International Journal of Engineering Science 42 (2004) 1645–1654

International Journal of Engineering Science

Motion due to ring source in ice-covered water

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Received 22 July 2003; accepted 18 October 2003
Available online 31 July 2004

Abstract

Motion due to a horizontal circular ring of wave sources of time-dependent strength submerged in water with an ice-cover, modelled as a thin elastic sheet, is investigated here by constructing the velocity potential. The problem is formulated as an initial value problem which is solved by Laplace transform technique. Water of infinite and uniform finite depth is considered. For the particular case of time-harmonic wave sources, the steady-state development of the potential is also obtained.

1. Introduction

The velocity potentials describing the motion due to two-dimensional line singularity, line multipoles and three-dimensional point sources submerged in water with a *free surface* were constructed by Thorne [1] in a systematic manner. These are useful in the mathematical study of water wave scattering or radiation problems involving obstacles of various geometrical shapes present in water with a free surface. However, if an obstacle is in the form of a vertical body of revolution having a common vertical axis of symmetry with the fluid motion, then one needs to consider potentials due to submerged horizontal circular rings of wave sources since the problem can then be formulated in terms of suitable distribution of rings of wave sources around the body (cf. [2]).

The potential due to a ring of wave sources of constant unit strength in an unbounded fluid is given by

$$\phi_0 = 2\pi a \int_0^\infty e^{-k|y-\eta|} J_0(ka) J_0(kR) \mathrm{d}k \tag{1.1}$$

where a is the radius of the ring with centre at $(0, 0, \eta)$ using a cylindrical co-ordinate system (R, θ, y), y-axis being taken as the axis of the ring. However, in a fluid with a boundary at its upper surface, the potential due to a ring source can be decomposed into two parts, the first part representing the potential due to a ring of wave sources present in an unbounded fluid while the second representing its image in the upper boundary and the bottom, if there be any, conditions.

Hulme [2] constructed the velocity potential due to a horizontal ring of wave sources of time-harmonic strength submerged in deep water with a *free surface* in terms of multi-valued toroidal harmonics. Rhodes-Robinson [3,4] earlier used a reduction technique to obtain the ring source potential for both deep water and finite depth water in the presence of surface tension at the free surface. Mandal and Kundu [5] obtained the velocity potential due to a ring source of time-dependent strength submerged in deep water with an *inertial surface* in the presence of surface tension, the inertial surface being composed of uniformly distributed non-interacting floating material. Here we consider the motion due to a submerged horizontal ring of wave sources of time-dependent strength present in water with an *ice-cover*, the ice-cover being modelled as a thin elastic sheet composed of elastic material of uniform area density. The problem is formulated as an initial value problem for the velocity potential describing the motion in the fluid, and the Laplace transform technique is employed to solve it. Three types of source strengths, namely impulsive initially but zero later, the classical case of constant strength and finally the important case of time-harmonic strength are considered. The steady-state development of the potential function for time-harmonic source strength shows the existence of outgoing progressive waves of *any frequency* under the ice-cover. This is in contrast with the case when the ice-cover is modelled as an *inertial surface* in which case outgoing time-harmonic progressive waves exist under the inertial surface only when the angular frequency is less than a certain constant which depends on the surface density of the inertial surface [6].

2. Mathematical formulation

A cylindrical co-ordinate system (R, θ, y) is chosen in which the y-axis is taken vertically downwards into the water which is assumed to be homogeneous with density ρ and inviscid. The upper surface of water is covered by a thin layer of ice modelled as an elastic sheet having uniform surface density $\epsilon\rho$, Young's modulus E and Poisson's ratio γ, ϵ being a constant having the dimension of length. A horizontal ring of radius a of uniformly distributed point sources, each of the same time-dependent strength $m(t)$, is present at a depth η below the mean position of the ice-cover, taken as the $y = 0$ plane. The axis of the ring coincides with the y-axis. The only external force acting on the system is the gravity g. The motion in water is generated when the point sources on the ring start operating at a given instant simultaneously. Since the motion in water starts from rest, it is irrotational and can be described by a potential function $\phi(R, y, t)$. Then ϕ satisfies

$$\frac{1}{R}(R\phi_R)_R + \phi_{yy} = 0 \tag{2.1}$$

in the fluid region except at points on the ring. If $\zeta(R,t)$ denotes the depression of the ice-cover below its mean position, then the linearised kinematic and dynamic conditions on the ice-cover are given by

$$\phi_y = \zeta_t \text{ on } y = 0 \tag{2.2}$$

and

$$(\phi - \epsilon\phi_y)_t = (D\nabla_R^4 + 1)g\zeta \text{ on } y = 0 \tag{2.3}$$

where $D = \frac{Eh_0^3}{12(1-\nu^2)\rho g}$ is a constant, h_0 being the very small thickness of the ice-cover and $\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial}{\partial R})$. Elimination of ζ between (2.2) and (2.3) produces the linearised ice-cover condition

$$(\phi - \epsilon\phi_y)_{tt} = (D\nabla_R^4 + 1)g\phi_y \text{ on } y = 0. \tag{2.4}$$

The initial conditions at the ice-cover are

$$\phi - \epsilon\phi_y = 0,\ (\phi - \epsilon\phi_y)_t = 0 \text{ on } y = 0 \text{ at } t = 0 \tag{2.5}$$

which are obtained due to continuity of ζ for all times. Also, ϕ must satisfy the bottom condition

$$\nabla\phi \to 0 \text{ as } y \to \infty \tag{2.6a}$$

for deep water, or

$$\phi_y = 0 \text{ on } y = h \tag{2.6b}$$

for water of uniform finite depth h. Also, at points near the ring

$$\phi \to m(t)\phi_0 \text{ as } \{(R-a)^2 + (y-\eta)^2\}^{1/2} \to 0 \tag{2.7}$$

where ϕ_0 is given by (1.1).

It may be noted that for time-harmonic motion of angular frequency σ, the ice-cover condition (2.4) becomes

$$K\phi + \left(D\nabla_R^4 + 1 - \epsilon K\right)\phi_y = 0 \text{ on } y = 0 \tag{2.8}$$

where $K = \sigma^2/g$. If ϕ has the time-harmonic progressive wave form given by

$$\phi = \mathrm{Re}\{\mathrm{e}^{-ky}H_0^{(1),(2)}(kR)\mathrm{e}^{-\mathrm{i}\sigma t}\}$$

for deep water, or

$$\phi = \mathrm{Re}\{\cosh k(h-y) H_0^{(1),(2)}(kR)\mathrm{e}^{-\mathrm{i}\sigma t}\}$$

for water of uniform finite depth h, then k satisfies the polynomial equation

$$\Delta(k) \equiv k(Dk^4 + 1 - \epsilon K) - K = 0 \tag{2.9}$$

for deep water, or the transcendental equation

$$\Delta_0(k) \equiv k(Dk^4 + 1 - \epsilon K)\sinh kh - K\cosh kh = 0 \tag{2.10}$$

for finite depth water. It can be easily verified that the nature of the zeros of $\Delta(k)$ and $\Delta_0(k)$ remains the same whether $1 - \epsilon K$ is positive or negative so long as $D \neq 0$, and that both $\Delta(k)$ and $\Delta_0(k)$ possess a unique positive real zero.

For $\Delta(k)$ we denote its positive real zero by λ. The other zeros of $\Delta(k)$ are two pairs of complex conjugate numbers denoted by $(\lambda_1, \overline{\lambda}_1)$ and $(\lambda_2, \overline{\lambda}_2)$ where $\mathrm{Re}\lambda_1 > 0$, $\mathrm{Im}\lambda_1 > 0$ and $\mathrm{Re}\lambda_2 < 0$, $\mathrm{Im}\lambda_2 > 0$. Chakrabarti et al. [7] gave an elementary proof for the nature of the zeros of $\Delta(k)$ for $\epsilon = 0$. However, for $\epsilon \neq 0$, the same elementary proof can be used to find the nature of the zeros of $\Delta(k)$ with obvious modifications.

Again, for $\Delta_0(k)$ we denote its positive real zero by μ. It can be shown that $\Delta_0(k)$ has a negative real zero at $k = -\mu$, two pairs of complex conjugate roots $\mu_1, \overline{\mu}_1$ and $-\mu_1, -\overline{\mu}_1$ with $Re\mu_1 > 0$, $\mathrm{Im}\mu_1 > 0$ and $\mathrm{Re}\mu_1 < \mathrm{Im}\mu_1$, and an infinite number of purely imaginary roots $\pm \mathrm{i}\alpha_n (\alpha_n > 0, n = 1, 2, \ldots)$ where $\alpha_n h \to n\pi$ as $n \to \infty$ (see [8]).

For the case $D = 0$, the ice-cover is no longer modelled as an elastic plate, and it becomes an *inertial surface*, and the ice-cover (inertial surface) condition becomes

$$K\phi + (1 - \epsilon K)\phi_y = 0. \tag{2.11}$$

This shows that progressive wave is possible only when $1 - \epsilon K > 0$ i.e. $\sigma < (g/\epsilon)^{1/2}$ (cf. [6]). For $\sigma \geqslant (g/\epsilon)^{1/2}$, the form (2.11) does not allow any progressive wave.

3. Solution

To solve the initial value problem for ϕ described above, we use Laplace transform defined by

$$\overline{\phi}(R, y, p) = \int_0^\infty \phi(R, y, t)\mathrm{e}^{-pt}\,\mathrm{d}t, \;\; p > 0, \tag{3.1}$$

then, $\overline{\phi}$ satisfies the boundary value problem described by

$$\frac{1}{R}(R\overline{\phi}_R)_R + \overline{\phi}_{yy} = 0 \tag{3.2}$$

in the fluid region except at points on the ring,

$$\overline{\phi} \to \overline{m}(p)\phi_0 \text{ as } \{(R-a)^2 + (y-\eta)^2\}^{1/2} \to 0, \tag{3.3}$$

$$p^2\overline{\phi} - \left(D\nabla_R^4 + 1 + \frac{\epsilon p^2}{g}\right)g\overline{\phi}_y = 0 \text{ on } y = 0, \tag{3.4}$$

$$\nabla\overline{\phi} \to 0 \text{ as } y \to \infty \tag{3.5a}$$

for deep water, or

$$\overline{\phi}_y = 0 \text{ on } y = h \tag{3.5b}$$

for finite depth water.

We now consider the cases of deep water and finite depth water separately.

Case (a): deep water

A solution for $\overline{\phi}$ satisfying (3.2), (3.3) and (3.5a) is constructed as

$$\overline{\phi}(R, y, p) = \overline{m}(p)\left\{\phi_0 + \int_0^\infty A(k)\mathrm{e}^{-ky}J_0(kR)\mathrm{d}k\right\} \tag{3.6}$$

where $A(k)$ is an unknown function of k to be determined such that the integral in (3.6) is convergent. Using the form of ϕ_0 given in (1.1), it is seen that the condition (3.4) is satisfied if we choose

$$A(k) = \frac{2\pi a J_0(ka)\{gk(1 + Dk^4) - (1 - \epsilon k)p^2\}\mathrm{e}^{-k\eta}}{(1 + \epsilon k)p^2 + gk(1 + Dk^4)}. \tag{3.7}$$

Thus $\overline{\phi}(R, y, p)$ in this case is obtained as

$$\overline{\phi}(R, y, p) = \overline{m}(p)X(R, y) + \overline{m}(p)\int_0^\infty \frac{\Omega^2}{\Omega^2 + p^2}Y(R, y; k)\mathrm{d}k \tag{3.8}$$

where

$$X(R, y) = 2\pi a\int_0^\infty \left\{\mathrm{e}^{-k|y-\eta|} - \frac{1 - \epsilon k}{1 + \epsilon k}\mathrm{e}^{-k(y+\eta)}\right\}J_0(ka)J_0(kR)\mathrm{d}k,$$

$$Y(R, y; k) = 4\pi a\frac{J_0(ka)}{1 + \epsilon k}J_0(kR)\mathrm{e}^{-k(y+\eta)} \tag{3.9}$$

and

$$\Omega^2(k) = \frac{gk(1 + Dk^4)}{1 + \epsilon k}. \tag{3.10}$$

Laplace inversion of (3.8) produces

$$\phi(R, y, t) = m(t)X(R, y) + \int_0^\infty \Omega(k)Y(R, y; k) \times \left\{\int_0^t m(\tau)\sin\Omega(t - \tau)\mathrm{d}\tau\right\}\mathrm{d}k. \tag{3.11}$$

For a ring source of *impulsive* strength we take $m(t) = \delta(t)$, and in this case (3.11) produces

$$\phi^{\text{imp}}(R,y,t) = \delta(t)X(R,y) + \int_0^\infty \Omega(k)Y(R,y;k)\sin\Omega t\,\mathrm{d}k \tag{3.12}$$

For large t, the expression in (3.12) vanishes. This has the interpretation that since the sources around the ring act instantaneously at $t = 0$, they have no effect on the fluid motion after a long lapse of time.

For a ring source of *constant* strength $m(t) = 1$, (3.11) gives

$$\phi^{\text{const}}(R,y,t) = X(R,y) + \int_0^\infty Y(R,y;k)(1-\cos\Omega t)\mathrm{d}k. \tag{3.13}$$

For a ring of wave sources of *time-harmonic* strength, we take $m(t) = \sin\sigma t$ where σ is the circular frequency. In this case, (3.11) gives

$$\phi(R,y,t) = \sin\sigma t X(R,y) + \int_0^\infty \Omega(k)Y(R,y;k)\frac{\Omega\sin\sigma t - \sigma\sin\Omega t}{\Omega^2-\sigma^2}\mathrm{d}k \tag{3.14}$$

To determine the form of (3.14) as $t \to \infty$, we introduce a Cauchy principal value at $k = \lambda$ which is the real positive zero of $\Omega^2 - \sigma^2$ i.e. $\Delta(k)$, in the integral in (3.14), and following Rhodes-Robinson [6], we obtain, as $t \to \infty$

$$\phi \to 2\pi a\sin\sigma t ⨍_0^\infty \left\{ \mathrm{e}^{-k|y-\eta|} + \frac{k(Dk^4+1-\epsilon K)+K}{\Delta(k)}\mathrm{e}^{-k(y+\eta)} \right\} J_0(ka)J_0(kR)\mathrm{d}k$$
$$-4\pi^2 a\cos\sigma t\frac{\lambda(D\lambda^4+1-\epsilon K)}{1-\epsilon K+5D\lambda^4}\mathrm{e}^{-\lambda(y+\eta)}J_0(\lambda a)J_0(\lambda R) \tag{3.15}$$

where the integral is in the sense of Cauchy principal value. This integral can be simplified by using the relation $2J_0(kR) = H_0^{(1)}(kR) + H_0^{(2)}(kR)$, and rotating the contour in the complex k-plane for the integral involving $H_0^{(1)}(kR)$ in the first quadrant and for the integral involving $H_0^{(2)}(kR)$ in the fourth quadrant. Thus an alternative representation for the expression in (3.15) is given by

$$\begin{aligned}\phi \to\ & 8a\sin\sigma t\int_0^\infty \frac{L(k,y)L(k,\eta)I_0(ka)}{k^2(1-\epsilon K+Dk^4)^2+K^2}K_0(kR)\mathrm{d}k \\ & + 2\pi^2 \mathrm{i}a\sin\sigma t\left\{ f(y,\eta;\lambda_1)H_0^{(1)}(\lambda_1,R) - f(y,\eta;\overline{\lambda}_1)H_0^{(2)}(\overline{\lambda}_1 R)\right\} \\ & - 2\pi^2 af(y,\eta;\lambda)\{\sin\sigma t Y_0(\lambda R) + \cos\sigma t J_0(\lambda R)\}\end{aligned} \tag{3.16}$$

where

$$L(k,y) = k(1-\epsilon K+Dk^4)\cos ky - K\sin ky, \tag{3.17}$$

$$f(y,\eta;k) = \frac{2k(1-\epsilon K + Dk^4)}{1-\epsilon K + 5Dk^4} \mathrm{e}^{-k(y+\eta)} J_0(ka). \tag{3.18}$$

It may be noted that the second term in the expression in (3.16) is real. For large R, we find from (3.16) that as $t \to \infty$.

$$\phi \to -4\pi^2 a \frac{\lambda(1-\epsilon K + D\lambda^4)}{1-\epsilon K + 5D\lambda^4} \mathrm{e}^{-\lambda(y+\eta)} J_0(\lambda a) \left(\frac{2}{\pi\lambda R}\right)^{1/2} \cos\left(\lambda R - \sigma t - \frac{\pi}{4}\right). \tag{3.19}$$

This shows that ϕ represents outgoing progressive waves as $R \to \infty$.

Case (b): finite depth water

In this case a solution for $\overline{\phi}$ satisfying (3.2) and (3.3) is constructed as

$$\overline{\phi} = \overline{m}(p)\left[\phi_0 - 2\pi a \int_0^\infty \mathrm{e}^{-k(y+\eta)} J_0(ka) J_0(kR)\mathrm{d}k + \int_0^\infty \{B(k)\cosh k(h-y) + C(k)\sinh ky\} \frac{J_0(ka)}{\cosh kh} J_0(kR)\mathrm{d}k\right] \tag{3.20}$$

where the functions $B(k)$ and $C(k)$, for the satisfaction of the conditions (3.4) and (3.5b), are chosen as

$$B(k) = 4\pi a \frac{gk\left(1 + Dk^4 + \epsilon\frac{p^2}{g}\right)}{M(k)(\Omega_0^2 + p^2)} \cosh k(h-\eta),$$

$$C(k) = 4\pi a \mathrm{e}^{-kh} \sinh k\eta \tag{3.21}$$

with

$$M(k) = \cosh kh + \epsilon k \sinh kh,$$

$$\Omega_0^2(k) = \frac{gk(1+Dk^4)\sinh kh}{M(k)}. \tag{3.22}$$

Thus $\overline{\phi}$ is obtained as

$$\overline{\phi} = \overline{m}(p)P(R,y) + \overline{m}(p)\int_0^\infty \frac{\Omega_0^2}{\Omega_0^2 + p^2} Q(R,y;k)\mathrm{d}k \tag{3.23}$$

where

$$P(R,y) = 2\pi a \int_0^\infty \left[\mathrm{e}^{-k|y-\eta|} - \mathrm{e}^{-k(y+\eta)} + \frac{2}{\cosh kh}\left\{\frac{\epsilon k}{M(k)} \cosh k(h-y)\cosh k(h-\eta) + \mathrm{e}^{-kh}\sinh ky \sinh k\eta\right\}\right] J_0(ka) J_0(kR)\mathrm{d}k \tag{3.24}$$

and

$$Q(R,y;k) = 4\pi a \frac{\cosh k(h-y)\cosh k(h-\eta)}{M(k)\sinh kh} J_0(ka)J_0(kR). \tag{3.25}$$

Laplace inversion of (3.23) gives

$$\phi(R,y,t) = m(t)P(R,y) + \int_0^\infty \Omega_0(k)Q(R,y;k)\left\{\int_0^t \sin\Omega_0(t-\tau)m(\tau)d\tau\right\}\mathrm{d}k. \tag{3.26}$$

For *impulsive* source strength $m(t) = \delta(t)$, and (3.26) gives

$$\phi^{\mathrm{imp}}(R,y,t) = \delta(t)P(R,y) + \int_0^\infty \Omega_0(k)Q(R,y;k)\sin\Omega_0 t\mathrm{d}t \tag{3.27}$$

which tends to zero as $t\to\infty$, as in the case of deep water.

For *constant* source strength $m(t) = 1$, and (3.26) gives

$$\phi^{\mathrm{c}}(R,y,t) = P(R,y) + \int_0^\infty Q(R,y;k)(1-\cos\Omega_0 t)\mathrm{d}k. \tag{3.28}$$

For *time-harmonic* source strength $m(t) = \sin\sigma t$ and in this case (3.26) produces

$$\phi(R,y,t) = \sin\sigma t P(R,y) + \int_0^\infty \Omega_0(k)Q(R,y;k)\frac{\Omega_0\sin\sigma t - \sigma\sin\Omega_0 t}{\Omega_0^2-\sigma^2}\mathrm{d}k. \tag{3.29}$$

As in the case of deep water, the steady-state development of ϕ, given by (3.29), can be obtained by introducing a Cauchy principal value at $k=\mu$ which is the real positive zero of $\Omega_0^2-\sigma^2$ i.e. $\Delta_0(k)$, in the integral in (3.29). Then as $t\to\infty$, we find

$$\begin{aligned}\phi \to \sin\sigma t\left[P(R,y) + 2\fint_0^\infty \frac{k(1+Dk^4)}{\Delta_0(k)M(k)}\cosh k(h-y)\cosh k(h-\eta)J_0(ka)J_0(kR)\mathrm{d}k\right]\\ - 8\pi^2 a\mu\frac{(1-\epsilon K+D\mu^4)\cosh\mu(h-y)\cosh\mu(h-\eta)J_0(\mu a)J_0(\mu R)}{2\mu h(1-\epsilon K+D\mu^4)+(1-\epsilon K+5D\mu^4)\sinh 2\mu h}\end{aligned} \tag{3.30}$$

where the integral is in the sense of CPV.

In the right-hand side of (3.30), combining the integral representation of $P(R,y)$ given in (3.24) and the CPV integral and changing the contour along the real axis with indentations above the pole at $k=-\mu$ and below the pole at $k=\mu$, the following alternative representation is obtained:

$$\begin{aligned}\phi \to 8\pi a\sin\sigma t\sum_{n=1}^\infty g(y,\eta;\mathrm{i}\alpha_n)K_0(\alpha_n R) + 4\pi^2\mathrm{i}a\sin\sigma t\{g(y,\eta;\mu_1)H_0^{(1)}(\mu_1 R)\\ - g(y,\eta;\overline{\mu}_1)H_0^{(2)}(\overline{\mu}_1 R)\} - 4\pi^2 a g(y,\eta;\mu)\{\sin\sigma t Y_0(\mu R) + \cos\sigma t J_0(\mu R)\}\end{aligned} \tag{3.31}$$

where

$$g(y,\eta;k) = \frac{2k(1-\epsilon K + Dk^4)\cosh k(h-y)\cosh k(h-\eta)J_0(ka)}{(1-\epsilon K + 5Dk^4)\sinh 2kh + (1-\epsilon K + Dk^4)2kh} \tag{3.32}$$

and the second term in (3.31) is real. For large R, we find that, as $t \to \infty$

$$\phi \to -8\pi^2 a\mu \frac{(1-\epsilon K + D\mu^4)\cosh \mu(h-y)\cosh \mu(h-\eta)J_0(\mu a)}{(1-\epsilon K + 5D\mu^4)\sinh 2\mu h + (1-\epsilon K + D\mu^4)2\mu h}\left(\frac{2}{\pi\mu R}\right)^{1/2}\cos\left(\mu R - \sigma t - \frac{\pi}{4}\right). \tag{3.33}$$

This shows that ϕ represents progressive outgoing waves as $R \to \infty$.

4. Conclusion

The velocity potential due to a horizontal circular ring of wave sources of time-dependent strength submerged in water with an ice-cover has been obtained for both infinite and finite depth of water. For the case of time-harmonic sources, the steady-state development of the potential function shows the existence of outgoing progressive waves at large distances from the ring source. If the elastic parameter D is put equal to zero, then the results for deep water coincide with the results obtained in [5] for deep water with an inertial surface in the absence of surface tension. If both D and ϵ are put equal to zero, then the results obtained above can be identified with the results obtained earlier in [3]. The effect of surface tension at the ice-cover can be incorporated in the above results.

Acknowledgement

This work is supported by CSIR, New Delhi.

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