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GROUP DIVISIBLE THIRD ORDER ROTATABLE DESIGNS (GDTORD)

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SUMMARY. In this paper we have considered the analysis of group divisible third order rotatable designs (GDTORD) when the factors are divided into s groups and the method of constructing these designs from balanced block designs with variable replication (BBD).

1. Introduction

The concept of rotatability was introduced by Box and Hunter (1957). Methods for constructing these designs have been considered by many authors. A review of the work done upto that time is available in Mead and Pike (1975).

Since in many situations, too many design points are required for satisfying rotatability conditions, attempts have been made by various authors (Herzberg, 1966, 1967; Das and Dey, 1967; Dey and Nigam, 1968; Adhikary and Sinha, 1976; Adhikary and Panda, 1980) to modify the concept of rotatability to group divisible rotatability. However these authors considered only group divisible second order rotatable designs (GDSORD) where the factors are divided into *two* groups. Adhikary and Panda (1980) considered the construction of GDSORD and GDTORD when the factors are divided into s groups.

These designs may be considered as particular cases of rotatable designs when the variance function remains constant for all orthogonal rotations of the type

$$P = \text{Diag.}(M_1, M_2, \ldots, M_s)$$

where M_μ is an orthogonal matrix of order $v_\mu \times v_\mu$, $\mu = 1, 2, \ldots, s$.

In this paper, we shall consider the analysis of GDTORD and the method of constructing these designs from balanced block designs with variable replications (BBD).

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2. Analysis of GDTORD

Let the v factors be divided into s groups as follows :

$$G_1 = (1, 2, \ldots, v_1),\ G_2 = (v_1+1, \ldots, v_1+v_2), \ldots$$
$$G_s = (v_1+v_2+\ldots+v_{s-1}+1, \ldots, v_1+v_2+\ldots+v_s),$$

where $v = v_1+v_2+\ldots+v_s$.

Considering the response surface to be of third order, we assume the model

$$y_u = b_0 + \sum_{\mu} \Sigma_\mu b_{i_\mu} x_{i_\mu u} + \sum_{\mu} \sum_{i_\mu \leqslant j_\mu} b_{i_\mu j_\mu} x_{i_\mu u} x_{j_\mu u}$$
$$+ \sum_{\mu<\mu'} \Sigma_\mu \Sigma_{\mu'} b_{i_\mu i_{\mu'}} x_{i_\mu u} x_{i_{\mu'} u} + \sum_{\mu} \sum_{i_\mu \leqslant j_\mu \leqslant k_\mu} x_{i_\mu u} x_{j_\mu u} x_{k_\mu u} b_{i_\mu j_\mu k_\mu}$$
$$+ \sum_{\mu \neq \mu'} \sum_{i_\mu \leqslant j_\mu} \Sigma_{\mu'} b_{i_\mu j_\mu k_{\mu'}} x_{i_\mu u} x_{j_\mu u} x_{k_{\mu'} u}$$
$$+ \sum_{\mu < \mu' < \mu''} \Sigma_\mu \Sigma_{\mu'} \Sigma_{\mu''} b_{i_\mu i_{\mu'} i_{\mu''}} x_{i_\mu u} x_{i_{\mu'} u} x_{i_{\mu''} u} + e_u \qquad \ldots (1)$$

where Σ_μ denotes the sum over the factors of μ-th group. In order that the design is a GDTORD, N design points must satisfy the following conditions :

A : $\sum_u x_{i_1 u}^{\alpha_1} x_{i_2 u}^{\alpha_2} \ldots x_{i_6 u}^{\alpha_6} = 0$ if any α_i is odd, $\Sigma\alpha_i \leqslant 6$.

B(i) : $\Sigma x_{iu}^2 = N\lambda_2^{(\mu)}$ if $i \in G_\mu$, $\mu = 1, 2, \ldots, s$.

B(ii) : $\Sigma x_{iu}^4 = N.3\lambda_4^{(\mu)}$ if $i \in G_\mu$, $\mu = 1, 2, \ldots, s$.

B_1 : $\Sigma x_{iu}^6 = N.15\lambda_6^{(\mu)}$ if $i \in G_\mu$

C : $\Sigma x_{iu}^2 x_{ju}^2 = N\lambda_4^{(\mu)}$ if $i \in G_\mu$, $j \in G_\mu$

$= N\theta^{\mu\mu'} (> 0)$ if $i \in G_\mu$, $j \in G_{\mu'}$, $\mu \neq \mu' = 1, 2, \ldots, s$.

C_1(i) : $\Sigma x_{iu}^2 x_{ju}^4 = N.3\lambda_6^{(\mu)}$ if $i \in G_\mu$, $j \in G_\mu$

$= N.3\theta^{\mu\mu'\mu'}$ if $i \in G_\mu$, $j \in G_{\mu'}$, $\mu \neq \mu' = 1, 2, \ldots, s$.

C_1(ii) : $\Sigma x_{iu}^2 x_{ju}^2 x_{ku}^2 = N\lambda_6^{(\mu)}$ if $i \in G_\mu$, $j \in G_\mu$ and $k \in G_\mu$

$= N\theta^{\mu\mu'\mu'}$ if $i \in G_\mu$, $j \in G_{\mu'}$, $k \in G_{\mu'}$

$= N\theta^{\mu\mu'\mu''}$ if $i \in G_\mu$, $j \in G_{\mu'}$, $k \in G_{\mu''}$,

$\mu \neq \mu' \neq \mu'' = 1, 2, \ldots, s$.

D : $\Sigma x_{iu}^4 = 3\Sigma x_{iu}^2 x_{ju}^2$
D_1(i) : $\Sigma x_{iu}^6 = 5\Sigma x_{iu}^2 x_{ju}^4$
(D and D_1(i)) if $i \in G_\mu$ and $j \in G_\mu$

D_1(ii) : $\Sigma x_{iu}^2 x_{ju}^4 = 3\Sigma x_{iu}^2 x_{ju}^2 x_{ku}^2$ if $i \in G_\mu$, $j \in G_\mu$, $k \in G_\mu$

or $i \in G_\mu$, $j \in G_{\mu'}$, $k \in G_{\mu'}$

$$\text{E} \quad : \frac{\lambda_4^{(\mu)}}{\{\lambda_2^{(\mu)}\}^2} > \frac{v_\mu}{v_\mu+2}$$

$$\frac{\lambda_2^{(\mu)}\lambda_6^{(\mu)}}{\{\lambda_4^{(\mu)}\}^2} > \frac{v_\mu+2}{v_\mu+4}; \frac{\lambda_2^{(\mu)}\theta^{\mu\mu'\mu'}}{((\theta)^{\mu\mu'})^2} > \frac{v_{\mu'}}{v_{\mu'}+2} \quad \text{non-singularity conditions}$$

Also
$$[(v_\mu+2)\lambda_4^{(\mu)}-v_\mu \{\lambda_2^{(\mu)}\}^2]\,[(v_\mu+2)\lambda_4^{(\mu')}-v_{\mu'} \{\lambda_2^{(\mu')}\}^2] -v_\mu v_{\mu'}\,[\lambda_2^{(\mu)}\lambda_2^{(\mu')}-\theta^{\mu\mu'}]^2 > 0$$

and
$$F_\mu^{-1} = [(v_\mu+4)\lambda_6^{(\mu)}\lambda_2^{(\mu)}-(v_\mu+2)\,\{\lambda_4^{(\mu)}\}^2]\,[(v_{\mu'}+2)\,\theta^{\mu\mu'\mu'}\,\lambda_2^{(\mu)}-v_{\mu'}(\theta^{\mu\mu'})^2] -v_{\mu'}(v_\mu+2)(\lambda_2^{(\mu)}\theta^{\mu'\mu\mu}-\lambda_4^{(\mu)}\,\theta^{\mu\mu'})^2 > 0.$$

The model (1) can be written in matrix notation as

$$Y = X\beta+\theta.$$

Assuming $X'X$ to be non-singular, the least squares estimator of β is given by

$$\beta = (X'X)^{-1}(X'Y)$$

with $V(\beta) = \frac{\sigma^2}{N}\left[\frac{1}{N}\,(X'X)\right]^{-1}$, where $V(\cdot)$ denotes the dispersion matrix.

For simplicity, we take $\lambda_2^{(\mu)} = 1 \;\forall\; \mu$. Following Gardiner, Grandage and Hader (1959),

$$\frac{1}{N}\,X'X = \text{Diag.}(G, H_1, H_2, K_{11}, \ldots, K_{sv_s}, M_1, M_2, M_3),$$

where

$$G = \begin{bmatrix} 1 & 1 & \ldots\ 1 & \ldots\ 1 \\ & X_{11} & \ldots\ X_{1\mu} & \ldots\ X_{1s} \\ & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ & & X_{\mu\mu} & \ldots\ X_{\mu s} \\ & & \cdots\cdots & \cdots\cdots \\ & & & X_{ss} \end{bmatrix},$$

$$X_{\mu\mu} = 2\lambda_4^{(\mu)}\,I_{v_\mu}+\lambda_4^{(\mu)}\,E_{v_\mu\times v_\mu}$$

$$X_{\mu\mu'} = \theta^{\mu\mu'}\,E_{v_\mu\times v_{\mu'}},\ \mu \neq \mu' = 1, 2, \ldots, s.$$

$$H_1 = \text{Diag.}\,(\lambda_4^{(1)}\ldots\lambda_4^{(1)}\ldots\lambda_4^{(\mu)}\ldots\lambda_4^{(\mu)}\ldots\lambda_4^{(s)}\ldots\lambda_4^{(s)})$$

where $\lambda_4^{(\mu)}$ is repeated $\binom{v_\mu}{2}$ times.

$$H_2 = \text{Diag.}\,(\theta^{12}\ldots\theta^{12}\ldots\theta^{(s-1)s}\ldots\theta^{(s-1)s})$$

where $\theta^{\mu\mu'}$ is repeated $v_\mu v_{\mu'}$ times, $\mu < \mu' = 1, 2, \ldots, s$.

$$K_{st} = \begin{bmatrix} 1 & 3\lambda_4^{(\mu)} & \lambda_4^{(\mu)} 1(1\times v_\mu - 1) & \theta^{1\mu} 1(1\times v_1) & \ldots & \theta^{\mu s} 1(1\times v_s) \\ & 15\lambda_4^{(\mu)} & 3\lambda_6^{(\mu)} 1(1\times v_\mu - 1) & 3\theta^{1\mu\mu} 1(1\times v_1) & \ldots & 3\theta^{\mu\mu s} 1(1\times v_s) \\ & & 2\lambda_6^{(\mu)} I_{v_\mu - 1} + \lambda_6^{(\mu)} E & \theta^{1\mu\mu} E & \ldots & \theta^{\mu\mu s} E \\ & & & 2\theta^{11\mu} I_{v_1} + \theta^{11\mu} E & \ldots & \theta^{1\mu s} E \\ & & & \ldots & \ldots & \ldots \\ & & & \ldots & \ldots & \ldots \\ & & & & & 2\theta^{\mu s s} I_{v_s} + \theta^{\mu s s} E \end{bmatrix}$$

where $\mu = 1, 2, \ldots, s$; $i = 1, 2, \ldots, v_\mu$.

$$M_1 = \text{Diag.}\,(\lambda_6^{(1)} \ldots \lambda_6^{(1)} \ldots \lambda_6^{(s)} \ldots \lambda_6^{(s)}),\ \lambda_6^{(\mu)}$$

is repeated $\binom{v_\mu}{3}$ times

$$M_2 = \text{Diag.}\,(\theta^{112} \ldots \theta^{112} \ldots \theta^{(s-1)ss} \ldots \theta^{(s-1)ss})$$

where $\theta^{\mu\mu'\mu'}$ is repeated $v_\mu \binom{v_{\mu'}}{2}$ times, $\mu \neq \mu' = 1, 2, \ldots, s$.

$$M_3 = \text{Diag.}\,(\theta^{123} \ldots \theta^{123} \ldots \theta^{(s-2)(s-1)s} \ldots \theta^{(s-2)(s-1)s})$$

where $\theta^{\mu\mu'\mu''}$ is repeated $v_\mu v_{\mu'} v_{\mu''}$ times, $\mu < \mu' < \mu'' = 1, 2, \ldots, s$.

It can be shown that

$$G^{-1} = \begin{bmatrix} a_0 & \boldsymbol{a}_1 & \ldots & \boldsymbol{a}_\mu & \ldots & \boldsymbol{a}_s \\ & X_{11}^* & \ldots & X_{1\mu}^* & \ldots & X_{1s}^* \\ & & & X_{\mu\mu}^* & \ldots & X_{\mu s}^* \\ & & & \ldots & \ldots & \ldots \\ & & & \ldots & \ldots & \ldots \\ & & & & & X_{ss}^* \end{bmatrix}$$

where $\quad \boldsymbol{a}_\mu = (a_\mu\, a_\mu \ldots a_\mu)\ X_{\mu\mu}^* = C_1^{(\mu)} I_{v_\mu} + C_2^{(\mu)} E\ X_{\mu\mu'}^* = C_3^{(\mu\mu')} E_{v_\mu \times v_{\mu'}}$,

$$a_0 = 1 - \sum_{\mu=1}^{s} v_\mu a_\mu$$

$$P^{s\times s} \boldsymbol{a}^{s\times 1} = -1^{s\times 1},\ \boldsymbol{a}' = (a_1\, a_2 \ldots a_s)$$

$$P^{s\times s} = (p_{\mu\mu'}),\ p_{\mu\mu'} = (v_\mu + 2)\lambda_4^{(\mu)} - v_\mu \text{ for } \mu = \mu'$$

$$= (\theta^{\mu\mu'} - 1)v_{\mu'} \text{ for } \mu \neq \mu'$$

$$C_1^{(\mu)} = 1/(2\lambda_6^{(\mu)})$$

$$C_3^{(\mu)} = (a_\mu + \sum_{\mu' \neq \mu} v_{\mu'} C_3^{(\mu\mu')} + 1/(2\lambda_4^{(\mu)}))/v_\mu .$$

$$Q \begin{bmatrix} C_3^{\mu 1} \\ \cdot \\ \cdot \\ \cdot \\ C_3^{\mu\mu-1} \\ C_3^{\mu\mu+1} \\ \cdot \\ \cdot \\ \cdot \\ C_3^{\mu s} \end{bmatrix} = a_\mu \begin{bmatrix} \theta^{\mu 1}-1 \\ \cdot \\ \cdot \\ \cdot \\ \theta^{\mu\mu-1}-1 \\ \theta^{\mu\mu+1}-1 \\ \cdot \\ \cdot \\ \cdot \\ C^{\mu s}-1 \end{bmatrix}$$

where $Q = (q_{\mu' \mu''})$,

$q_{\mu'\mu'} = (v_{\mu'}+2)\lambda_4^{(\mu')} - v_{\mu'}\theta^{\mu\mu'}$, $\mu'(\neq \mu) = 1, 2, \ldots, s$

$q_{\mu'\mu''} = v_{\mu''}(\theta^{\mu''\mu'} - \theta^{\mu\mu'})$, $\mu' \neq \mu''(\neq \mu) = 1, 2, \ldots, s$.

For $K_{\mu t}^{-1}$ to be positive definite, the following conditions must be satisfied :

(i) $\lambda_{6}^{(\mu)}/\{\lambda_4^{(\mu)}\}^2 > (v^\mu+2)/(v_\mu+4)$

(ii) $\theta^{\mu\mu'\mu'}/(\theta^{\mu\mu'})^2 < v_{\mu'}/(v_{\mu'}+2)$

(iii) $F_\mu^{-1} = [(v_\mu+4)\lambda_6^{(\mu)} - (v_\mu+2)(\lambda_4^{(\mu)})^2][(v_{\mu'}+2)\theta^{\mu\mu'\mu'} - v_{\mu'}(\theta^{\mu\mu'})^2]$
$\quad - v_{\mu'}(v_\mu+2)(\theta^{\mu'\mu\mu} - \lambda_4^{(\mu)}\theta^{\mu\mu'})^2 > 0.$

Then,

$$K_{\mu t}^{-1} = \begin{bmatrix} f^{(\mu)} & g_\mu^{(\mu)} & g_\mu^{(\mu)}\mathbf{1}(1\times v_{\mu-1}) & g_1^{(\mu)}\mathbf{1}(1\times v_1) & \ldots & g_s^{(\mu)}\mathbf{1}(1\times v_s) \\ & h^{(\mu)} & m_\mu^{(\mu)}\mathbf{1} & m_1^{(\mu)}\mathbf{1} & \ldots & m_s^{(\mu)}\mathbf{1} \\ & & w_{\mu 1}^{(\mu)} I_{v_{\mu-1}} + m_\mu^{(\mu)} E & m_1^{(\mu)} E & \ldots & m_s^{(\mu)} E \\ & & & w_{11}^{(\mu)} I_{v_1} + w_{12}^{(\mu)} E & \ldots & t_{1s}^{(\mu)} E \\ & & & \cdots & \cdots & \cdots \\ & & & & & w_{s1}^{(\mu)} I_{v_s} + w_{s2}^{(\mu)} E \end{bmatrix}$$

The solution for $f^{(\mu)}$, $g_{(\mu)}^{\mu}$'s, $t_{\mu'\mu''}^{(\mu)}$'s, $w_{\mu'1}^{(\mu)}$'s and $m_{(\mu)}^{(\mu)}$'s, $w_{\mu'2}^{(\mu)}$'s are obtained from

$$K_{\mu t} K_{\mu t}^{-1} = I_{v+1}.$$

Unfortunately, in general the solution is complicated. However, it can be easily seen that

$$h^{(\mu)} = m_\mu^{(\mu)} + \left(\frac{1}{6}\right) / \lambda_6^{(\mu)}$$

$$w_{\mu 1}^{(\mu)} = \left(\frac{1}{2}\right) / \lambda_6^{(\mu)}$$

$$w_{\mu' 1}^{(\mu)} = \left(\frac{1}{2}\right) / \theta^{\mu\mu'\mu'}.$$

The variance function is given by

$$V(\hat{y}_u) = \frac{\sigma^2}{N}\Big[a_0 + \sum_{\mu}(f^{(\mu)}+2a_\mu)\rho_\mu^2 + \sum_{\mu}(C_1^{(\mu)}+C_2^{(\mu)})\rho_\mu^4 + \sum_{\mu<\mu'}(2C_3^{\mu\mu'}+\tfrac{1}{2}/\theta^{\mu\mu'})$$

$$\times \rho_\mu^2\rho_{\mu'}^2 + 2\sum_{\mu}\sum_{\mu'} g_{\mu'}^{(\mu)}\rho_\mu^2\rho_{\mu'}^2 + \sum_{\mu}\left(\frac{1}{6}\,/\,\lambda_3^{(\mu)}+m_\mu^{(\mu)}\right)\rho_\mu^6$$

$$+2\sum_{\mu\neq\mu'}\sum m_{\mu'}^{(\mu)}\rho_{\mu'}^2\rho_\mu^4 + \sum_{\mu\neq\mu'}\sum\left(\frac{1}{2}\,/\,\theta^{\mu\mu'\mu'}+w_{\mu'2}^{(\mu)}\right)\rho_\mu^2\rho_{\mu'}^4$$

$$+2\sum_{\mu\neq\mu'\neq\mu''}\sum\sum t_{\mu'\mu''}^{(\mu)}\rho_\mu^2\rho_{\mu'}^2\rho_{\mu''}^2\Big] = f(\rho_1^2, \ldots, \rho_s^2).$$

3. Construction of GDTOBD using balanced block designs (BBD)

Let $N_2 : v\times b$ be the incidence matrix of the BBD (Adhikary, 1965) with the parameters $v, b, k, r_1 r_2 \ldots r_s, \lambda_{\mu\mu'}; \mu, \mu' = 1, 2, \ldots, s$ and $\bar{N}_2 : v\times b$ be the incidence matrix of the complementary design with block size $v-k$

For convenience, we assume that $k \geqslant v-k$.

Let $\otimes$ denote the restricted kronecker product of two matrices of appropriate order as defined by Adhikary and Panda (1977, p. 61). Construct

$$N_{2(1)} : v\times 2^{p_1}b = N_2 \otimes N_{31}, \text{ a matrix of order } v\times 2^{p_1}b$$

and $$N_{2(2)} : v\times 2^{p_2}b = \bar{N}_2 \otimes N_{32}$$

where $N_{3i} : k_i\times 2^{p_i}$, $i = 1, 2$ is the matrix of treatment combinations of a $1/2^{k_i-p_i}$ replicate of 2^{k_i}–expt. with levels ± 1 such that the fraction is of resolution VI with $k_1 = k$ and $k_2 = v-k$. Form

$$N_2^* : v\times 2^{p_1+1}b = (N_{2(1)}, N_{2(1)} \text{ replicated } 2^{p_1-p_2} \text{ times}).$$

Take $$N^* : v\times 2^{p_1+1}b = N_1 \otimes N_2^*$$

where $N_1' = (a_1 \ldots a_1\, a_s \ldots a_s)$, a_μ is repeated v_μ times.

Here v factors are divided into s groups of factors of which μ-th group consists v_μ-factors.

It can be shown that if $i \in G_\mu, j \neq k \in G_{\mu'}$, then

D$_1$(ii) : $$\sum x_{iu}^2 x_{ju}^4 \geqslant 3\sum x_{iu}^2 x_{ju}^2 x_{ku}^2 \Longleftrightarrow 2b \lesseqgtr 2(r_\mu + r_{\mu'} - 2\lambda_{\mu\mu'}) + 3(r_{\mu'} - \lambda_{\mu'\mu'}) = D_{\mu\mu'}, \text{ say}.$$

If $2b = D_{\mu\mu'}$, then condition D$_1$(ii) will be satisfied. But if $2b > D_{\mu\mu'}$, then we add points given by

$$N_{\mu\mu'}^{**} : v\times 4v_\mu v_{\mu'}.$$

where $N^{**}_{\mu\mu'} = N_1 \otimes N_{\mu\mu'}$ with $N_1' = (0 \ldots 0\, b_\mu \ldots b_\mu \ldots b_{\mu'} \ldots b_{\mu'}\, 0 \ldots 0)$

$$N_{\mu\mu'} : v \times 4v_\mu v_{\mu'} = N_{1\mu\mu'} \otimes N_{2\mu\mu'}$$

where $N_{1\mu\mu'} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$.

$N_{2\mu\mu'}$: $v \times v_\mu v_{\mu'}$ is the incidence matrix of the BBD with the parameters $v_1, \ldots, v_s$, $b = v_\mu v_{\mu'}$, $k = 2$, $r_\mu = v_{\mu'}$, $r_{\mu'} = v_{\mu'}$, $r_{\mu''} = 0$ for $\mu''(\neq \mu, \mu') = 1, 2, \ldots, s$, $\lambda_{\mu\mu'} = 1$ and other $\lambda_{\mu\mu'}$'s are zero.

b_μ and $b_{\mu'}$ are choosen so that condition D_1(ii) is satisfied. If $2b < D_{\mu\mu'}$ then $N^{**}_{\mu\mu'} : v \times 2^{p_{\mu\mu'}}$ is constructed as follows :—

$$N_{\mu\mu'} : v_\mu + v_{\mu'} \times 2^{p_{\mu\mu'}} = N_{1\mu\mu'} \otimes N_{2\mu\mu'}$$

where $N_{1\mu\mu'} = (b_\mu \ldots b_\mu\, b_{\mu'} \ldots b_{\mu'})$, b_μ is repeated v_μ times.

$N_{2\mu\mu'} : v_\mu + v_{\mu'} \times 2^{p_{\mu\mu'}}$ is the matrix of treatment combinations of $1/2^{v_\mu + v_{\mu'} - p_{\mu\mu'}}$ replicate of $2^{v_\mu + v_{\mu'}}$—expt. with the levels ± 1, the fraction being of resolution VI.

Let
$$N_{\mu\mu'} = \begin{pmatrix} N_{\mu\mu'(1)} : v_\mu \times 2^{p_{\mu\mu'}} \\ N_{\mu\mu'(2)} : v_{\mu'} \times 2^{p_{\mu\mu'}} \end{pmatrix}$$

Then
$$N^{**}_{\mu\mu'} : v \times 2^{p_{\mu\mu'}} = \begin{pmatrix} 0 : \sum_{1}^{\mu-1} v_\mu \times 2^{p_{\mu\mu'}} \\ N_{\mu\mu'(1)} \\ 0 \\ N_{\mu\mu'(2)} \\ 0 \end{pmatrix}$$

Let $N^{**} = (N^{**}_{\mu\mu'}\ \forall\ \mu \neq \mu'$ for which $2b \neq D_{\mu\mu'})$. Suppose for the points given by the columns of $(N^* N^{**})$, we have

$$\left.\begin{array}{ll} \text{B(i)} & : \sum x_{iu}^2 = \alpha_{\mu 1}, \text{ say} \\ \text{B(ii)} & : \sum x_{iu}^4 = \alpha_{\mu 2} \\ \text{B}_1 & : \sum x_{iu}^6 = \alpha_{\mu 3} \end{array}\right\} \quad i \in G_\mu$$

$$\left.\begin{array}{ll} \text{C} & : \sum x_{iu}^2 x_{ju}^2 = \beta_{\mu\mu'} \\ \text{C}_1\text{(i)} & : \sum x_{iu}^2 x_{ju}^4 = \beta_{\mu\mu'\mu'} \end{array}\right\} \quad i \in G_\mu,\ j \in G_{\mu'}$$

$$\text{C}_1\text{(ii)} : \sum x_{iu}^2 x_{ju}^2 x_{ku}^2 = \nu_{\mu\mu'\mu''} \qquad \text{if } i \in G_\mu,\ j \in G_{\mu'},\ k \in G_{\mu''}.$$

We now add points to sytisfy the conditions D, D_1(i) and D_1(ii). For simplicity we may assume that

$$\Sigma x_{iu}^2 x_{ju}^4 < 3\Sigma x_{iu}^2 x_{ju}^2 x_{ku}^2,\ \Sigma x_{iu}^6 < 5\Sigma x_{iu}^2 x_{ju}^4,\ \Sigma x_{iu}^4 < 3\Sigma x_{iu}^2 x_{ju}^2 \quad \text{for } \mu = 1, 2, \ldots, s_1$$

$$= \qquad\qquad < \qquad\qquad < \qquad \text{for } \mu = s_1+1, \ldots, s_1+s_2$$

$$> \qquad\qquad\qquad\qquad\qquad \text{for } \mu = s_1+s_2+1, \ldots, s.$$

Construct

$$N_\mu^{(1)} = (N_{\mu 1}^{(1)} \ldots N_{\mu t_\mu}^{(1)}),\ \mu = 1, 2, \ldots, s_1, s_1+s_2+1, \ldots, s.$$

where

$$N_{\mu i}^{(1)} : v_\mu \times 4\binom{v_\mu}{2} = \begin{pmatrix} C_{\mu i} & C_{\mu i} & -C_{\mu i} & -C_{\mu i} \\ C_{\mu i} & -C_{\mu i} & C_{\mu i} & -C_{\mu i} \end{pmatrix} \otimes N,\ i = 1, 2, \ldots, t_\mu;$$

$N : v_\mu \times \binom{v_\mu}{2}$ is the incidence matrix of BIB design with the parameters

$$v_\mu\ \binom{v_\mu}{2},\ 2,\ v_\mu - 1,\ 1.$$

Form

$$N^{(1)} = \begin{bmatrix} N_{11} : \sum_1^{s_1} v_\mu \times 4 \sum_1^{s_1} t_\mu \binom{v_\mu}{2} & 0 \\ 0 \ : \sum_{s_1+1}^{s_1+s_2} v_\mu \times 4 \Sigma t_\mu \binom{v_\mu}{2} & 0 \\ 0 & N_{33} : \sum_{s_1+s_2+1}^{s} v_\mu \times 4 \Sigma t_\mu \binom{v_\mu}{2} \end{bmatrix}$$

where

$$N_{11} = \text{Diag}[N_1^{(1)} \ldots N_{s_1}^{(1)}],\ N_{33} = \text{Diag}[N_{s_1+s_2+1}^{(1)} \ldots N_s^{(1)}].$$

Construct

$$N^{(2)} : v \times \sum_{s_1+s_2+1}^{s} w_\mu 2^{q_\mu} = \begin{pmatrix} 0 \\ R : \sum_{s_1+s_2+1}^{s} v_\mu \times \sum_{s_1+s_2+1}^{s} w_\mu 2^{q_\mu} \end{pmatrix}$$

where $R = \text{Diag.}(N_{s_1+s_2+1}^{(2)} \ldots N_s^{(2)})$ with

$$N_\mu^{(2)} : v_\mu \times w_\mu 2^{q_\mu} = (N_{\mu 1}^{(2)} \ldots N_{\mu w_\mu}^{(2)}),\ \mu = s_1+s_2+1, \ldots, s.$$

$N_{\mu k}^{(2)} : v_\mu \times 2^{q_\mu}$ is the matrix of the treatment combinations of a $1/2^{v_\mu - q_\mu}$ replicate of 2^{v_μ}-exp. with levels $\pm d_{\mu k}$ such that fraction is of resolution VI, $k = 1, 2, \ldots, w_\mu$. Also construct

$$N^{(3)} : v \times 2 \sum_1^{s} p_\mu v_\mu = \text{Diag.}(N_1^{(3)} \ldots N_s^{(3)})$$

where $N_\mu^{(3)} = (N_{\mu 1}^{(3)} \ldots N_{\mu p_\mu}^{(3)})$ with $N_{\mu j}^{(3)} = (e_{\mu j} - e_{\mu j}) \otimes I^{v_\mu}$

For the design points given by the columns of $(N^{*}N^{**}\ N^{(1)}N^{(2)}N^{(3)})$, conditions D, D_1(i), D_0(ii) will be satisfied if $C_{\mu i}$'s, $d_{\mu k}$'s and $e_{\mu j}$'s are choosen so that

$$\left.\begin{array}{ll} D_1(ii) & : \beta_{\mu\mu\mu}+4\Sigma C^6_{\mu_i} = 3v_{\mu\mu\mu} \\ D_1(i) & : \alpha_{\mu 3}+4(v_\mu-1)\Sigma C^6_{\mu_i}+2\Sigma e^6_{\mu_j} = 5(\beta_{\mu\mu\mu}+4\Sigma C^6_{\mu_i}) \\ D & : \alpha_{\mu 2}+4(v_\mu-1)\Sigma C^4_{\mu_i}+2\Sigma e^4_{\mu_j} = 3(\beta_{\mu\mu}+4\Sigma C^4_{\mu_i}) \end{array}\right\} \mu = 1, 2, \ldots, s_1$$

$$\left.\begin{array}{ll} D_1(i) & : \alpha_{\mu 3}+2\Sigma e^6_{\mu_j} = 5\beta_{\mu\mu\mu} \\ D & : \alpha_{\mu 2}+2\Sigma e^4_{\mu_j} = 3\beta_{\mu\mu} \end{array}\right\} \mu = s_1+1, \ldots, s_1+s_2$$

and

$$\left.\begin{array}{ll} D_1(ii) & : \beta_{\mu\mu\mu}+4\Sigma C^6_{\mu_i}+2^{q_\mu}\Sigma d^6_{\mu_k} = 3(v_{\mu\mu\mu}+2^{q_\mu}\Sigma d^6_{\mu_k}) \\ D_1(i) & : \alpha_{\mu 3}+2\Sigma e^6_{\mu_j}+4(v_\mu-1)\Sigma C^6_{\mu_i}+2^{q_k}\Sigma d^6_{\mu_k} \\ & = 5(\beta_{\mu\mu\mu}+4\Sigma C^6_{\mu_i}+2^{q_\mu}\Sigma d^6_{\mu_k}) \\ D & : \alpha_{\mu 2}+2\Sigma e^4_{\mu_j}+4(v_\mu-1)\Sigma C^4_{\mu_i}+2^{q_k}\Sigma d^4_{\mu_k} \\ & = 3(\beta_{\mu\mu}+4\Sigma C^4_{\mu_i}+2^{q_\mu}\Sigma d^4_{\mu_k}) \end{array}\right] \mu = s_1+s_2+1, \ldots, s.$$

[For detailed calculation we refer to Technical Report, I.S.I., No. ASC/81/14 by Adhikary and Panda].

Example : Here $N_2 : 4\times 3$ is the incidence matrix of the BBD with the parameters $v_1 = 3,\ v_2 = 1,\ b = 3,\ k = 3,\ r_1 = 2,\ r_2 = 3,\ \lambda_{11} = 1,\ \lambda_{12} = 2$. The blocks of the BBD are (124) (134) (234).

Here $G_1 = (123),\ G_2 = (4)$

$N_{31} : 3\times 8$ is the matrix of 2^3—expt. with level combinations ± 1

$$N^*_2 : 4\times 24 = N_{31}\otimes N_2.$$

Take $N'_1 = (a_1\ a_1\ a_1\ a_2)$. Construct

$$N^*(4\times 24) = N_1\otimes N^*_2.$$

Here $$2b > 2(r_1+r_2-2\lambda_{12})+3(r_1-\lambda_{11}) = 5.$$

So $$\Sigma x^2_{iu}x^4_{ju} < 3\Sigma x^2_{iu}x^2_{ju}x^2_{ku} \quad \text{for } i\in G_2,\ j\neq k\in G_1$$

So to satisfy the condition D_1(ii) for factors belonging to different groups, add 12 points given by N^{**} where

$$N^{**} : 4\times 12 = (b_1\, b_1\, b_1\, b_2)' \otimes N_{12};\ N_{12} = N_{112} \otimes N_{212}$$

where $N_{112} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$.

N_{212} is the incidence matrix of the BBD with parameters $v_1 = 3$, $v_2 = 1$, $b = 3$, $r_1 = 1$, $r_2 = 3$, $\lambda_{11} = 0$, $\lambda_{12} = 1$.

If $a_1^2 = b_1^2$ and $a_2^2 = 2b_2^2$ then

D_1(ii) : $\Sigma x_{iu}^2 x_{ju}^4 = 3\Sigma\, x_{ju}^2 x_{ju}^2 x_{ku}^2$ is satisfied if $i \in G_2$, $j \neq k \in G_1$.

For 36 design points given by the columns, we have

B(i) : $\Sigma\, x_{iu}^2 = 16a_1^2 + 4b_1^2 = 20a_1^2$ if $i \in G_1$

$= 24a_2^2 + 12b_2^2 = 48a_2^2$ if $i = 4$

B(ii) : $\Sigma\, x_{iu}^4 = 16a_1^4 + 4b_1^4 = 20a_1^4$ if $i \in G_1$

$= 24a_2^4 + 12b_2^4 = 72a_2^4$ if $i = 4$

B_1 : $\Sigma\, x_{iu}^6 = 16a_1^6 + 4b_1^6 = 20a_1^6$ if $i \in G_1$

$= 24a_2^6 + 12b_2^6 = 120a_2^6$ if $i = 4$

C : $\Sigma\, x_{iu}^2 x_{ju}^2 = 8a_1^4$ if $i \neq 1 \in G_1$

$= 16a_1^2 a_2^2 + 4b_1^2 b_2^2 = 24a_1^2 a_2^2$, $i \in G_1$, $j = 4$

C_1(i) : $\Sigma\, x_{iu}^2 x_{ju}^4 = 8a_1^6$ if $i \neq j \in G_1$

$= 16a_1^4 a_2^2 + 4b_1^4 b_2^2 = 24a_1^4 a_2^2$ if $j \in G_1$, $i = 4$

$= 16a_1^2 a_2^4 + 4b_1^2 b_2^4 = 32a_1^2 a_2^4$ if $i \in G_1$, $j = 4$

C_1(ii) : $\Sigma\, x_{iu}^2 x_{ju}^2 x_{ku}^2 = 0$ if $i \neq j \neq k \in G_1$

$= 8a_1^4 a_2^2$ if $i \neq j \in G_1$, $k \in G_2$.

So for $i \neq j \neq k \in G_1$, $\Sigma\, x_{iu}^2 x_{ju}^4 > 3\, \Sigma\, x_{iu}^2 x_{ju}^2 x_{ku}^2$.

Take

$$N^{(2)} : 4\times 8 = \begin{pmatrix} N_1^{(2)} : & 3\times 8 \\ 0 \quad : & 1\times 8 \end{pmatrix}$$

where $N_1^{(2)}$ is the design matrix of 2^3—exp. with the levels $\pm d_1$.

$$N^{(3)} : 4\times 6 = \begin{pmatrix} N_1^{(3)} : & 3\times 6 \\ 0 \quad : & 1\times 6 \end{pmatrix}$$

where $N_1^{(3)} = (e_1 - e_1) \otimes I_3$.

For these 50 design points given by the columns of $[N^{*}N^{**}N^{(2)}N^{(3)}]$ and for $d_1^2 = 7968a_1^2$, $e_1^2 = 2{\cdot}64a_1^2$, we have

$$\text{B(i)} \quad : \quad \Sigma x_{iu}^2 = 20a_1^2+8d_1^2+2e_1^2 = 31{\cdot}6544a_1^2 \qquad \text{if } i \in G_1$$
$$= 48a_2^2$$

$$\text{B(ii)} \quad : \quad \Sigma x_{iu}^4 = 39a_1^4 \qquad \text{if } i \in G_1$$
$$= 72a_1^4 \qquad \text{if } i = 4$$

$$\text{B}_1 \quad : \quad \Sigma x_{iu}^6 = 60a_1^6 \qquad \text{if } i \in G_1$$
$$= 120a_2^6 \qquad \text{if } i = 4$$

$$\text{C} \quad : \quad \Sigma x_{iu}^2 x_{ju}^2 = 8a_1^4+8d_1^4 = 13a_1^4 \qquad \text{if } i \neq j \in G_1$$
$$= 24a_1^2a_2^2 \qquad \text{if } i \in G_1,\ j \in G_2$$

$$\text{C}_1\text{(i)} \quad : \quad \Sigma x_{iu}^2 x_{ju}^4 = 8a_1^6+8d_1^6 = 12a_1^6 \qquad \text{if } i \neq j \in G_1$$
$$= 24a_1^4a_2^2 \qquad \text{if } j \in G_1,\ i \in G_2$$
$$= 24a_1^2a_2^4 \qquad \text{if } i \in G_1,\ j \in G_2$$

$$\text{C}_1\text{(iii)} \quad : \quad \Sigma x_{iu}^2 x_{ju}^2 x_{ku}^2 = 8d_1^6 = 4a_1^6 \qquad \text{if } i \neq j \neq k \in G_1$$
$$= 8a_1^4a_2^2 \qquad \text{if } i \neq j \in G_1,\ k \in G_2.$$

It can be easily verified that the non-singularity conditions E and E_1 are also satisfied.

So all the conditions of GDTORD are satisfied. So we have a GDTORD on 4 factors in 50 design points. It may be remarked that the minimum number of design points for a TORD on 4 factors is 72 (Das and Narasimham, 1962; Herzberg, 1964).

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