22 April 2002

PHYSICS LETTERS A

Physics Letters A 296 (2002) 187–191

Gazeau–Klauder coherent state for the Morse potential and some of its properties

B. Roy, P. Roy *

Physics and Applied Mathematics Unit, Indian Statistical Institute, Calcutta 700035, India

Received 8 January 2002; accepted 28 January 2002

Communicated by V.M. Agranovich

Abstract

Using the Gazeau–Klauder formalism we construct coherent state corresponding to the Morse potential. Some properties of this coherent state have also been examined.

1. Introduction

Coherent states of the harmonic oscillator were introduced as the most classical quantum state and such states have many interesting properties as well as applications [1]. However, the concept of coherent state is not confined to the harmonic oscillator only but it has been generalized to various systems, for example, those with Lie algebraic $su(1,1)$ or $su(2)$ symmetry [2] or even nonlinear algebraic symmetry [3–5].

Coherent states are generally constructed by (i) using the displacement operator technique or defining them as (ii) lowering operator eigenstates or (iii) minimum uncertainty states. However, even when such operators do not exist different approaches have been utilized to construct coherent states corresponding to various quantum mechanical potentials [6–10]. Indeed it has been shown that in principle coherent states can be constructed corresponding to arbitrary potentials [11]. However, in practise it is not always easy to obtain a closed form expressions for coherent states for arbitrary potentials. In a recent paper Gazeau and Klauder determined a set of criteria which a coherent state should satisfy [12] and subsequently a number of exactly solvable nonlinear problems has been treated using the Gazeau–Klauder approach [13–15]. In the present Letter we shall employ the Gazeau–Klauder formalism to construct coherent states of the Morse potential and examine some of their properties.

2. Gazeau–Klauder coherent state of the Morse potential

According to Gazeau and Klauder [12] a two parameter set of coherent states $|J,\gamma\rangle$, $J \geqslant 0$ and $-\infty < \gamma < +\infty$ should satisfy the following requirements:

(1) Continuity of labelling: $(J',\gamma') \to (J,\gamma) \Rightarrow |J',\gamma'\rangle \to |J,\gamma\rangle$;
(2) Resolution of unity: $\mathbf{1} = \int |J,\gamma\rangle\langle J,\gamma|\, d\mu(J,\gamma)$;

* Corresponding author.
E-mail addresses: barnana@isical.ac.in (B. Roy), pinaki@isical.ac.in (P. Roy).

(3) Temporal stability: $e^{-iHt}|J,\gamma\rangle = |J,\gamma+\omega t\rangle$;
(4) Action identity: $\langle J,\gamma|H|J,\gamma\rangle = \omega J$.

Let us now consider the case when the Hamiltonian possesses only a discrete spectrum:

$$H|n\rangle = E_n|n\rangle = \omega e_n|n\rangle, \quad 0 = E_0 < E_1 < E_2 < \cdots. \tag{1}$$

Then the GK coherent state is defined as

$$|J,\gamma\rangle = N\sum_{n=0}^{\infty}\frac{J^{n/2}\exp(-i\gamma e_n)}{\sqrt{\rho_n}}|n\rangle, \tag{2}$$

where N is a normalization constant given by:

$$N = \left[\sum_{n=0}^{\infty}\frac{J^n}{\rho_n}\right]^{-1/2}, \quad 0 < J < R = \lim_{n\to\infty}(\rho_n)^{1/n}. \tag{3}$$

In (3) ρ_n denotes the moments of a probability distribution $\rho(x)$ and are given by [12]

$$\rho_n = \int_0^R x^n\rho(x)\,dx = \prod_{i=1}^{n} e_i, \qquad \rho_0 = 1. \tag{4}$$

It can be shown that the coherent state in (2) satisfy all the four criteria stated above.

Let us now apply the above formalism to construct coherent state of the Morse potential. The Morse potential and the corresponding energy spectrum are given by

$$V(x) = (M+1)^2 - (2M+3)e^{-x} + e^{-2x}, \tag{5}$$

$$E_n = (M+1)^2 - (M+1-n)^2, \quad n = 0,1,2,\ldots,<(M+1), \tag{6}$$

where for the sake of convenience we have taken $M =$ a positive integer. Note that unlike the harmonic oscillator (and similar potentials) the spectrum of the Morse potential is finite dimensional. Thus the Morse potential coherent state is given by

$$|\psi\rangle_{\text{Morse}} = N\sum_{n=0}^{M}\frac{J^{n/2}e^{-i\gamma E_n}}{\sqrt{\rho_n}}|n\rangle, \tag{7}$$

where the normalization constant N and the moments ρ_n are given, respectively, by

$$N = \left[\sum_{n=0}^{M}\frac{J^n}{\rho_n}\right]^{-1/2}, \tag{8}$$

$$\rho_0 = 1, \qquad \rho_n = \frac{\Gamma(n+1)\Gamma(2M+2)}{\Gamma(2M+2-n)}. \tag{9}$$

We note that the series (8) determining the normalization constant N is a finite series and thus it exists for all values of J. Next it is necessary to find a probability distribution which has only a finite number of moments given by (9). To this end we first note the following result [16]:

$$\int_0^\infty x^\mu J_\nu(ax)\,dx = 2^\mu a^{-\mu-1}\frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)}, \quad -\operatorname{Re}\nu - 1 < \operatorname{Re}\mu < \frac{1}{2}. \tag{10}$$

Now using (10) the probability distribution is found to be

$$\rho(x) = \Gamma(2M+2)x^{-(M+1)}J_{2M+2}\left(2\sqrt{x}\right). \tag{11}$$

It can be easily checked that the probability distribution (11) has exactly $(M+1)$ moments given by (9). Also the distribution (11) cannot have more than $(M+1)$ moments since in that case the conditions in (10) and (5) will become incompatible. Now it can be shown that the coherent state (7) satisfies the criteria (1)–(3) mentioned earlier. The fourth criteria, i.e., the action identity is also satisfied albeit in a slightly modified way: ${}_{\text{Morse}}\langle\psi|H|\psi\rangle_{\text{Morse}} = f(J)$ where f is a certain function of J. We note that this modification is not characteristic of the Morse potential coherent state only but of all coherent states corresponding to systems with a finite dimensional energy spectrum. In the next section we shall examine some properties of the Morse coherent state (7).

3. Properties of the Morse coherent state

In this section we shall examine some properties of the Morse coherent state (7). As mentioned earlier coherent states can be constructed in different ways. For example, they can be defined as minimum uncertainty states [11]. In contrast the GK coherent states may or may not be minimum uncertainty states. So it is interesting to examine whether or not the Morse coherent states are minimum uncertainty states with respect to some symmetry algebra. In order to do so it is necessary to determine the raising and lowering operators.

For the Morse potential such operators exist but usually depend on the quantum number n [11]. In other words such operators connect two consecutive states. In order to circumvent this difficulty we define raising and lowering operators with the help of projection operators suitable for treating finite dimensional systems [17,18]. Thus we define

$$A=\sum_{n=1}^{M}\chi_n|n-1\rangle\langle n|,$$
$$A^\dagger=\sum_{n=1}^{M}\chi_n^*|n\rangle\langle n-1|. \tag{12}$$

In order that $A^\dagger$ and A can be interpreted as raising and lowering operator, respectively, it is necessary that χ_n should satisfy the condition $\chi_0=\chi_{M+1}=0$. This will ensure that a state beyond $|M\rangle$ cannot be created by application of the raising operator. Clearly there can be various choices for the function χ_n (and consequently for $A^\dagger$ and A) satisfying the above condition. Here we choose a simple form of χ_n:

$$\chi_n=\sqrt{n(M-n+1)}\,e^{i\gamma(E_n-E_{n-1})}. \tag{13}$$

The action of the raising and lowering operators $A^\dagger$ and A act on the eigenstates in the following way:

$$A|n\rangle=\chi_n|n-1\rangle,\qquad A^\dagger|n\rangle=\chi_{n+1}^*|n+1\rangle,$$
$$A|0\rangle=A^\dagger|M\rangle=0, \tag{14}$$

and it can be shown that together with an operator A_0 (defined below) they satisfy the $su(2)$ algebra:

$$\left[A^\dagger,A\right]=2A_0=(2N-M),\qquad \left[A_0,A^\dagger\right]=A^\dagger,$$
$$[A_0,A]=-A,\qquad N=\sum_{n=1}^{M}n|n\rangle\langle n|. \tag{15}$$

We note that the algebra (15) would be different for a different choice of χ_n. In the present case the Morse Hamiltonian is a nonlinear function of the generator A_0:

$$H=M(A_0+M)-\frac{(A_0+M)^2}{4}. \tag{16}$$

Let us now introduce the following Hermitian operators:

$$X=\frac{A+A^\dagger}{2},\qquad Y=\frac{A-A^\dagger}{2i}. \tag{17}$$

The uncertainty relation between X and Y reads

$$\left\langle(\Delta X)^2\right\rangle\left\langle(\Delta Y)^2\right\rangle\geqslant\frac{1}{4}\left|\left\langle[X,Y]\right\rangle\right|^2,$$
$$\left\langle(\Delta X)^2\right\rangle=\left\langle X^2\right\rangle-\langle X\rangle^2. \tag{18}$$

The states which saturate the uncertainty relation (18) are called minimum uncertainty states or more generally intelligent states. We shall now examine the behaviour of Morse coherent states with respect to the inequality (18). In order to do this we consider the functional $F(J)$ defined by

$$F(J)=\left\langle(\Delta X)^2\right\rangle\left\langle(\Delta Y)^2\right\rangle-\frac{1}{4}\left|\left\langle[X,Y]\right\rangle\right|^2. \tag{19}$$

Thus from (18) it follows that $F(J)\geqslant 0$ and the equality sign will indicate that the coherent state is a minimum uncertainty or intelligent state. We now compute $F(J)$ with the help of the following expectation values:

$$\langle A\rangle=\left\langle A^\dagger\right\rangle$$
$$=N^2\sum_{n=0}^{M-1}\frac{J^{n+1/2}}{\rho_n}\sqrt{\frac{M-n}{2M-n+1}},$$
$$\left\langle A^2\right\rangle=\left\langle A^{\dagger 2}\right\rangle$$
$$=N^2\sum_{n=0}^{M-2}\frac{J^{n+1}}{\rho_n}\sqrt{\frac{(M-n)(M-n-1)}{(2M-n+1)(2M-n)}},$$
$$\left\langle A^\dagger A\right\rangle=N^2\sum_{n=0}^{M-1}\frac{J^{n+1}}{\rho_n}\frac{(M-n)}{(2M-n+1)},$$
$$\left\langle AA^\dagger\right\rangle=N^2\sum_{n=0}^{M-1}\frac{J^n}{\rho_n}(n+1)(M-n), \tag{20}$$

and the results are plotted in Fig. 1 against J for $M=8$ and $M=10$. At this point we note that $F(J)\neq 0$ for the coherent state (7) for any value of the parameters so that it is not an *exact* minimum uncertainty state. However, if $F(J)$ is small enough then the coherent state may be regarded as an *approximate* minimum uncertainty or intelligent state. From Fig. 1 we find that for both $M=8$ and $M=10$, $F(J)\approx 0$ for a relatively low range of values of J (depending, of course, on the order of smallness we accept as zero). Thus in this range of J the coherent state (7) is an *approximate* minimum uncertainty state. Also $F(J)$ increases with J and for larger values of J,

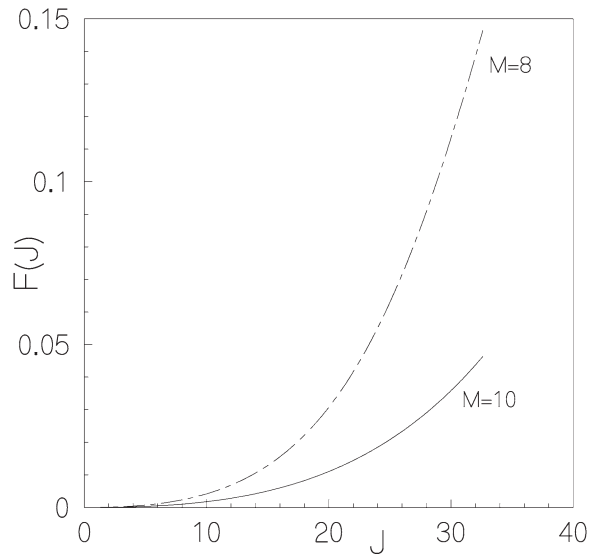

Fig. 1. Graph of $F(J)$ against J for $M=10$ (solid curve) and $M=8$ (broken curve).

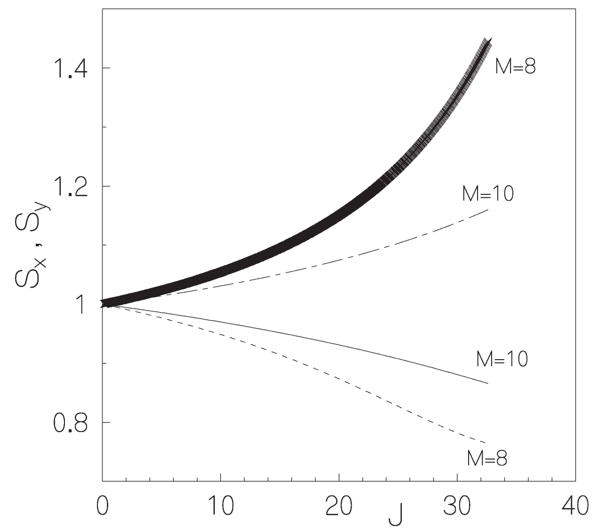

Fig. 2. Graph of S_x for $M=10$ (solid curve), $M=8$ (dotted curve) and S_y for $M=10$ (broken curve), $M=8$ (thick curve) against J.

it is appreciably different from zero. Thus for larger values of J, (7) cannot be regarded as an *approximate* minimum uncertainty state. It is also interesting to observe that $F(J)$ is larger for $M=8$ compared to $M=10$. In other words $F(J)$ can be made smaller by increasing M.

As mentioned before $F(J)>0$ and it is an increasing function of J. This raises the possibility that there might be squeezing for certain values of J and M. In order to examine such a possibility we introduce squeezing parameters S_x and S_y:

$$S_x = \frac{2\langle(\Delta X)^2\rangle}{|\langle[X,Y]\rangle|}, \qquad S_y = \frac{2\langle(\Delta Y)^2\rangle}{|\langle[X,Y]\rangle|}. \tag{21}$$

Then the state is said to be squeezed if $S_x<1$ or $S_y<1$ and the smaller the value of S_x (or S_y) the larger is the squeezing. We have evaluated S_x and S_y using (20) and the results are plotted in Fig. 2 for different values of M. From Fig. 2 we find that S_x is a decreasing function of J and S_y has an increasing trend. We also find that $S_x<1$ while $S_y>1$ over the same range. From the Fig. 2 we also find that S_x is smaller for $M=8$ than for $M=10$ while the converse is true for S_y. We have also verified that S_x (S_y) becomes smaller (larger) if M is reduced and/or J is increased. Thus we conclude that the Morse coherent state exhibits squeezing and the squeezing can be increased or decreased by tuning the parameters.

4. Conclusion

Here we have constructed coherent state of the Morse potential following the Gazeau–Klauder formalism. Some properties of this coherent state have also been examined. In particular, it has been shown that the Morse coherent state possesses both classical like feature *(approximate* minimum uncertainty property) as well as nonclassical feature (squeezing property) in different parameter regimes. Finally, we would like to point out that the properties studied here are not characteristic of the coherent state (7) only but are shared by GK coherent states of several other potentials such as Rosen–Morse potential, soliton potential etc. This is due to the fact that although these potentials have different wave functions they have the same spectral properties as the Morse potential.

References

[1] W.M. Zhang, D.H. Feng, R. Gilmore, Rev. Mod. Phys. 62 (1990) 867.
[2] A.M. Perelomov, Generalised Coherent States and Their Applications, Springer, Berlin, 1996.

[3] C.D.J. Fernandez, L.M. Nieto, O.R. Oritz, J. Phys. A 28 (1995) 2693.
[4] V.I. Manko, G. Marmo, E.C.G. Sudarshan, F. Zaccaria, Phys. Scr. 55 (1997) 528.
[5] G. Junker, P. Roy, Phys. Lett. A 227 (1999) 113.
[6] M.M. Nieto, L.M. Simmons, V.P. Gutschick, Phys. Rev. D 23 (1981) 927, and references therein.
[7] I.L. Cooper, J. Phys. A 25 (1992) 1671.
[8] B. Molnar, M.G. Benedict, Phys. Rev. A 60 (1999) R1737.
[9] A.B. Balantekin, M.A.C. Rebeiro, A.N.F. Alexio, J. Phys. A 32 (1999) 2785.
[10] B.F. Samsonov, J. Phys. A 33 (2000) 591.
[11] M.M. Nieto, L.M. Simmons, Phys. Rev. Lett. 41 (1978) 207.
[12] J.-P. Gazeau, J.R. Klauder, J. Phys. A 32 (1999) 123.
[13] J.-P. Antoine, J.-P. Gazeau, P. Monceau, J.R. Klauder, K.A. Penson, J. Math. Phys. 42 (2001) 2349.
[14] J.M. Hollingworth et al., J. Phys. A 34 (2001) 9463.
[15] A.H. El Kinani, M. Daoud, Phys. Lett. A 283 (2001) 291.
[16] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals and Series, Academic Press, 1980.
[17] S.M. Barnett, D.T. Pegg, J. Mod. Phys. 39 (1989) 7.
[18] P. Roy, B. Roy, J. Phys. A 31 (1998) 1307.