

**ISOSPECTRAL PARTNERS OF A COMPLEX  
 $\mathcal{PT}$ -INVARIANT POTENTIAL**

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## Abstract

We construct isospectral partner potentials of a complex  $\mathcal{PT}$ -invariant potential, viz.,  $V(x) = -V_1 \operatorname{sech}^2 x - iV_2 \operatorname{sech} x \tanh x$  using Darboux's method. One set of isospectral potentials are obtained which can be termed 'Satellite potentials', in the sense that they are of the same form as the original potential. In a particular case, the supersymmetric partner potential has the same spectrum, including the zero energy ground state, a fact which cannot occur in conventional supersymmetric quantum mechanics with real potential. An explicit example of a non-trivial set of isospectral potential is also obtained.

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**Key words** :  $\mathcal{PT}$ - symmetry, non-Hermitian Hamiltonian, isospectral potential, Darboux's method, supersymmetry.

# I. Introduction

Quantum systems characterized by non-Hermitian Hamiltonians have been studied from time to time, because of their applications in scattering problems [1]. For this reason, complex potentials were also termed as *optical* or *average nuclear potentials*. The interest in this field has been revived fairly recently after it was conjectured by Basis, Bender and Boettcher, and others, that Hermiticity of the Hamiltonian is not essential for a real spectrum [2,3]. Several non-Hermitian  $\mathcal{PT}$ -invariant complex potentials have been found to possess a real spectrum, though the corresponding Hamiltonians are non-Hermitian [4,5]. Moreover,  $\mathcal{PT}$ -invariant models admit some of the properties of the usual Hermitian ones, *viz.*, supersymmetry, potential algebra, quasi-solvability, etc. [6-17]. Such non-Hermitian  $\mathcal{PT}$ -invariant Hamiltonians have found applications in many areas of theoretical physics — nuclear physics, field theories when studying Lee-Yang zeros, localization-delocalization transitions in superconductors, theoretical description of defraction of atoms by standing light waves, and also the study of solitons on a complex Toda lattice [7].

In this study, our aim is to construct new complex potentials via the Darboux's method [7-10], for which the corresponding eigenvalue problem can be solved exactly. The constructed potentials are not necessarily  $\mathcal{PT}$ -invariant, but still give rise to a real and discrete spectrum, provided the original potential admits real energies only.

The organization of the present note is as follows. To make it self-contained, we give a brief review of  $\mathcal{PT}$ -symmetry in Section II. In Section III, we give an outline of the Darboux's method for constructing isospectral partner potentials of a given known  $\mathcal{PT}$ -symmetric potential. In Section IV, we illustrate our approach with the help of an explicit example, *viz.*, the one-dimensional  $\mathcal{PT}$ -invariant potential

$$V(x) = -V_1 \operatorname{sech}^2 x - iV_2 \operatorname{sech} x \tanh x \quad V_1 > 0 \quad (1)$$

Section V is kept for conclusions and discussions.

## II. $\mathcal{PT}$ -symmetry

The Hamiltonian  $\mathcal{H}$  for a particle of mass  $m$ , in a complex potential  $V(x) = V_R(x) + iV_I(x)$  is given by

$$\mathcal{H} = -\frac{1}{2m} \frac{d^2}{dx^2} + V(x) \quad (2)$$

$\mathcal{H}$  is said to be  $\mathcal{PT}$ -symmetric when

$$\mathcal{PT}\mathcal{H} = \mathcal{H}\mathcal{PT} \quad (3)$$

Here  $\mathcal{P}$  is the *Parity operator* acting as spatial reflection, and  $\mathcal{T}$  stands for *Time Reversal*, acting as the complex conjugation operator. Their action on the position and momentum operators are given by :

$$\begin{aligned}\mathcal{P} &: x \rightarrow -x, \quad p \rightarrow -p \\ \mathcal{T} &: x \rightarrow x, \quad p \rightarrow -p, \quad i \rightarrow -i\end{aligned}$$

Hence, in explicit form, the condition for a potential to be  $\mathcal{PT}$ -symmetric is

$$V^*(-x) = V(x) \tag{4}$$

The commutation relation

$$[x, p] = i\hbar \tag{5}$$

remains invariant under  $\mathcal{PT}$  for both real as well as complex  $x$  and  $p$ .

It is worth mentioning here that though the Hermiticity of the Hamiltonian may be replaced by the weaker condition of  $\mathcal{PT}$ -symmetry, the latter is not sufficient for the reality of the spectrum. Various authors have studied several one-dimensional non-Hermitian  $\mathcal{PT}$ -symmetric models and shown that such Hamiltonians exhibit 2 types of behaviour —

- (i) In the unbroken  $\mathcal{PT}$ -symmetry phase, the eigenfunctions of  $\mathcal{H}$  are also eigenfunctions of  $\mathcal{PT}$ , and the energy spectrum is real and discrete.
- (ii) In spontaneous breakdown of  $\mathcal{PT}$ -symmetry, though the potential retains  $\mathcal{PT}$ -symmetry, the corresponding wavefunctions do not, and the energy eigenvalues exist as complex conjugate pairs.

### III. Darboux's method

The Darboux's method [7-10] relates the spectral properties of a pair of standard Schrödinger Hamiltonians

$$H_{\pm} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_{\pm}(x) \tag{6}$$

We assume that the spectral properties of one of these Hamiltonians, say  $H_+$ , is exactly known. Thus

$$H_+ \phi_n(x) = E_n \phi_n(x) \tag{7}$$

where the eigenvalues  $E_n$  and the corresponding eigenfunctions  $\phi_n(x)$  are known explicitly. The spectrum is assumed to be discrete such that  $E_0 < E_1 < E_2 < \dots$ .

Let there exist a linear operator  $A$ , such that it obeys an intertwining relationship

$$AH_+ = H_-A \tag{8}$$

Hence the functions  $\psi_n = A\phi_n$  are eigenfunctions of  $H_-$  with the same eigenvalues  $E_n$

$$H_- \psi_n(x) = E_n \psi_n(x) \tag{9}$$

A general form for an intertwining operator  $A$  obeying (3) may be given by [7,8]

$$A = \sum_{k=0}^N f_k(x) \frac{d^k}{dx^k} \quad (10)$$

where  $f_k (k = 0, 1, 2, \dots, N - 1)$  are at least twice differential functions and  $f_N$  is an arbitrary constant.

For simplicity of calculations, we work in units  $\hbar = 2m = 1$ .

We take the simplest non-trivial choice for  $A(N = 1)$

$$A = -\frac{d}{dx} + f(x) \quad (11)$$

Putting (6) in (3) yields

$$[V_- - V_+ + 2f'] \frac{d}{dx} - [(V_- - V_+) f + V'_+ - f''] \cdot 1 = 0 \quad (12)$$

where prime denotes differentiation with respect to  $x$ . This relationship (12) puts certain restrictions on the functions  $V_+(x)$ ,  $V_-(x)$ , and  $f(x)$ , viz.,

$$V_-(x) = V_+(x) - 2f'(x) \quad (13)$$

$$[V_-(x) - V_+(x)] f(x) + V'_+(x) - f''(x) = 0 \quad (14)$$

Substituting (14) in (13), and integrating, we obtain

$$f^2(x) + f'(x) - V_+(x) = -\epsilon \quad (15)$$

where  $\epsilon$  is an arbitrary integration constant, in general complex.

Putting

$$f(x) = \frac{u'(x)}{u(x)} \quad (16)$$

(15) can be cast in the form of Schrödinger-like equation

$$-\frac{d^2 u(x)}{dx^2} + V_+ u(x) = \epsilon u(x) \quad (17)$$

$\epsilon$  is sometimes called the *factorization energy*. It is worth noting here that  $u(x)$  need not be square-integrable. So we are not restricted to normalizable solutions of (17). However, for  $A$  to be well defined,  $u$  must not have any zeroes on the real line. For this, it requires the condition  $\epsilon < E_0$ .

In terms of the function  $f(x)$ , the two potentials are expressed as

$$V_{\pm}(x) = f^2(x) \pm f'(x) + \epsilon \quad (18)$$

At this point it is obvious that for  $\epsilon = 0$ , the two potentials  $V_{\pm}(x)$  are the supersymmetric (SUSY) partner potentials, and  $H_{\pm}$  are the SUSY partner Hamiltonians. From (18), the partner Hamiltonians may be expressed in terms of the linear operator  $A$  as,

$$H_+ = A^+ A + \epsilon \quad (19)$$

$$H_- = A A^+ + \epsilon \quad (20)$$

We shall apply the above approach to construct a new complex potential  $V_-(x)$ , which is isospectral with a known  $\mathcal{PT}$ -symmetric potential  $V_+(x)$  with real eigenvalues, obeying the relationship

$$V_-(x) = 2 \left\{ \frac{u'(x)}{u(x)} \right\}^2 - V_+(x) + 2\epsilon \quad (21)$$

Though  $\mathcal{PT}$ -symmetry may be either restored or broken in the new potential, the partner Hamiltonian  $H_-$  has the same eigenvalues as  $H_+$  (with the possible exception of the ground state). The corresponding eigenfunctions are given by

$$\psi_n(x) = c_n \left\{ -\frac{d\phi_n(x)}{dx} + \frac{u'(x)}{u(x)} \phi_n(x) \right\} \quad (22)$$

Here  $c_n$  is the normalization constant, given by

$$\begin{aligned} |c_n|^{-2} &= \langle A\phi_n | A\phi_n \rangle \\ &= E_n - \epsilon \end{aligned} \quad (23)$$

We shall make our ideas clear with the help of an explicit example in the next section.

## IV. Explicit example

As an explicit example, we shall consider the one-dimensional  $\mathcal{PT}$ -invariant potential [10,18]

$$V(x) = -V_1 \operatorname{sech}^2 x - iV_2 \operatorname{sech} x \tanh x \quad V_1 > 0 \quad (24)$$

This non-Hermitian Hamiltonian has the interesting property of having both real and complex discrete spectrum, depending on the relative magnitudes of its real and imaginary parts, viz.,  $V_1$  and  $V_2$ .

(i) For  $|V_2| \leq V_1 + \frac{1}{4}$

$\mathcal{PT}$ -symmetry is unbroken and energies are real.

(ii) For  $|V_2| > V_1 + \frac{1}{4}$

$\mathcal{PT}$ -symmetry is spontaneously broken and energies are complex conjugate pairs.

We shall restrict our attention to the unbroken  $\mathcal{PT}$ -symmetric phase, with real energies only.

To solve the Schrödinger-like equation

$$-\frac{d^2 u(x)}{dx^2} + V(x)u(x) = \epsilon u(x)$$

with  $V(x)$  given by (24), we make the following substitutions :

$$z = \frac{1 - i \sinh x}{2} \quad (25)$$

$$u(x) = z^{-p} (1 - z)^{-q} \chi(z) \quad (26)$$

Then the differential equation satisfied by  $\chi(z)$  is

$$z(1 - z)\chi'' + \left[-2p + \frac{1}{2} + z(2p + 2q - 1)\right]\chi' - [(p + q)^2 - \lambda^2]\chi = 0 \quad (27)$$

provided

$$p = -\frac{1}{4} \pm \frac{s}{2} \quad (28)$$

$$q = -\frac{1}{4} \pm \frac{t}{2} \quad (29)$$

$$t = \sqrt{V_1 - V_2 + \frac{1}{4}} \quad (30)$$

$$s = \sqrt{V_1 + V_2 + \frac{1}{4}} \quad (31)$$

$$\lambda^2 = -\epsilon \quad (32)$$

The most general solution of (27) is

$$\chi(z) = \alpha F(a, b, c, z) + \beta z^{1-c} (1 - z)^{c-a-b} F(1 - a, 1 - b, 2 - c, z) \quad (33)$$

where

$$a = -p - q + \lambda \quad (34)$$

$$b = -p - q - \lambda \quad (35)$$

$$c = -2p + \frac{1}{2} \quad (36)$$

So the general form of  $u$  is

$$u = \alpha z^{-p} (1 - z)^{-q} F(a, b, c, z) + \beta z^{p+1/2} (1 - z)^{q+1/2} F(1 - a, 1 - b, 2 - c, z) \quad (37)$$

Since  $u$  must not have any real zero, so  $\alpha$  must be non-zero. This allows us to put  $\alpha = 1$ . However, the proper choice of the sign in the expressions for  $p$  and  $q$  is extremely important.

For the potential given in (24), the real energy eigenvalues and the corresponding eigenfunctions are explicitly given by [18],

$$E_n = - \left\{ n + \frac{1}{2} - \frac{1}{2}(s+t) \right\}^2 \quad n = 0, 1, 2, \dots < \frac{s+t-1}{2} \quad (38)$$

$$\phi_n(x) = \binom{n}{n-2p-\frac{1}{2}} \left( \frac{1-i \sinh x}{2} \right)^{-p} \left( \frac{1+i \sinh x}{2} \right)^{-q} P_n^{-2p-\frac{1}{2}, -2q-\frac{1}{2}}(i \sinh x) \quad (39)$$

where  $P_n^{a,b}(z)$  are the Jacobi polynomials expressed as [19]

$$P_n^{a,b}(z) = 2^{-n} \sum_{m=0}^n \binom{n+a}{m} \binom{n+b}{n-m} (z-1)^{n-m} (z+1)^m \quad (40)$$

### Case (i)

First let us consider the simplest case given by

$$\beta = 0$$

$$b = c$$

Hence

$$\lambda = p - q - \frac{1}{2} \quad (41)$$

$$\epsilon = -\lambda^2 = -[p - q - 1/2]^2 \quad (42)$$

Using the relationship [20]

$$F(a, b, b, z) = (1-z)^{-a} \quad (43)$$

$u(x)$  reduces to the simple form

$$u(x) = \alpha \left( \frac{1-i \sinh x}{2} \right)^{-p} \left( \frac{1+i \sinh x}{2} \right)^{q+1/2} \quad (44)$$

Thus  $f(x) = u'(x)/u(x)$  is obtained to be

$$f(x) = i \left[ p + \left( q + \frac{1}{2} \right) \right] \operatorname{sech} x - \left[ p - \left( q + \frac{1}{2} \right) \right] \tanh x \quad (45)$$

The isospectral partner potential of (24) as obtained from (21) turns out to be

$$V_-(x) = - \left[ 4 \left\{ p^2 + \left( q + \frac{1}{2} \right)^2 \right\} - V_1 \right] \operatorname{sech}^2 x$$



$$- i \left[ 4 \left\{ p^2 - \left( q + \frac{1}{2} \right)^2 \right\} - V_2 \right] \operatorname{sech} x \tanh x \quad (46)$$

with the corresponding eigenfunction (from (22))  $\psi_n$  as

$$\begin{aligned} \psi_n = & \left[ \left( \frac{1 - i \sinh x}{2} \right)^{-p} \left( \frac{1 + i \sinh x}{2} \right)^{-q} \right] \\ & \left[ \left( \binom{n}{n - 2p - \frac{1}{2}} \right) \left( 2q + \frac{1}{2} \right) [i \operatorname{sech} x + \tanh x] P_n^{-2p - \frac{1}{2}, -2q - \frac{1}{2}}(i \sinh x) \right. \\ & \left. - \frac{(p + q)^2 - E_n}{-2p + 1/2} \left( \binom{n - 1}{n - 2p - \frac{1}{2}} \right) \left( 2q + \frac{1}{2} \right) [i \operatorname{sech} x + \tanh x] P_{n-1}^{-2p + \frac{1}{2}, -2q + \frac{1}{2}}(i \sinh x) \right] \end{aligned} \quad (47)$$

This is the special case of the so-called satellite potentials, with  $V_+(x)$  and  $V_-(x)$  having exactly the same form, but with different coefficients.

We illustrate this with the help of some simple examples.

Let  $V_1 = 25$ ,  $V_2 = 5$

Thus

$$V_+(x) = -25 \operatorname{sech}^2 x - 5i \operatorname{sech} x \tanh x \quad (48)$$

with energies

$$E_n = - \left\{ n + \frac{1}{2} - 5 \right\}^2, \quad n = 0, 1, 2, 3, 4 \quad (49)$$

Then using equations (28) - (31), the values of  $p$  and  $q$  are

$$\begin{array}{ll} p = \frac{5}{2} & \text{or} \quad -3 \\ q = 2 & \text{or} \quad -\frac{5}{2} \end{array}$$

In order that the wave function vanishes asymptotically, only the positive sign of the discriminant is allowed for  $p$ . Therefore,  $p = 5/2$ . The two values of  $q$  give rise to two interesting cases.

**a)**  $q = -5/2$

$f(x)$  for this particular case becomes

$$f(x) = \frac{11}{2} \tanh x - \frac{i}{2} \operatorname{sech} x \quad (50)$$

and the isospectral partner potential of (48) turns out to be

$$V_-(x) = -16 \operatorname{sech}^2 x - 4i \operatorname{sech} x \tanh x \quad (51)$$

with energies

$$E_n = - \left\{ n + \frac{1}{2} - 4 \right\}^2, \quad n = 0, 1, 2, 3, \quad (52)$$

It is easy to check that  $V_{\pm}(x)$  share exactly the same energy spectrum with the exception of the ground state, as shown in Table 1 below.

**Table 1**

$n$	$E_n[V_+(x)]$	$E_n[V_-(x)]$
0	$-81/4$	$-49/4$
1	$-49/4$	$-25/4$
2	$-25/4$	$-9/4$
3	$-9/4$	$-1/4$
4	$-1/4$	—

b)  $q = 2$

This is the special case when  $p = q + 1/2$ , so that  $\epsilon = 0$  and  $f(x)$  becomes purely imaginary.

$$f(x) = i 5 \operatorname{sech} x \quad (53)$$

The isospectral partner reduces to

$$V_-(x) = -25 \operatorname{sech}^2 x + 5 i \operatorname{sech} x \tanh x \quad (54)$$

Thus

$$V_{\pm}(x) = f^2(x) \pm f'(x) \quad (55)$$

The partner potentials (48) and (54) are totally degenerate. They share identical energies, including the zero energy ground state, and supersymmetry (SUSY) is broken. This scenario is quite different from conventional SUSY breaking. In the conventional supersymmetric quantum mechanics (SUSYQM), SUSY is broken when the zero state energy does not exist and both  $V_{\pm}(x)$  have the same spectrum. The ground state wavefunctions  $\phi_0(x)$  and  $\psi_0(x)$  of the partner potentials (48) and (54), are respectively given by

$$\begin{aligned} \phi_0(x) &= \left( \frac{1 - i \sinh x}{2} \right)^{-\frac{5}{2}} \left( \frac{1 + i \sinh x}{2} \right)^{-2} \\ &= \operatorname{sech}^4 x \left[ \operatorname{sech} x \sqrt{1 + \cosh x} + i \frac{\tanh x}{\sqrt{1 + \cosh x}} \right] \end{aligned} \quad (56)$$

$$\begin{aligned} \psi_0(x) &= \left[ -\frac{d\phi_0}{dx} + \frac{u'(x)}{u(x)} \phi_0(x) \right] \\ &= \frac{9}{2} [\tanh x + i \operatorname{sech} x] \phi_0(x) \\ &= \operatorname{sech}^5 x \left[ \tanh x \left( \sqrt{1 + \cosh x} \frac{1}{\sqrt{1 + \cosh x}} \right) \right. \\ &\quad \left. + i \operatorname{sech} x \left( \frac{\operatorname{sech}^2 x}{\sqrt{1 + \cosh x}} + \sqrt{1 + \cosh x} - \frac{1}{\sqrt{1 + \cosh x}} \right) \right] \end{aligned} \quad (57)$$

apart from normalization constants.

**Case (ii)**

Next we consider the particular case

$$\beta = 0, \quad b = a + \frac{1}{2}, \quad c = 2a$$

This gives the following values for

$$\lambda = -\frac{1}{4}, \quad \epsilon = -\frac{1}{16}, \quad q = -\frac{1}{2}$$

Hence, for this particular case,  $V_1 = V_2 = V_0$  (say) in (24). Using the relationship [20]

$$F\left(a, a + \frac{1}{2}, 2a, z\right) = 2^{2a-1}(1-z)^{-1/2} \left[1 + (1-z)^{1/2}\right]^{1-2a} \quad (58)$$

$u(x)$  reduces to

$$u(x) = \alpha 2^{-2p-1/2} \left(\frac{1-i \sinh x}{2}\right)^{-p} \left[1 + \left(\frac{1+i \sinh x}{2}\right)^{1/2}\right]^{2p+1/2} \quad (59)$$

Thus  $f(x) = u'(x)/u(x)$  is obtained to be

$$f(x) = \frac{1}{4} \tanh x - \frac{i}{4} \operatorname{sech} x + i \left(2p + \frac{1}{2}\right) \operatorname{sech} x \left[\frac{1+i \sinh x}{2}\right]^{1/2} \quad (60)$$

The isospectral partner potential of (24) viz.,

$$V_+(x) = -V_0 \operatorname{sech}^2 x - iV_0 \operatorname{sech} x \tanh x \quad (61)$$

is obtained from (21), and turns out to be

$$\begin{aligned} V_-(x) = & -\left[\frac{1}{4} + \left(2p + \frac{1}{2}\right)^2 - V_0\right] \operatorname{sech}^2 x - i \left[\frac{1}{4} + \left(2p + \frac{1}{2}\right)^2 - V_0\right] \operatorname{sech} x \tanh x \\ & + \left(2p + \frac{1}{2}\right) \left[\operatorname{sech}^2 x + i \operatorname{sech} x \tanh x\right] \left(\frac{1+i \sinh x}{2}\right)^{1/2} \end{aligned} \quad (62)$$

or more explicitly,

$$V_-(x) = -V_R(x) \operatorname{sech}^2 x - iV_I(x) \operatorname{sech} x \tanh x \quad (63)$$

with

$$V_R(x) = \sigma + \left(p + \frac{1}{4}\right) \left\{ \sqrt{\cosh x + 1} + \frac{1}{\sqrt{\cosh x + 1}} - \frac{\cosh^2 x}{\sqrt{\cosh x + 1}} \right\} \quad (64)$$

$$V_I(x) = \sigma - \left(p + \frac{1}{4}\right) \left\{ \sqrt{\cosh x + 1} + \frac{1}{\sqrt{\cosh x + 1}} \right\} \quad (65)$$

where

$$\sigma = 4p^2 + 2p + \frac{1}{2} - V_0 \quad (66)$$

and

$$p = -\frac{1}{4} \pm \frac{1}{2} \sqrt{2V_0 + \frac{1}{4}} \quad (67)$$

It is worth noting here that the coefficients of  $(\text{sech}^2 x)$  and  $(\text{sech } x \tanh x)$  are pure numbers in the expression for  $V_+(x)$ , whereas they are functions of  $x$  in  $V_-(x)$ . From equation (38) it is essential that  $s$  and hence  $V_0$  be large for sufficient number of energy levels, as  $t = 1/4$ . This prevents  $p$  from taking the value  $-1/4$ . So the isospectral partners  $V_{\pm}(x)$  in equations (61) and (63), cannot be reduced to satellite potentials for any physical value of  $p$  in this particular case.

The corresponding eigenfunction is calculated to be

$$\begin{aligned} \psi_n(x) = & \left[ -i \left( p + \frac{3}{4} \right) \text{sech } x + \left( p - \frac{1}{4} \right) \tanh x + i \left( 2p + \frac{1}{2} \right) \text{sech } x \left( \frac{1 + i \sinh x}{2} \right)^{1/2} \right] \\ & \left( \begin{matrix} n \\ n - 2p - 1/2 \end{matrix} \right) \left( \frac{1 - i \sinh x}{2} \right)^{-p} \left( \frac{1 + i \sinh x}{2} \right)^{\frac{1}{2}} P_n^{-2p-\frac{1}{2}, \frac{1}{2}}(i \sinh x) \\ & - \frac{(p - 1/2)^2 - E_n}{-2p + 1/2} \left( \frac{1 - i \sinh x}{2} \right)^{-p} \left( \frac{1 + i \sinh x}{2} \right)^{\frac{1}{2}} \\ & \left( \begin{matrix} n - 1 \\ n - 2p - 1/2 \end{matrix} \right) P_n^{-2p+\frac{1}{2}, \frac{3}{2}}(i \sinh x) \end{aligned} \quad (68)$$

## V. Conclusions and Discussions

To conclude, it is shown in this paper how to generate equivalent SUSY partners of the complex  $\mathcal{PT}$ -invariant potential, viz.,

$$V(x) = -V_1 \text{sech}^2 x - i V_2 \text{sech } x \tanh x$$

with the help of Darboux's method. The form of the newly constructed potentials depend on the choice of  $\beta$  and  $\epsilon$ , but they share the same energy spectrum as the original potential, with the possible exception of the ground state. If the original potential is so chosen that it admits real eigenvalues only, then we can obtain a series of non-trivial complex potentials generating the same real-valued spectrum. The new wavefunctions are also easily obtained by this approach.

Depending on particular values of  $V_1$  and  $V_2$ , the isospectral partners may be of similar nature with just the coupling constants taking different values. We term these as *satellite potentials*. In the special case  $p = 5/2, q = 2, \epsilon = -(p - q - 1/2)^2 - 0$ , the SUSY partner potentials have the same spectrum including the zero energy ground state, a situation which cannot occur in conventional SUSYQM with real potential.

We have also constructed a non-trivial partner potential, and compared its real and imaginary parts graphically with those of the original potential, in Figures 1 and 2 respectively. It is interesting to note that even in the non-trivial case, the partner potentials have similar geometric form.

Further constructions with non-zero  $\beta$  and other values of the integration constant  $\epsilon$  are left as a future exercise.

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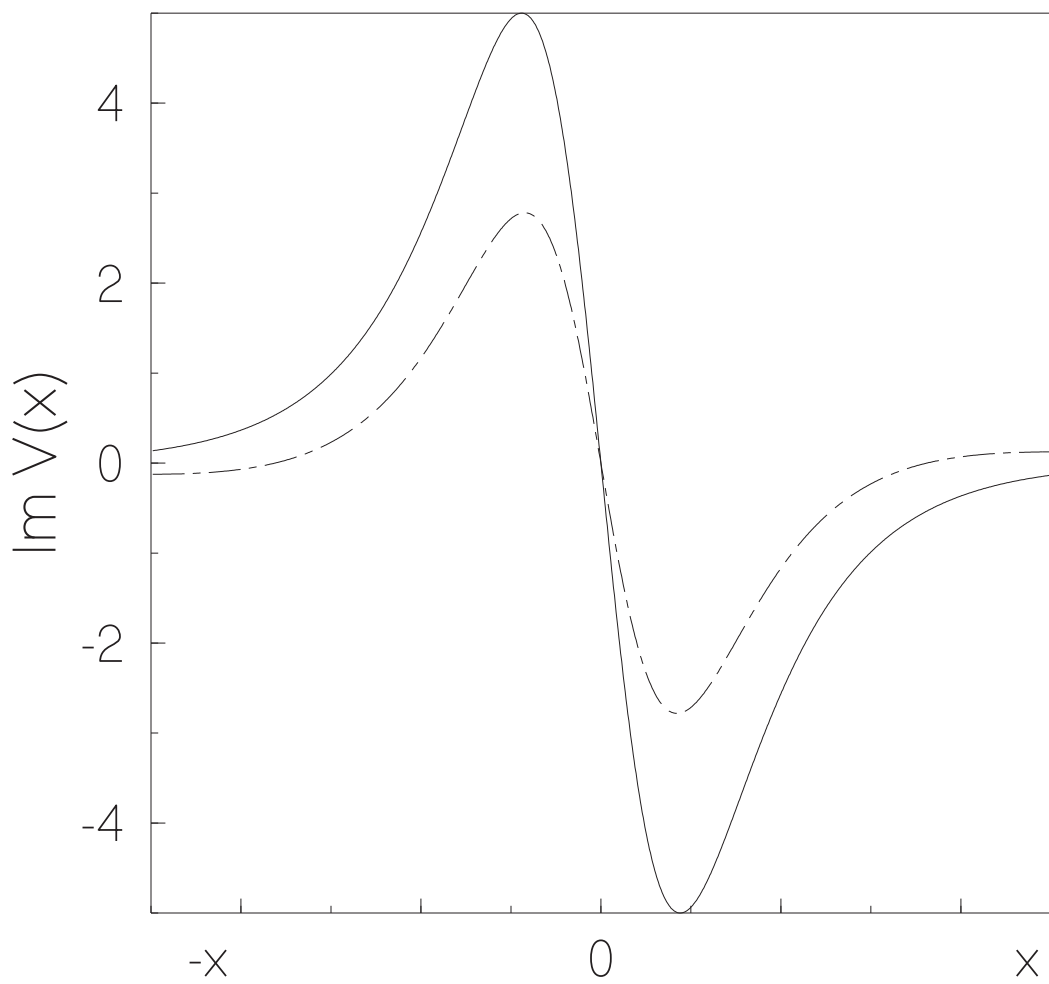


Fig. 1

solid line : Imaginary part of  $V_-(x)$  [eq.(61)]  
broken line : Imaginary part of  $V_+(x)$  [eq.(63)]

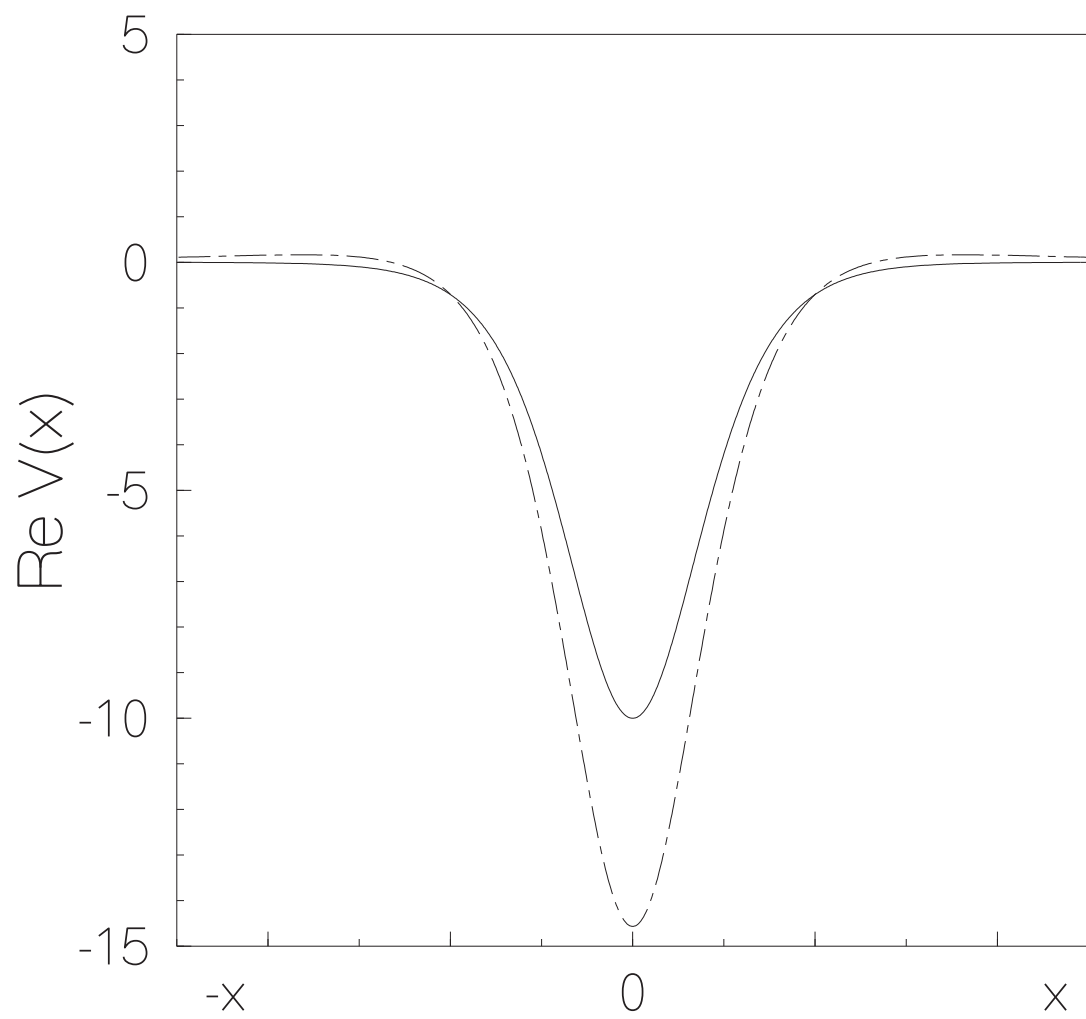


Fig. 2

solid line : Real part of  $V_+(x)$  [eq.(61)]

broken line : Real part of  $V_-(x)$  [eq.(63)]