpha. \tag{3.6}$$

The terminal fixed-width confidence interval corresponding to Eq. (3.5) can also be given by

$$(\hat{\theta}_{\hat{\nu}^*} - d, \hat{\theta}_{\hat{\nu}^*} + d), \tag{3.7}$$

where $\hat{\nu}^*$ is obtained by Eq. (3.4) with u_α replaced by τ_α satisfying

$$\Phi(\tau_\alpha) = 1 - \alpha/2.$$

Then, as in Eq. (3.6), we have, using the note under Theorem 2.3, as $m \to \infty$,

$$P(|\hat{\theta}_{\hat{\nu}^*} - \theta| < d) = P(N^*(d) < \hat{\nu}^*) \to 1 - \alpha. \tag{3.8}$$

Now, to compare Eq. (3.5) with Eq. (3.7), we, for a specific F and G, compute the ratios ν/m and ν^*/m for some selected b and α. These are given in Sec. 5. The following ratio

$$\rho = \lim_{m\to\infty} \frac{E(N(d))}{E(N^*(d))} = \frac{\lim_{m\to\infty} E(d^2 N(d))}{\lim_{m\to\infty} E(d^2 N^*(d))} \tag{3.9}$$

can also be used as a measure of Asymptotic Relative Efficiency (ARE) of Eq. (3.5) relative to Eq. (3.7). We study the convergence of these expectations in the next section.

4. CONVERGENCE OF EXPECTATIONS

We have studied the limiting distributions of $d^2N(d)$ and $d^2N^*(d)$. The present section provides the convergence of $E(d^2N(d))$ and $E(d^2N^*(d))$. This can be achieved by using the following theorem.

Theorem 4.1. *Suppose r satisfies Eq.* (2.2). *Then, for every a* (> 0), $m(\geq 1)$ *and for some K* (> 0), *we have*

$$P\left(\sup_{n\geq r}|\hat{\theta}_n-\theta|\geq\frac{a}{\sqrt{r}}\right)\leq\frac{K}{a^4}.$$

Proof. Using Eq. (2.8), we can write

$$P\left(\sup_{n\geq r}|\hat{\theta}_n-\theta|\geq\frac{2a}{\sqrt{r}}\right)\leq P\left(\sup_{n\geq r}|\bar{U}_n|\geq\frac{a}{\sqrt{r}}\right)+P\left(|\bar{U}_{1m}|\geq\frac{a}{\sigma_1\sqrt{r}}\right). \tag{4.1}$$

As $E(\bar{U}_{1m}^4)\leq(4/m^2)$, we have

$$P\left(|\bar{U}_{1m}|\geq\frac{a}{\sigma_1\sqrt{r}}\right)\leq\frac{r^2\sigma_1^4}{a^4}E|\bar{U}_{1m}|^4\leq\frac{4\sigma_1^4}{a^4}\cdot\left(\frac{r}{m}\right)^2. \tag{4.2}$$

Let k be a positive integer such that, for a given m, $2^k\leq r<2^{k+1}$. Then, writing $n\bar{U}_n=S_n$. we note that S_n, given $\mathbf{X_m}$, is the sum of i.i.d. bounded random variables having zero expectations. Hence, utilizing a generalizations of Kolmogorov's inequality to the conditional probability for given $\mathbf{X_m}$, we get

$$\begin{aligned}P\left(\sup_{n\geq r}|\bar{U}_n|\geq\frac{a}{2\sqrt{r}}\right)&\leq\sum_{i=k}^{\infty}P\left(\sup_{n\leq 2^{i+1}}|S_n|\geq\frac{a}{2\sqrt{r}}2^i\right)\\&\leq\sum_{i=k}^{\infty}\frac{E(S_{2^{i+1}}^4|\mathbf{X_m})}{((a/\sqrt{r})\times 2^i)^4}\\&\leq\frac{4r^2}{a^4}\sum_{i=k}^{\infty}\frac{(2^{i+1})^2}{2^{4i}}\leq\frac{64}{3a^4}.\end{aligned} \tag{4.3}$$

Thus, by Eqs. (4.2) and (4.3), the right hand side of Eq. (4.1) is

$$\leq\frac{1}{a^4}\left[\frac{64}{3}+\left(\frac{r}{m}\right)^2 4\sigma_1^4\right]. \tag{4.4}$$

Now, by Eq. (2.2), there exists $K\ (>0)$ such that Eq. (4.4) cannot exceed K/a^4 for all m (≥ 1). Hence the result. □

Using Theorem 2.3 and the above theorem we get

$$E(d^2N(d)) = \int_0^\infty P(d^2N(d) \geq v)\,dv \to \int_0^\infty P(V \geq v)\,dv$$
$$= \int_0^\infty \left[\mathbf{1} - \psi\left(\frac{\sqrt{v}}{\sqrt{\sigma_2^2 + (\sigma_1^2/b^2)v}}\right)\right] dv. \tag{4.5}$$

Similarly, from the note under Theorem 2.3, we get

$$E(d^2N^*(d)) \to \int_0^\infty \left[2\left(1 - \Phi\left(\frac{\sqrt{v}}{\sqrt{\sigma_2^2 + (\sigma_1^2/b^2)v}}\right)\right)\right] dv. \tag{4.6}$$

Hence, for given F, G, and b, we can compute Eqs. (4.5) and (4.6) and hence ρ through numerical integration. These are given in Sec. 5.

5. NUMERICAL COMPUTATIONS

We provide some related computational results in this section. We have done a detailed computations taking $\alpha = 0.01, 0.05, 0.1$, and different distributions and different possible values of m. But, for brevity, we present only a few tables which will be sufficient to illustrate the behavior of our methodology. Table 1 provides the u_α and τ_α values for different α. For the Tables 2–3, we consider $\alpha = 0.05$, $m = 100$ and we take $F \equiv N(0, 1)$. If we take $G \equiv N(0, 1)$, we get $\theta = 0.5$ and $\sigma_1^2 = \sigma_2^2 = 0.0833$. Table 2 provides the values of ν and ν^* for different b for $F \equiv N(0, 1)$ and Case (1): $G \equiv N(0, 1)$, Case (2): $G \equiv N(0, 2^2)$ (where $\theta = 0.5$ also, and $\sigma_1^2 = 0.03205$ and $\sigma_2^2 = 0.14758$), Case (3): $G \equiv N(1, 2^2)$ (where $\theta = 0.67264$, $\sigma_1^2 = 0.02671$ $\sigma_1^2 = 0.12786$) and Case (4): $G \equiv N(3, 2^2)$ (where the values are respectively 0.91014, 0.00624 and 0.04145), respectively. In Table 3, the choices of G are respectively Case (5): $C(0,1)$, Case (6): $C(0, 2^2)$, Case (7): $C(1, 2^2)$ and Case (8): $C(3, 2^2)$, where $C(\mu, \sigma^2)$ is the Cauchy distribution with location parameter μ and scale parameter σ. For Case (5), $\theta = 0.5$, $\sigma_1^2 = 0.04556$, and $\sigma_2^2 = 0.12680$; for Case (6), the values are respectively 0.5, 0.01820, and 0.17257; for Case (7), these are 0.62876, 0.01455, and 0.16564, respectively; and for Case (8), these values are 0.80079, 0.00340, and 0.12885, respectively. We present the computations of ρ for (d, m) such that $b = 1.2$. For example, with $F \equiv N(0, 1)$ and $G \equiv N(0, 1)$, and for b=1.2, the

Table 1. The values of u_α and τ_α for different α.

α	u_α	τ_α
0.01	2.8073	2.5758
0.05	2.2415	1.9600
0.1	1.9600	1.6449

Table 2. The values of ν and ν^* for $F \equiv N(0, 1)$, and case (1): $G \equiv N(0, 1)$, case (2): $G \equiv N(0, 2^2)$ and case (3): $G \equiv N(1, 2^2)$, and case (4): $G \equiv N(3, 2^2)$.

	Case (1)		Case (2)		Case (3)		Case (4)	
b	ν	ν^*	ν	ν^*	ν	ν^*	ν	ν^*
0.2							2404	993
0.4					2489	856	162	118
0.5			834	447				
0.6			373	240	285	191	64	48
0.7	587	189						
0.8	190	101	155	110	127	92	35	26
0.9	107	66						
1.0	73	48	89	65	75	55	22	17
1.1	53	36						
1.2	41	29	58	44	50	37	15	12

Table 3. The values of ν and ν^* for $F \equiv N(0, 1)$, and case (5): $G \equiv C(0, 1)$, case (6): $G \equiv C(0, 2^2)$, case (7): $G \equiv C(1, 2^2)$, and case (8): $G \equiv C(3, 2^2)$.

	Case (5)		Case (6)		Case (7)		Case (8)	
b	ν	ν^*	ν	ν^*	ν	ν^*	ν	ν^*
0.2							2822	1837
0.3					4928	1867		
0.4			1265	735	958	612		
0.5	3022	650						
0.6	487	264	323	229	291	210	189	143
0.8	155	105	159	117	147	109	104	79
1.0	83	60	96	72	90	68	66	51
1.2	53	39	65	49	61	46	46	35

Table 4. Numerator and denominator of Eq. (3.9) with their ratio ρ with $F \equiv N(0, 1)$ and $b = 1.2$.

G	Numerator of Eq. (3.9)	Denominator of Eq. (3.9)	ρ
$N(0, 1)$	0.154782	0.205476	1.327519
$N(0, 2^2)$	0.158748	0.292472	1.842367
$N(1, 2^2)$	0.135739	0.249829	1.840510
$N(3, 2^2)$	0.042001	0.077023	1.833837
$C(0, 1)$	0.141369	0.261153	1.847314
$C(0, 2^2)$	0.179568	0.329987	1.837672
$C(1, 2^2)$	0.170934	0.313918	1.836483
$C(3, 2^2)$	0.129774	0.237871	1.832972

numerator and the denominator of Eq. (3.9) are respectively 0.20548 and 0.15478. In Table 4 we present the values of the numerator and denominator of Eq. (3.9) and also their ratio ρ. Most of the values of ρ are quite close. Quite expectedly, the required sample size of the proposed procedure is larger.

6. CONCLUDING REMARKS

We illustrate our proposed procedure by one real dataset. The data has been collected on two hourly basis from an aluminium factory in August 1999. Grindability is an important characteristic of aluminium powder which is basically the granual size. Aluminium hydrate is heated at some predetermined temperature for providing aluminium powder. Several process parameters were there. For our purpose we consider mineral dose, a particular chemical mixed with aluminium hydrate at the rate of kg/h. In the first phase of the study, the mineral dose was 6.66–6.72 kg/h. There were $m = 26$ observations. Then the second sample involved the mineral dose of 4.44–4.92 kg/h, and 75 observations were collected. Our objective is to find the value of $\hat{\nu}$ and $\hat{\nu}^*$ if the second sample were collected using our present sampling scheme to achieve a fixed-width confidence interval. At $\alpha = 0.05$, we observe that for $b = 0.3$ the value of $\hat{\nu}$ is 40 and $\hat{\nu}^*$ is 30. If we set $b = 0.4$, $\hat{\nu} = 22$, and $\hat{\nu}^* = 18$. Thus the method could be easily used in such a situation to plan the experiment and obtain the exact number of samples required to achieve a fixed-width confidence interval.

The proposed procedure can be interpreted as a fixed-width confidence interval for the stress-strength reliability, where X and Y are the stress on and strength of a particular component. Thus the application of the proposed technique can be extended to the reliability set up also. The procedure is useful only when we require that the difference between the parametric value and its estimator will not exceed a preassigned value for a long duration of time. Specifically in quality control application of delicate and expensive tools we may require such kind of protection.

APPENDIX

Result 1. Let $W = \{W(t),\ 0 \le t \le \mathbf{1}\}$ be a Wiener process distributed independently of Z having an $N(0, 1)$ distribution. Then, the d.f. $\psi(u)$ of the random variable $U = \sup\{|\sqrt{p}W(t) + \sqrt{1-p}Z|,\ 0 \le t \le \mathbf{1}\}$, where $p \in (0,1]$ can be obtained from the relation:

$$\psi(u) = \sum_{k=-\infty}^{\infty} [\Phi((4k+1)u) + \Phi((4k-3)u) - 2\Phi((4k-1)u)], \quad u > 0.$$

Proof. We have

$$\begin{aligned}\psi(u) &= P(U \le u)\\ &= P\Big(|\sqrt{p}W(t) + \sqrt{1-p}Z| \le u \text{ for all } t: 0 \le t \le 1\Big)\\ &= \int_{-\infty}^{\infty} P\Bigg(-\frac{u}{\sqrt{p}+} - z\sqrt{\frac{1-p}{p}} \le W(t) \le \frac{u}{\sqrt{p}}\\ &\quad -z\sqrt{\frac{1-p}{p}} \text{ for all } t: 0 \le t \le 1\Bigg)\, d\Phi(z),\end{aligned}$$

which, from Sen,[14] (p. 42) (see also Parthasarathy,[15] p. 230), can be written as

$$\begin{aligned}&\int_{-\infty}^{\infty}\Bigg[\sum_{k=-\infty}^{\infty} P\Bigg(-\frac{u}{\sqrt{p}} - z\sqrt{\frac{1-p}{p}} + 4k\frac{u}{\sqrt{p}} \le N \le \frac{u}{\sqrt{p}} - z\sqrt{\frac{1-p}{p}} + 4k\frac{u}{\sqrt{p}}\Bigg)\\ &\quad - \sum_{k=-\infty}^{\infty} P\Bigg(-\frac{3u}{\sqrt{p}} + z\sqrt{\frac{1-p}{p}} + 4k\frac{u}{\sqrt{p}} \le N \le -\frac{u}{\sqrt{p}}\\ &\quad -z\sqrt{\frac{1-p}{p}} + 4k\frac{u}{\sqrt{p}}\Bigg)\Bigg]\, d\Phi(z),\end{aligned} \tag{A.1}$$

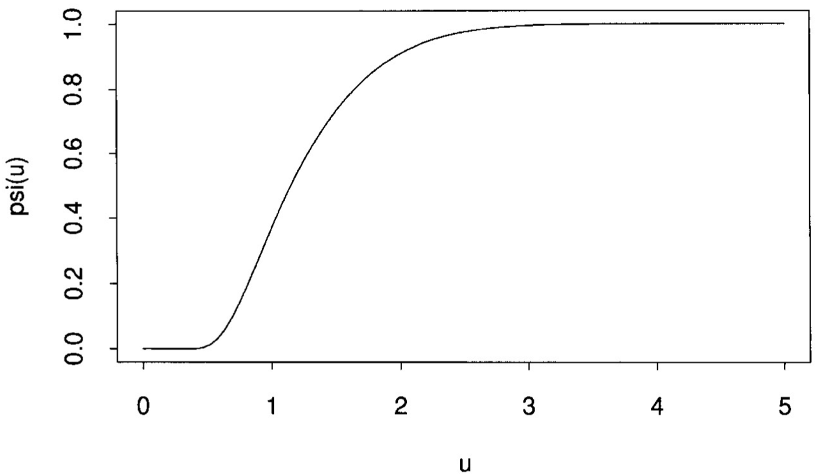

Figure 1. Plot of $\psi(u)$ against u.

where N is an N (0, 1) random variable. If we take the integral under the sums of (A.1), we get the required result using the relation

$$\int_{-\infty}^{\infty} \Phi(az+b)\, d\Phi(z) = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right), \qquad -\infty < a,\ b < \infty,$$

and making some routine calculations. Note that the above operation is valid since both the series converge uniformly in z for each fixed u. □

Remark. Figure 1 provides a plot of $\psi(u)$ against u. It is observed that $\psi(u)$ is almost equal to zero up to $u = 0.5$, then it rises sharply and reaches almost up to unity between $u = 2.5$ and 3.

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