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$L^1(\mu, X)$ as a constrained subspace of its bidual

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Abstract. In this note we consider the property of being constrained in the bidual, for the space of Bochner integrable functions. For a Banach space X having the Radon–Nikodym property and constrained in its bidual and for $Y \subset X$, under a natural assumption on Y, we show that $L^1(\mu, X/Y)$ is constrained in its bidual and $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X)$. As an application of these results, we show that, if $L^1(\mu, X)$ admits generalized centers for finite sets and if $Y \subset X$ is reflexive, then $L^1(\mu, X/Y)$ also admits generalized centers for finite sets.

Keywords. Spaces of Bochner integrable functions; vector measures; proximinal subspaces; generalized centers.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let X be a Banach space that is constrained in its bidual i.e., X is the range of a norm one projection when canonically embedded in its bidual. Any dual space, thanks to the canonical projection of Dixmier (See [H], p. 213) is contrained in its bidual. It is well known that the space of integrable functions $L^1(\mu)$ satisfies this property. It is easy to see that X is constrained in its bidual iff X is isometric to the range of a norm one projection in some dual space (see [Lin]). Thus this property is preserved by ranges of norm one projections. In this note we consider the question, when is the space of Bochner integrable functions, $L^1(\mu, X)$, constrained in its bidual? That X should be constrained in its bidual is clearly a necessary condition. c_0, the space of sequences converging to zero, is not constrained in its bidual (it is not even complemented in ℓ^∞, see [H], p. 232). The author in [R3] noted the isometric version of a result of Emmanuele [E1] that, if X is any Banach space that has an isometric copy of c_0, then c_0 is isometric to the range of a norm one projection in $L^1(\mu, X)$. Thus by taking $X = \ell^\infty$ we see that $L^1(\mu, X)$ is not constrained in its bidual. If X is constrained in its bidual and has the Radon–Nikodym property (RNP), it was proved in [R1, R2], that $L^1(\mu, X)$ is constrained in its bidual. Let cabv(μ, X) denote the space of X-valued countably additive measures of bounded variation on $\mathcal{A}$ that are absolutely continuous with respect to μ. Emmanuele and Rao in 1994 (see [E2, R4]) have obtained an internal characterization by showing that when X is constrained in its bidual, $L^1(\mu, X)$ is constrained in its bidual iff it is constrained in cabv(μ, X). Using this, in this short note we exhibit more examples of Banach spaces for which $L^1(\mu, X)$ is constrained in its bidual. Our work is motivated by the recent exposition of the Lindenstrauss' lifting principle (LLP) by Kalton and Pełczyński in [KP]. A closed subspace $Y \subset X$ is said to be proximinal if given $x \in X$ there exists a $y \in Y$ such that $d(x,Y) = \|x-y\|$. A natural question that can be asked in the context of spaces of Bochner integrable functions is that, if $Y \subset X$ is a proximinal subspace, then is $L^1(\mu,Y)$ a proximinal subspace of $L^1(\mu,X)$? Only recently a counterexample was obtained by Mendoza in [M]. As an interesting consequence of our approach we exhibit new classes of proximinal subspaces Y for which $L^1(\mu,Y)$ is proximinal in $L^1(\mu,X)$. In this process we obtain new proofs of some well known result on proximinality in $L^1(\mu,X)$.

In the concluding part of the paper we apply these results to study a weaker geometric notion called GC (defined later in the paper) for quotient spaces of Bochner integrable functions. Here our result states that, if $L^1(\mu,X)$ has GC and $Y \subset X$ is reflexive then $L^1(\mu,X/Y)$ has GC.

Our notation and terminology is fairly standard and can be found in [DU, H, L, HWW].

2. Main results

The new examples we exhibit come from quotient spaces and an isometric version of LLP is used to achieve this. These results also give a unified approach to Corollary 1 and Proposition in [R4]. We first recall the isomorphic version of LLP from [KP].

Lindenstrauss lifting principle: Let $Y \subset X$ and Y be complemented in its bidual. Let F be any $\mathcal{L}_1$ space. Every bounded linear operator $T: F \to X/Y$ admits a lifting, i.e., there exists a bounded linear operator $T^\wedge: F \to X$ such that $\pi T^\wedge = T$.

In this paper we assume that X, Y and X/Y are constrained in their bidual and will consider the space $L^1(\mu,X/Y)$. We assume that $L^1(\mu,X)$ is constrained in its bidual and most often we assume that Y has the RNP, so that $L^1(\mu,Y)$ is also constrained in its bidual. Here it may be worth recalling from [DU] that $L^1(\mu,X/Y)$ can be identified with the quotient space $L^1(\mu,X)/L^1(\mu,Y)$. We consider the question, when is $L^1(\mu,X/Y)$ constrained in its bidual?

In what follows we will need norm preserving liftings from $L^1(\mu)$. Even under the assumptions of the above paragraph we do not know if this can always be achieved. Thus we need an extra assumption on the projection which is satisfied in several naturally occurring situations.

Assumption. Let $P: X^{**} \to X$ be a norm one projection. Let $Y \subset X$ be a closed subspace such that $P(Y^{\perp\perp}) = Y$.

Remark 1. This clearly is the case when X is constrained in its bidual and Y is reflexive. Also if $M \subset X^*$ then the Dixmier projection, $Q: X^{***} \to X^*$ defined by $Q(\Lambda) = \Lambda/X$, satisfies the condition in the '*assumption*' precisely when M is weak* closed (Lemma IV.1.1 of [HWW]). We recall from Chapter IV of [HWW] that a Banach space X is a L-summand in its bidual if when X is canonically embedded in its bidual, there exists a projection $P: X^{**} \to X$ such that $\|P(\Lambda)\| + \|\Lambda - P(\Lambda)\| = \|\Lambda\|$ for all $\Lambda \in X^{**}$. Clearly such a X is constrained in its bidual and is also a proximinal subspace of its bidual. It was shown in [Li] that if both X, Y are L-summands in their biduals then the '*assumption*' is satisfied. In Proposition 3 below we exhibit some more situations where the '*assumption*' is satisfied.

The following proposition which identifies Q as the only projection with the above property is probably known but as we are not aware of a reference to it in the literature, we give below its easy proof.

PROPOSITION 1

Suppose $P : X^{***} \to X^*$ is a bounded linear projection such that for every weak closed subspace $M \subset X^*$, $P(M^{\perp\perp}) = M$. Then $P = Q$.*

Proof. Since P and Q are projections with the same range, it is clear enough to show that $\mathrm{Ker}P \subset X^{\perp}$. Let $\Lambda \in \mathrm{Ker}P$ and let $x \in X$. Write $\Lambda = f + \gamma$ where $f \in X^*$ and $\gamma \in X^{\perp}$. Let $M = \mathrm{Ker}(x)$. Then M is a weak* closed subspace of X^* and $M^{\perp\perp} = \{\lambda \in X^{***} : \lambda(x) = 0\}$. Thus $\gamma \in M^{\perp\perp}$. By hypothesis, $P(M^{\perp\perp}) = M$. Therefore $P(\gamma) \in M$. Now $P(\Lambda) = 0 = f(x)$. Hence $\Lambda(x) = 0$. Therefore $\Lambda \in X^{\perp}$.

It follows from Lemma 1.1 in Chapter IV of [HWW] that the '*assumption*' implies that Y is a proximinal subspace of X. This is one of the motivations for considering this approach.

We need the following elementary lemma.

Lemma 1. *Under the 'assumption'*, X/Y is constrained in its bidual.

Proof. Define $Q : X^{**}/Y^{\perp\perp} \to X/Y$ by $Q(\pi^{**}(x^{**})) = \pi(P(x^{**}))$. That Q is well defined is guaranteed by the '*assumption*'. Clearly Q is a projection. Also

$$\begin{aligned} \|Q(\pi^{**}(x^{**}))\| &= \|\pi(P(x^{**}))\| \\ &= d(P(x^{**}), Y) \\ &\leq \|P(x^{**}) - P(\lambda)\| \quad \text{for any} \quad \lambda \in Y^{\perp\perp} \\ &\leq \|x^{**} - \lambda\|. \end{aligned}$$

Therefore $\|Q(\pi^{**}(x^{**}))\| \leq \|\pi^{**}(x^{**})\|$. Hence X/Y is constrained in its bidual.

Before proceeding further we prove a proposition of independent interest that shows the limitations of the '*assumption*'.

PROPOSITION 2

Let Y be a Banach space that is constrained in its bidual. Suppose whenever Y is isometrically embedded in a Banach space X which is constrained in its bidual, the 'assumption' is satisfied. Then Y is reflexive.

Proof. Let X be any Banach space containing (isometrically) Y. We shall show that Y is proximinal in X. It then follows from a result of Pollul [CW] that Y is reflexive. Note that since $Y \subset X \subset X^{**}$, the hypothesis on Y implies that it is proximinal in X^{**}. Therefore Y is proximinal in X and hence reflexive.

We do not know an answer to the corresponding subspace formulation.

Question. Characterize spaces X with the property that the '*assumption*' is satisfied for all $Y \subset X$ that are constrained in their bidual.

The following lemma is the isometric version of the lifting theorem we need and we give below its simple proof for the sake of completeness.

Lemma 2. *Let $Y \subset X$ be such that the 'assumption' is satisfied. For any bounded linear operator $T : L^1(\mu) \to X/Y$, there exists a $T^\wedge : L^1(\mu) \to X$ such that $\pi T^\wedge = T$ and $\|T^\wedge\| = \|T\|$.*

Proof. It follows from Theorem 8 on p. 178 of [L] that there is a $S : L^1(\mu) \to X^{**}$ such that $\|S\| = \|T\|$ and $\pi^{**}S = T$. Put $T^\wedge = PS$. Clearly $\|T^\wedge\| = \|T\|$. Let $f \in L^1(\mu)$. If $T(f) = \pi(x)$ for some $x \in X$, then $x - S(f) \in Y^{\perp\perp}$. By hypothesis we get $P(x - S(f)) \in Y$. Thus $\pi T^\wedge = T$.

Theorem 1. *Let $Y \subset X$ be such that the 'assumption' is satisfied. Suppose Y has the RNP and $L^1(\mu, X)$ is constrained in its bidual, then $L^1(\mu, X/Y)$ is constrained in its bidual.*

Proof. We follow the arguments given during the proof of Proposition in [R4]. It is thus enough to show that $L^1(\mu, X/Y)$ is the range of a norm one projection in $\text{cabv}(\mu, X/Y)$. Since Y has the RNP, and since $L^1(\mu, X)$ is constrained in $\text{cabv}(\mu, X)$ by Theorem 5 of [E2] the required conclusion follows once we show that elements of $\text{cabv}(\mu, X/Y)$ can be lifted to elements of $\text{cabv}(\mu, X)$ in a norm preserving way. As in the proof of Proposition in [R4] this can be achieved by using the correspondence between vector measures and operators on L^1-spaces and the above Lifting result for operators.

Remark 2. Even under the '*assumption*' of the above theorem it not clear if the inclusion $L^1(\mu, Y) \subset L^1(\mu, X)$ satisfies the '*assumption*' (see Proposition 3 below).

In the following Proposition we consider two situations where the explicit knowledge of the projection shows that the '*assumption*' is satisfied in the space of Bochner integrable functions and hence the quotient space result of our Theorem can be obtained by simply using Lemma 1 instead of the Lifting and other result of Emmanuele [E2] on quotient space valued measures, see also Remark 6 below.

PROPOSITION 3

Suppose X has the RNP and is constrained in its bidual.

1. *Consider the embedding of X as constant functions in $L^1(\mu, X)$.*
2. *Let $Y \subset X$ and the 'assumption' is satisfied. Consider the inclusion $L^1(\mu, Y) \subset L^1(\mu, X)$.*

The 'assumption' is satisfied in both these cases. Thus the subspaces are proximinal and the quotient spaces are constrained in the bidual.

Proof. Let $P : X^{**} \to X$ be a norm one projection such that $P(Y^{\perp\perp}) = Y$. We assume w.l.o.g that μ is a category measure on the Borel σ-field of a compact hyperstonean space K. Consider the canonical embedding $X^{**} \subset L^1(\mu, X)^{**}$. We recall from [R1, R2] that a norm one projection $P^\wedge : L^1(\mu, X)^{**} \to L^1(\mu, X)$ was defined by $P^\wedge(\lambda) = (\lambda/C(K, X^*) \circ P)_a/\mathrm{d}\mu$. Here we used Singer's theorem that identified $C(K, X^*)^*$ as $\text{cabv}(X^{**})$ and the suffix a indicates the absolutely continuous (w.r.t. μ) part of the measure.

Now if $\lambda \in X^{**}$ then the vector measure under consideration is the Dirac measure at $P(\lambda)$ and thus $P^\wedge$ extends P.

Let $Y \subset X$ satisfy the '*assumption*' and let $\lambda \in L^1(\mu, Y)^{\perp\perp}$. We claim that λ can be naturally restricted to $C(K, Y^*)$. To see this, let $f \in C(K, Y^*)$. We first observe that it can be in a norm preserving way extended to a $f^\wedge \in C(K, X^*)$.

Since K is hyperstonean one can do this easily using properties of extremally disconnected spaces and Stone–Čech compactification (see § 11 of [L]). Equivalently one can treat a $f \in C(K, Y^*)$ as a compact operator $T : Y \to C(K)$ and use Theorem 1 on p. 205 of [L] to get a norm preserving extension $T^\wedge : X \to C(K)$.

Now one defines the restriction by

$$\lambda(f) = \lambda(f^\wedge).$$

This is well defined since $\lambda \in L^1(\mu, Y)^{\perp\perp}$. This is what is meant by the natural restriction.

By the uniqueness part of Singer's representation theorem, we see that the corresponding vector measure actually takes values in $Y^{\perp\perp}$ and thus $P^\wedge(L^1(\mu, Y)^{**}) = L^1(\mu, Y)$. Hence the conclusion follows.

As a corollary to the proof of the above proposition we have the following result that extends several classical proximinality situations in the space of Bochner integrable functions.

COROLLARY 1

Suppose $Y \subset X$ satisfies the 'assumption' and Y has the RNP. Then $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X)$.

Proof. We note that the projection $P^\wedge$ defined as above, now has $\mathrm{cabv}(\mu, X)$ as its range. Since Y has the RNP, $P^\wedge(L^1(\mu, Y)^{**}) = L^1(\mu, Y)$. Therefore $L^1(\mu, Y) \subset \mathrm{cabv}(\mu, X)$ is proximinal and in particular it is proximinal in $L^1(\mu, X)$.

Remark 3. Let $Y \subset X^*$ be a weak* closed subspace having the RNP. Since the '*assumption*' is satisfied via the canonical projection, we get that $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X^*)$. If $Y \subset X$ is a reflexive subspace then again since Y is a weak* closed subspace of X^{**} we get a new proof of the classical result that $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X)$ (see [LC], Theorem 2.13), see also [R5].

If X is a Banach space that is a L-summand in its bidual it is still not known whether $L^1(\mu, X)$ will always be a L-summand in its bidual. It follows from the above corollary that if $Y \subset X$ are both L-summands in their bidual (as mentioned in Remark 1, the '*assumption*' is satisfied in this case) and Y has the RNP, then $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X)$.

Remark 4. We take this opportunity to point out that Corollary 3.5 of [M] does not lead to a new class of proximinal subspaces, since the author's '*assumption*' "each separable subspace Y is proximinal in X" already implies that X is reflexive. To see this, note that it is enough to show that every separable $Y \subset X$ is reflexive. Now for such a Y, for each closed subspace $Z \subset Y$, the hypothesis implies that Z is proximinal in X and hence in Y. It now follows from the proof of the Theorem on p. 161 of [H] that Y and hence X is reflexive.

Remark 5. Suppose X, Y satisfy the '*assumption*' and X/Y is isometric to some $L^1(\nu)$. Since the identity map on X/Y can now be lifted in a norm preserving way, we get that Y is also the kernel of a norm one projection. Thus $L^1(\mu, Y)$ being the kernel of a norm one projection, is again a proximinal subspace of $L^1(\mu, X)$.

Remark 6. Suppose X, Y and X/Y are all constrained in their biduals. If X has the RNP then $L^1(\mu, X/Y)$ is constrained in its bidual. To see this we only have to observe that the hypothesis implies that X/Y has the RNP, then the conclusion follows from [R2]. In view of Lewis and Stegall characterization of the RNP in terms of factorization of operators on

L^1 spaces (Theorem III.1.8 of [DU]) and the LLP we see that when X has the RNP so does X/Y.

The following corollary which covers the isomorphic case, is an immediate consequence of the LLP and the arguments given during the proof of the above theorem. We note that since we will be applying the LLP when the domain is a $L^1(\nu)$ space, $\sigma = 1$ in the proof of the LLP in [KP].

COROLLARY 2

Suppose X, Y and X/Y are complemented in their bidual. Suppose $L^1(\mu, X)$ is complemented in its bidual and Y has the RNP. $L^1(\mu, X/Y)$ is complemented in its bidual.

It follows from Proposition 2.3 of [KP] that if X is a $L^1(\nu)$ space and $Y \subset X$ is such that X/Y has the RNP and constrained in its bidual (thus $L^1(\mu, X)$ and $L^1(\mu, X/Y)$ are both constrained in their biduals) then there is a projection $Q: X^{**} \to X$ such that $Q(Y^{\perp\perp}) = Y$. However such a projection in general need not be of norm one, as can be seen by taking $Y = \mathrm{Ker}(x^*)$ where $x^* \in X^*$ does not attain its norm.

In the concluding part of the paper, as an application of the proximinality results proved here, we consider a weaker geometric notion for quotient spaces of Bochner integrable functions. We first recall the notion of generalized center (GC) due to Veselý that is related to the existence of weighted Chebyshev centers, from [BR].

DEFINITION

A Banach space X is said to have GC, if every finite collection of closed balls in X^{**} with centers from X (as before, X is canonically embedded in X^{**}) and having non-empty intersection, has an element of X in the intersection.

It is easy to see that if X is constrained in its bidual then it has GC. c_0 has GC and more generally any Banach space whose dual is isometric to a $L^1(\mu)$ has GC (see [BR]).

To facilitate the study of GC the authors of [BR] have introduced the notion of a central subspace. $Y \subset X$ is said to be a central subspace if for every finite collection of elements $\{y_1, \ldots, y_n\}$ in Y and $x \in X$, there exists a $y_0 \in Y$ such that $\|y_i - y_0\| \leq \|y_i - x\|$. It is easy to see that a Banach space X has GC iff it is a central subspace of X^{**} (see [BR]). We also have from [BR] that if $Y \subset Z \subset X$ and Y is proximinal in X and Z is a central subspace of X, then Z/Y is a central subspace of X/Y.

Using the arguments given in [R4] the following proposition is easy to prove.

PROPOSITION 4

$L^1(\mu, X)$ has GC iff X has GC and $L^1(\mu, X)$ is a central subspace of $\mathrm{cabv}(\mu, X^{**})$.

We use the above proposition and give an application of the proximinality results.

Theorem 2. *Let X be such that $L^1(\mu, X)$ has GC. Let $Y \subset X$ be reflexive. Then $L^1(\mu, X/Y)$ has GC.*

Proof. We note that since Y is reflexive, $\mathrm{cabv}(\mu, X^{**}/Y)$ can be identified with the quotient space, $\mathrm{cabv}(\mu, X^{**})/L^1(\mu, Y)$. From the above proposition and the hypothesis, we have that $L^1(\mu, X)$ is a central subspace of $\mathrm{cabv}(\mu, X^{**})$. From the proof of Proposition 3 and the subsequent remark, we have that $L^1(\mu, Y)$ is a proximinal subspace of $\mathrm{cabv}(\mu, X^{**})$. Therefore $L^1(\mu, X/Y)$ is a central subspace of $\mathrm{cabv}(\mu, X^{**}/Y)$ and hence has GC.

References

[BR] P Bandyopadhyay and T S S R K Rao, Central subspaces of Banach spaces, *J. Approx. Theory* (to appear)

[CW] E W Cheney and D E Wulbert, The existence and unicity of best approximations, *Math. Scand.* **24** (1969) 113–140; **27** (1970) 145

[DU] J Diestel and J J Uhl Jr, Vector Measures, *Math. Surveys* **15**; *Amer. Math. Soc.* (Providence, R.I.) (1977)

[E1] G Emmanuele, On complemented copies of c_0 in L_X^p, *Proc. Amer. Math. Soc.* **104** (1988) 785–786

[E2] G Emmanuele, Remarks on the complementability of spaces of Bochner integrable functions in spaces of vector measures, *Comm. Math. Univ. Carolinae* **37** (1996) 217–228

[HWW] P Harmand, D Werner and W Werner, *M-ideals in Banach spaces and Banach algebras*, LNM 1547 (Springer-Verlag) (1993)

[H] R B Holmes, *Geometric functional analysis and its applications* (GTM 24) (Springer-Verlag) (1975)

[KP] N J Kalton and A Pełczyński, Kernels of surjections from $\mathcal{L}_1$-spaces with an application to Sidon sets, *Math. Ann.* **309** (1997) 135–158

[L] H E Lacey, Isometric theory of classical Banach spaces, *Band* **208** (Springer, Berlin) (1973)

[Li] D Li, Espaces L-facteurs de leurs biduaux: bonne disposition, milleure approximation et propriété de Radon–Nikodym, *Q. J. Math. Oxford* **38** (1987) 229–243

[LC] W A Light and E W Cheney, *Approximation theory in tensor product spaces*, LNM 1169 (Springer, Berlin) (1985)

[Lin] J Lindenstrauss, On a certain subspace of ℓ^1, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **12** (1964) 539–542

[M] J Mendoza, Proximinality in $L_p(\mu, X)$, *J. Approx. Theory* **93** (1998) 331–343

[R1] T S S R K Rao, A K Roy and K Sundaresan, Intersection properties of balls in tensor products of some Banach spaces, *Math. Scand.* **65** (1989) 103–118

[R2] T S S R K Rao, A note on the $R_{n,k}$ property for $L^1(\mu, E)$, *Canad. Math. Bull.* **32** (1989) 74–77

[R3] T S S R K Rao, Intersection properties of balls in tensor products of some Banach spaces II, *Indian J. Pure Appl. Math.* **21** (1990) 275–284

[R4] T S S R K Rao, $L^1(\mu, X)$ as a complemented subspace of its bidual, *Proc. Indian. Acad. Sci.* **104** (1994) 421–424

[R5] T S S R K Rao, *Some remarks on proximinality in Bochner function spaces*, Preprint (1997)