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Flow of a power-law fluid film on an unsteady stretching surface

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Abstract

Flow of a thin liquid film of a power-law fluid caused by the unsteady stretching of a surface is investigated by using a similarity transformation. This transformation reduces the unsteady boundary-layer equations to a non-linear ordinary differential equation governed by a nondimensional unsteadiness parameter S. The effect of S on the film thickness is explored numerically for different values of the power-law index n. A physical explanation for the findings is also provided.

Keywords: Flow; Power-law fluid film; Unsteady stretching surface

1. Introduction

The prime aim in almost every extrusion application is to maintain the surface quality of the extrudate. All coating processes demand a smooth glossy surface to meet the requirements for best appearance and optimum service properties such as low friction, transparency and strength. Frequently, a textured surface is required, perhaps to prevent layers of thin films from adhering together or to produce a combination of high translucency with high diffusion in a light fitting. The problem of the coextrusion of thin surface layers needs special attention to gain some knowledge for controlling the coating product efficiently. Crane [1] and McCormack and Crane [2] studied the steady two-dimensional flow of a Newtonian fluid induced by the stretching of an elastic flat sheet in its own plane with a velocity varying linearly with the distance from a fixed point due to the application of a uniform stress. This problem was extended to a class of visco-elastic fluids known as second-order fluids by Rajagopal et al. [3] and they solved the boundary layer equations numerically. Later on Dandapat and Gupta [4] examined the same problem with heat transfer and obtained an exact analytical solution of the non-linear equation governing the self-similar flow, which is consistent with the numerical results in Ref. [3]. Recently Andersson and Dandapat [5] examined this stretching sheet problem initiated in Ref. [1] for fluids obeying the power-law model. It is interesting to note that in all these above studies the fluid on the stretching sheet is considered to extend to infinity when in reality the fluid will adhere to the sheet as a thin liquid film. Wang [6] first considered the finite film on the stretching surface and solved the unsteady Newtonian fluid flow problem induced by an accelerating stretching surface. Later on Usha and Sridharan [7] studied the analogous axisymmetric flow problem.

Needless to say, most of the paints or protective coatings applied on an extrudate are in general non-Newtonian fluids. In the present study we therefore examine the behaviour of a liquid film of an incompressible non-Newtonian fluid obeying the Ostwald-de-Waele power-law model due to unsteady stretching of the surface. It will be demonstrated that a similarity transformation exists, which exactly transforms the non-Newtonian boundary layer problem into an ordinary differential equation. Numerical solutions will elucidate the effects of non-Newtonian fluid behaviour.

2. Formulation of the problem

Consider the flow of a thin liquid film of an incompressible fluid obeying the power-law model. The flow arises due to the stretching of an elastic sheet parallel with the x-axis at $y = 0$. Two equal and opposite increasing forces are applied along the x-axis, so that the stretching of the wall is accelerated but the origin remains fixed. The basic equations governing the resulting boundary layer flow are, in usual notation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial y}\right) \tag{2}$$

where u and v are the velocity components along the x and y directions respectively, ρ is the density and τ_{xy} represents the shear stress. In the present problem, we have $\partial u/\partial y \leq 0$, this gives the shear stress as

$$\tau_{xy} = -K\left(-\frac{\partial u}{\partial y}\right)^n \tag{3}$$

where K is called the consistency coefficient and n is the power-law index. For the particular parameter value $n = 1$, one can retrieve the Newtonian fluid model with dynamic coefficient of viscosity K. As n deviates from unity, deviations from Newtonian behaviour occur. For example, $n < 1$ and $n > 1$ correspond to pseudoplastic and dilatant fluids respectively.

Combining Eqs. (20) and (3) we have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\left(\frac{K}{\rho}\right)\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^{n} \tag{4}$$

The flow is caused solely by the stretching of the elastic surface at $y=0$ with a velocity

$$U=\frac{cx}{(1-\alpha t)} \tag{5}$$

where c and α are positive constants both with dimensions time^{-1}. It should be pointed out here that the analysis is valid only for time $t<(1/\alpha)$. Further it should be noted that the end effects and gravity are negligible and that surface tension is sufficiently large such that the film surface remains smooth and stable throughout the motion. Eqs. (1) and (4) have the similarity solution

$$\begin{aligned} u&=cx(1-\alpha t)^{-1}f'(\eta)\\ v&=-\left(\frac{c^{1-2n}}{K/\rho}\right)^{-1/(1+n)}x^{(n-1)/(n+1)}(1-\alpha t)^{(1-2n)/(1+n)}\left[\left(\frac{2n}{n+1}\right)f(\eta)+\left(\frac{1-n}{1+n}\right)\eta f'(\eta)\right] \end{aligned} \tag{6}$$

where the similarity variable η is given by

$$\eta=\left(\frac{c^{2-n}}{K/\rho}\right)^{1/(n+1)}x^{(1-n)/(1+n)}(1-\alpha t)^{(n-2)/(1+n)}y \tag{7}$$

Clearly, the u and v values given in Eq. (6) satisfy the mass continuity equation (1) and after substitution in Eq. (4) we get

$$n(-f'')^{n-1}f'''-(f')^2+\left(\frac{2n}{n+1}\right)ff''-S\left[f'+\left(\frac{2-n}{n+1}\right)\eta f''\right]=0 \tag{8}$$

where S ($\equiv\alpha/c$) is a dimensionless measure of unsteadiness and a prime denotes differentiation with respect to η.

The boundary conditions are: no-slip on the stretching boundary ($\eta=0$) and vanishing of the shear stress at the free boundary ($y=h(x,t)$ or $\eta=\beta$) along with the constraint that the motion must satisfy the kinematic condition

$$v=u\frac{\partial h}{\partial x}+\frac{\partial h}{\partial t} \tag{9}$$

at the free surface. In dimensionless form these boundary conditions become

$$f'(0)=1,\quad f(0)=0 \tag{10}$$

$$f''(\beta)=0 \tag{11}$$

and

$$\left(\frac{n}{n+1}\right)f(\beta)+\left(\frac{1-n}{n+1}\right)\beta f'(\beta)=\frac{S}{2}\left(\frac{2-n}{n+1}\right)\beta \tag{12}$$

Clearly, for $n = 1$, the above problem simplifies considerably and we revert to the study of Wang [6].

3. Numerical results

The highly non-linear differential equation (8) subject to the boundary conditions (10)–(12) constitutes a two-point boundary-value problem, which can be solved by a standard shooting method. The third-order ODE is formulated as a set of three first-order equations and integrated as an initial value problem using the ODEX routine [8]. Trial values of $f''(0)$ are adjusted iteratively by Powell's method using the DNSQE routine [9] until the outer condition (11) is satisfied to within a tolerance error of 10^{-10} for a tentative position of the free boundary β. The numerical solution thus obtained generally does not satisfy the kinematic free surface condition (12). The estimated value of β is therefore systematically adjusted until Eq. (12) is satisfied to within 10^{-10}.

Converged numerical results were obtained for a pseudo-plastic fluid ($n = 0.8$) and a dilatant fluid ($n = 1.2$) as well as for the Newtonian case $n = 1.0$ over a wide range of the unsteadiness parameter S. Velocity similarity profiles $f'(\eta)$ are shown in Fig. 1 for two representative values of S. The variation of the dimensionless film thickness β with S is presented in Fig. 2, while dimensionless forms of the free-surface velocity $f'(\beta)$ and the velocity gradient $f''(0)$ on the stretching sheet are displayed in Figs. 3 and 4 respectively. The velocity gradient $f''(0)$ is related to the dimensionless skin-friction coefficient C_f according to its definition

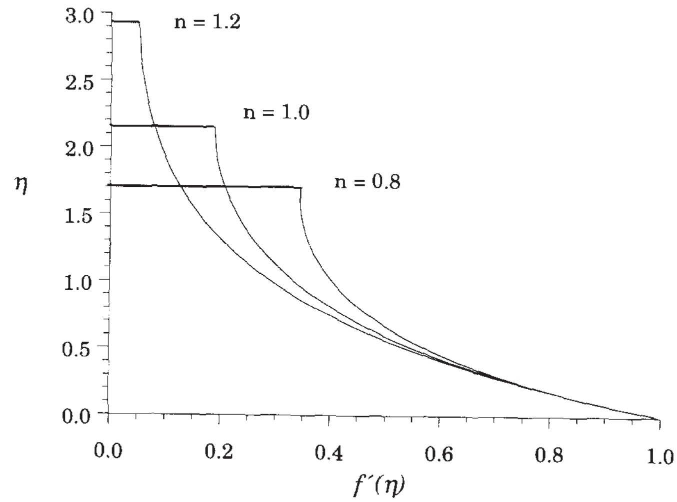
$$C_f \equiv \frac{-2\tau_{xy}(0)}{\rho U^2} = 2[-f''(0)]^n \cdot \mathrm{Re}_x^{-1/(n+1)} \tag{13}$$

where $\mathrm{Re}_x = \rho U^{2-n} x^n / K$ is a local Reynolds number based on the sheet velocity U defined in Eq. (5).

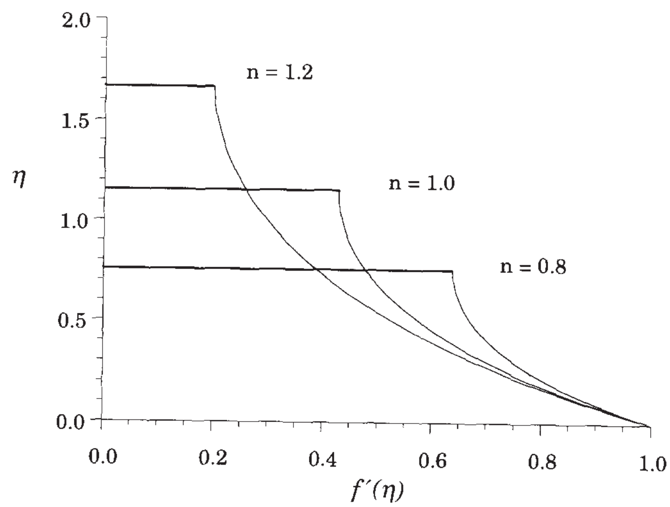
4. Discussion and conclusions

Let us first recall the major findings of Wang [6] for the particular parameter value $n = 1$ of a Newtonian fluid. Wang observed that, for positive values of the unsteadiness parameter, solutions exist only for $0 \leq S \leq 2$. Moreover, when S tends to zero the solution approaches the analytical solution of Crane [1], which corresponds to an infinitely thick layer of fluid, while the limiting solution as $S \to 2.0$ represents a liquid film with infinitesimal thickness. It was also observed that the film thickness β decreased monotonically with increasing S, whereas the magnitude of the wall gradient $-f''(0)$ increased with S until it reached a maximum of 1.283 near $S = 1.11$ and thereafter rapidly decreased to zero as S approached 2.0. The numerical results presented by Wang [6] were supplemented by asymptotic solutions for thin and thick films, i.e. for $S \approx 2$ and $S \approx 0$ respectively. These asymptotes are shown as broken lines in Figs. 2 and 4 in order to demonstrate the accuracy of the present computations for $n = 1$.

The velocity profiles presented in Fig. 1 show that even moderate deviations from Newtonian rheology ($n = 1$) have a significant influence on the variation of the horizontal velocity



(a)



(b)

Fig. 1. Similarity velocity profiles $f'(\eta)$ for different values of the power-law index n: (a) $S = 0.8$; (b) $S = 1.2$.

component u across the liquid film. For a given value of the unsteadiness parameter, the pseudo-plastic film ($n = 0.8$) is thinner and exhibits a greater surface velocity than a Newtonian film, while the opposite behaviour is observed for the dilatant fluid ($n = 1.2$). Pseudo-plastic fluids are commonly known as shear-thinning fluids since the viscosity function becomes progressively reduced with increasing shear rates, whereas dilatant substances have viscosity functions that increase with the shear rate and thus become progressively more viscous and thicken with increasing rates of shear. It is therefore not surprising to observe that the shear-thinning pseudo-plastics are more amenable to flow nearly as an inviscid layer on top of the stretching sheet than are the shear-thickening or dilatant fluids. For $S = 1.2$, for example, the velocity varies by 36.5% across the film for $n = 0.8$ and by as much as 80.2% for $n = 1.2$; cf. Fig. 1b.

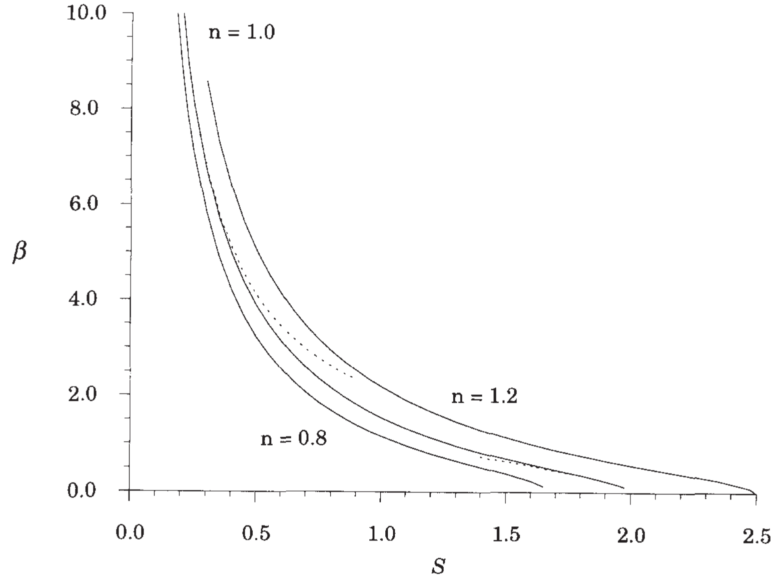


Fig. 2. Film thickness β vs. unsteadiness parameter S. The broken lines denote asymptotic solutions for $n = 1$ by Wang (Eqs. (19) and (33) in Ref. [6]).

The variations with S of the dimensionless film thickness β and the surface velocity $f'(\beta)$ in Figs. 2 and 3 respectively, show the same tendency for all values of S, but the effect of the power-law index is more pronounced for the higher S values. For a particular value S_0 of S, the film became infinitely thin ($\beta \to 0$) and the surface velocity $f'(\beta)$ approached one. This critical

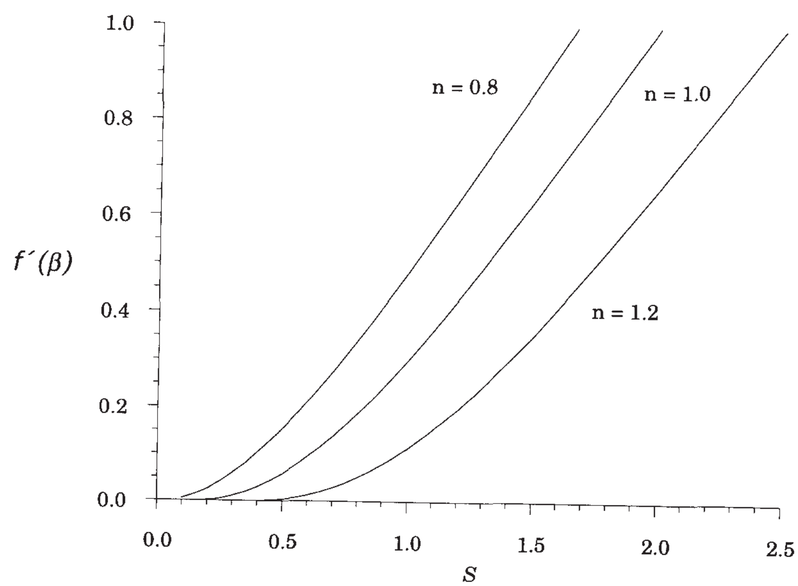


Fig. 3. Free-surface velocity $f'(\beta)$ vs. unsteadiness parameter S.

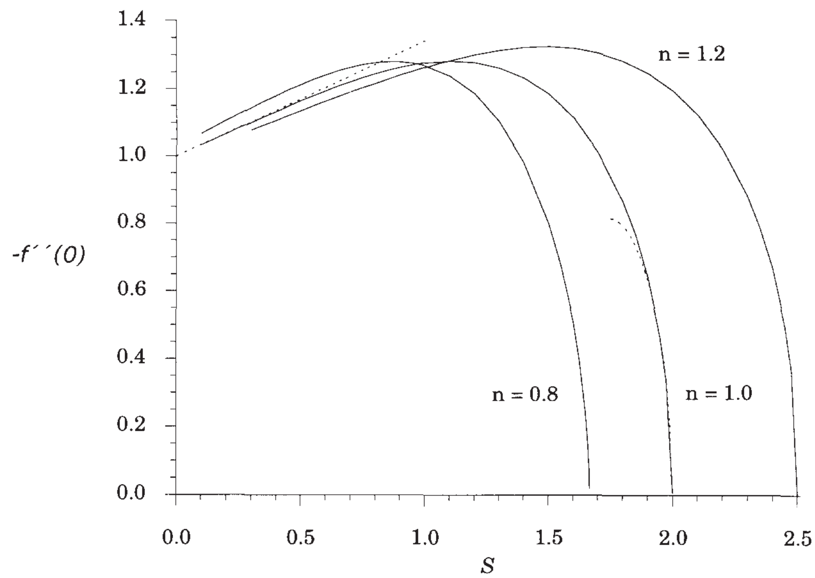


Fig. 4. Surface shear stress $-f''(0)$ vs. unsteadiness parameter S. The broken lines denote asymptotic solutions for $n = 1$ by Wang (Eqs. (21) and (34) in Ref. [6]).

parameter value, above which no solutions could be obtained, turned out to be strongly fluid dependent. While Wang [6] found $S_0 = 2.0$ for a Newtonian fluid, the present calculations show that S_0 is about 1.67 for $n = 0.8$ and 2.50 for $n = 1.2$. Admittedly, the computations are difficult to perform in the extreme limits when $\beta \to \infty$ and $\beta \to 0$, and the reported values of S_0 are meant as estimates; cf. also Fig. 2.

The variation with S of the velocity gradient at the stretching surface $f''(0)$ in Fig. 4 shows that the same trend as reported by Wang [6] for $n = 1$ is maintained for non-Newtonian films. However, the maximum of $-f''(0)$ appears to increase slightly with n, and so do the S values for which the maximum is reached. It is interesting to observe from Fig. 4 that the trend of the variation of $f''(0)$ with n for given S changes from $S = 0.8$ to $S = 1.2$. For $S = 0.8$ the magnitude of the surface gradient is greater for the pseudo-plastic fluid than for the dilatant fluid, whereas the opposite holds for $S = 1.2$. This observation indicates that the velocity profiles for $S = 0.8$ in Fig. 1a intersect each other close to the η-axis.

Finally, it should be emphasized that although the dimensionless film thickness β is a constant (which depends on S and n), the actual thickness of the liquid layer

$$h(x, t) = \beta \left[\frac{K/\rho}{c^{2-n}} \right]^{1/(n+1)} \cdot x^{(n-1)/(n+1)} \cdot (1 - \alpha t)^{(2-n)/(n+1)} \tag{14}$$

is a function of time and position. In the Newtonian case $n = 1$, however, h becomes a function only of time, whereas for non-Newtonian films the thickness decreases with x for pseudo-plastics ($n < 1$) while the film thickens in the streamwise direction for dilatant fluids ($n > 1$).

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