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Numerical solution of some classes of integral equations using Bernstein polynomials

B.N. Mandal [a,*], Subhra Bhattacharya [b]

[a] *Physics and Applied Mathematics Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata 700 108, India*
[b] *Department of Mathematics, Jadavpur University, Kolkata 700 032, India*

Abstract

This paper is concerned with obtaining approximate numerical solutions of some classes of integral equations by using Bernstein polynomials as basis. The integral equations considered are Fredholm integral equations of second kind, a simple hypersingular integral equation and a hypersingular integral equation of second kind. The method is explained with illustrative examples. Also, the convergence of the method is established rigorously for each class of integral equations considered here.

Keywords: Fredholm integral equation; Hypersingular integral equation; Bernstein polynomials

1. Introduction

Bernstein polynomials have been used recently to solve some linear as well as nonlinear differential equations approximately by Bhatti and Bracken [1] and Bhatta and Bhatti [2]. References of other works in which these polynomials have been used can be found in Bhatti and Bracken [1]. These polynomials defined on an interval form a complete basis over the interval. Each of these polynomials are positive and their sum is unity.

There exists in the literature a number of approximate methods for solving numerically various classes of integral equations. Here, a numerical method for solving some classes of integral equations by approximating the solution in the Bernstein polynomial basis is proposed. Two classes of integral equations is considered, one class involves second kind Fredholm integral equations with regular kernel while the other class involves hypersingular kernels.

As illustrative examples, two Fredholm integral equations of second kind, a simple hypersingular integral equation and a second kind hypersingular integral equation, whose exact solutions are known, have been considered for approximate numerical solutions. Numerical solution of each equation based on the exact and approximate solutions are compared, and excellent agreement is seen to have been achieved. The absolute difference between the exact and approximate solutions for each example is plotted graphically to determine the accuracy of numerical solutions. Also convergence of the method for the two classes of integral equations is established rigorously.

2. The general method

We consider the integral equation of first kind given by

$$\int_a^b \kappa(x,t)\phi(t)\mathrm{d}t = f(x), \quad a < x < b, \tag{2.1}$$

where $\phi(t)$ is an unknown function to be determined, $\kappa(x,t)$ is the kernel and $f(x)$ is a known function. If $\kappa(x,t)$ is a continuous and square integrable function, then the integral in (2.1) will be assumed to exist in the usual sense. However, if $\kappa(x,t)$ has singularity at $t = x$ within the range (a,b), then the integral in (2.1) will be assumed to be defined in some appropriate manner.

To find an appropriate solution of (2.1), $\phi(t)$ is approximated in the Bernstein polynomial basis in $[a,b]$ as

$$\phi(t) = \sum_{i=0}^{n} a_i B_{i,n}(t), \tag{2.2}$$

where $B_{i,n}(x)(i = 0,1,2,\ldots,n)$ are Bernstein polynomials of degree n defined on $[a,b]$ as

$$B_{i,n}(x) = \binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^n}, \quad i = 0,1,2,\ldots,n \tag{2.3}$$

and $a_i(i = 0,1,\ldots,n)$ are unknown constants to be determined. Substituting (2.2) in (2.1), we obtain

$$\sum_{i=0}^{n} a_i \int_a^b \kappa(x,t)B_{i,n}(t)\mathrm{d}t = f(x), \quad a < x < b. \tag{2.4}$$

Multiplying both sides by $B_{j,n}(x)(j = 0,1,\ldots,n)$ and integrating both sides with respect to x between $x = a$ and $x = b$, we obtain the linear system

$$\sum_{i=0}^{n} a_i c_{ij} = b_j, \quad j = 0,1,2\ldots,n, \tag{2.5}$$

where

$$c_{ij} = \int_a^b \left[\int_a^b \kappa(x,t)B_{i,n}(t)\mathrm{d}t\right] B_{j,n}(x)\mathrm{d}x, \quad i,j = 0,1,\ldots,n \tag{2.6}$$

and

$$b_j = \int_a^b f(x)B_{j,n}(x)\mathrm{d}x. \tag{2.7}$$

The linear system (2.5) can be solved by any standard method to produce $a_i(i = 0,1,\ldots,n)$. These a_is when substituted in (2.2) produce $\phi(t)$ approximately.

Instead of the first kind integral equation (2.1), if we have the second kind integral equation given by

$$a(x)\phi(x) + \int_a^b \kappa(x,t)\phi(t)\mathrm{d}t = f(x), \quad a < x < b \tag{2.8}$$

then the modification is obvious. In this case c_{ij} in (2.5) is given by

$$c_{ij} = \int_a^b \left[a(x) + \int_a^b \kappa(x,t)B_{i,n}(t)\mathrm{d}t\right] B_{j,n}(x)\mathrm{d}x, \quad i,j = 0,1,\ldots,n \tag{2.9}$$

while $b_j(j = 0,1,\ldots,n)$ remains the same as in (2.7).

If $\kappa(x,t)$ is hypersingular at $t=x$ in the sense that $\kappa(x,t)=\frac{L(x,t)}{(x-t)^2}$ where $L(x,t)$ is regular in x and t, then the integral in (2.1) or (2.8) exists in the sense of Hadamard finite part of order two. In this case, $\phi(x)$ must be such that it vanishes at the end points and has the behavior

$$\phi(x)\sim|x-c|^{\frac{1}{2}},\quad c=a,b.$$

Then $\phi(x)$ can be written as $\phi(x)=\{(x-a)(b-x)\}^{\frac{1}{2}}\Psi(x)$, and $\Psi(t)$ will be assumed to have representation in the form (2.2). The aforesaid method then can be applied with appropriate modification. The details are given in the following section when two types of hypersingular integral equations are considered for finding approximate numerical solutions.

3. Illustrative examples

Here, we illustrate the above method (or its modification) to obtain approximate numerical solutions of two Fredholm integral equations of second kind and two hypersingular integral equations, one of the first kind known as simple hypersingular integral equation and the other is of the second kind.

Example 1. We consider a Fredholm integral of the second kind given by

$$\phi(x)-\int_{-1}^{1}(xt+x^2t^2)\phi(t)\mathrm{d}t=1,\quad -1\leqslant x\leqslant 1 \tag{3.1}$$

having the exact solution (cf. [3, p. 12])

$$\phi(x)=1+\frac{10}{9}x^2. \tag{3.2}$$

Using the method illustrated in Section 2, if we approximate $\phi(x)$ as

$$\phi(x)=\sum_{i=0}^{n}a_iB_{i,n}(x), \tag{3.3}$$

then $a_i(i=0,1,\ldots,n)$ satisfy the linear system

$$\sum_{i=0}^{n}a_ic_{ij}=b_j,\quad j=0,1,\ldots,n, \tag{3.4}$$

where

$$\begin{aligned}c_{ij}&=\int_{-1}^{1}B_{i,n}(x)B_{j,n}(x)\mathrm{d}x-\int_{-1}^{1}\left\{\int_{-1}^{1}(xt+x^2t^2)B_{i,n}(t)\mathrm{d}t\right\}B_{j,n}(x)\mathrm{d}x\\&=\frac{1}{2^{2n}}\binom{n}{i}\binom{n}{j}\left[\sum_{k=0}^{2n}\frac{1+(-1)^k}{1+k}d_k^{i+j,2n}-\sum_{k=0}^{n}\sum_{r=0}^{n}\left\{\frac{1-(-1)^k}{2+k}\frac{1-(-1)^r}{2+r}+\frac{1+(-1)^k}{3+k}\frac{1+(-1)^r}{3+r}\right\}d_k^{i,n}d_r^{j,n}\right]\end{aligned} \tag{3.5}$$

with

$$d_k^{i,n}=\sum_{s}(-1)^{k-s}\binom{i}{s}\binom{n-i}{k-s}, \tag{3.6}$$

the summation over s being taken as follows: for $i<n<n-i$, (i) $s=0$ to k for $k\leqslant i$, (ii) $s=0$ to i for $i<k\leqslant n-i$, (iii) $s=k-(n-i)$ to $n-i$ for $n-i<k\leqslant n$ while for $i=n-i$ (n being an even integer) (i) $s=0$ to k for $k\leqslant i$, (ii) $s=k-i$ to i for $i<k\leqslant n$; for $i>n-i$, i and $n-i$ above are to be interchanged, and

$$b_j=\frac{1}{2^n}\binom{n}{j}\sum_{k=0}^{n}\frac{1+(-1)^k}{1+k}d_k^{j,n}. \tag{3.7}$$

The linear system (3.4) can be solved for $a_i (i = 0, 1, \ldots, n)$ by standard method and hence $\phi(x)$ is obtained approximately. In our numerical calculations, n is chosen as 4 and $a_0, a_1, \ldots, a_4$ are obtained numerically. Values of $\phi(x)$, calculated by using the approximate expression (3.3) for $n = 4$, and also calculated by using the exact expression (3.2) at the points $x = 0, \pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8$ are presented in Table 1. It is seen that the approximate and exact values coincide up to five decimal places. In Fig. 1, a plot of the absolute difference between the exact and approximate solutions is displayed. It is observed from this figure that the accuracy is of the order 10^{-13} with only five Bernstein polynomials.

Example 2. We consider another Fredholm integral equation of the second kind given by

$$\phi(x) - \int_{-1}^{1} (x^4 - t^4)\phi(t)\mathrm{d}t = x, \quad -1 \leqslant x \leqslant 1 \tag{3.8}$$

having the exact solution (cf. [4])

$$\phi(x) = x. \tag{3.9}$$

If $\phi(x)$ is approximated by (3.3), then $a_i (i = 0, 1, \ldots, n)$ satisfy the linear system (3.4) where now

$$c_{ij} = \frac{1}{2^{2n}} \binom{n}{i} \binom{n}{j} \left[\sum_{k=0}^{2n} \frac{1 + (-1)^k}{1 + k} d_k^{i+j,2n} - \sum_{k=0}^{n} \sum_{r=0}^{n} \left\{ \frac{1 + (-1)^k}{5 + k} \frac{1 + (-1)^r}{1 + r} - \frac{1 + (-1)^k}{1 + k} \frac{1 + (-1)^r}{5 + r} \right\} d_k^{i,n} d_r^{j,n} \right], \tag{3.10}$$

$d_k^{i,n}$ being same as in (3.6) above, and

$$b_j = \frac{1}{2^n} \binom{n}{j} \sum_{k=0}^{n} \frac{1 - (-1)^k}{2 + k} d_k^{j,n}. \tag{3.11}$$

Table 1
Approximate and exact solutions of Eq. (3.1)

x	0	± 0.2	± 0.4	± 0.6	± 0.8
$\phi(x)$ (approx)	0.99999	1.04444	1.17777	1.40000	1.71111
$\phi(x)$ (exact)	1.00000	1.04444	1.17777	1.40000	1.71111

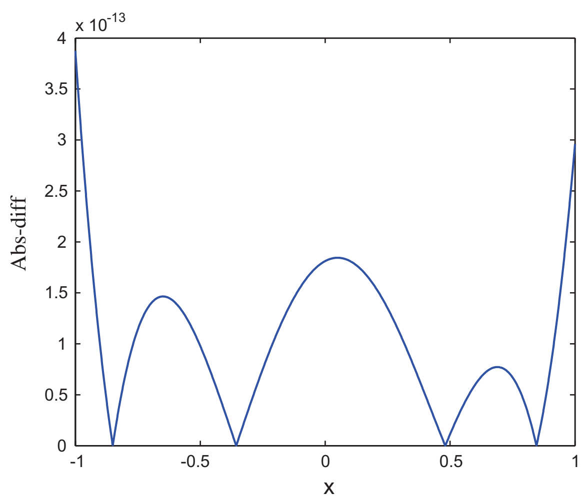

Fig. 1. Absolute difference between exact and approximate solutions of Eq. (3.1).

Choosing $n = 4$, $a_i (i = 0, 1, \ldots, 4)$ are obtained here and thus $\phi(x)$ is obtained approximately. In Table 2, approximate and exact values of $\phi(x)$ at $x = 0, \pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8$ are shown. It is seen that the approximate and exact values almost coincide. In Fig. 2, the absolute difference between the exact and approximate solution is plotted. This figure shows that here also the accuracy is of order 10^{-13}.

Example 3. Here, we consider a simple hypersingular integral equation of the form

$$\oint_{-1}^{1} \frac{\phi(t)}{(t-x)^2} \mathrm{d}t = f(x), \quad -1 \leqslant x \leqslant 1 \tag{3.12}$$

with the additional requirement that $\phi(\pm 1) = 0$. In (3.12), the integral is in the sense of Hadamard finite part of order 2 and is defined by

$$\oint_{-1}^{1} \frac{\phi(t)}{(t-x)^2} \mathrm{d}t = \lim_{\epsilon \to 0+} \left[\int_{-1}^{x-\epsilon} \frac{\phi(t)}{(t-x)^2} \mathrm{d}t + \int_{x+\epsilon}^{1} \frac{\phi(t)}{(t-x)^2} \mathrm{d}t - \frac{\phi(x-\epsilon) + \phi(x+\epsilon)}{\epsilon} \right], \quad -1 < x < 1. \tag{3.13}$$

The exact solution of (3.12) is given by (cf. [5], [6]):

$$\phi(x) = \frac{1}{\pi^2} \int_{-1}^{1} f(t) \ln \left| \frac{x-t}{1 - xt - \{(1-x^2)(1-t^2)\}^{\frac{1}{2}}} \right| \mathrm{d}t, \quad -1 \leqslant x \leqslant 1. \tag{3.14}$$

For the special case when $f(x) = 1$, the exact solution of (3.12) is found to be

$$\phi(x) = -\frac{1}{\pi} (1 - x^2)^{\frac{1}{2}}. \tag{3.15}$$

To use Bernstein polynomials to solve (3.12) we first represent the unknown function $\phi(x)$ as

$$\phi(x) = (1 - x^2)^{\frac{1}{2}} \Psi(x), \quad -1 \leqslant x \leqslant 1 \tag{3.16}$$

Table 2
Approximate and exact solutions of Eq. (3.8)

x	0	±0.2	±0.4	±0.6	±0.8
$\phi(x)$ (approx)	0	±0.20000	±0.40000	±0.60000	±0. 80,000
$\phi(x)$ (exact)	0	±0.2	±0.4	±0.6	±0.8

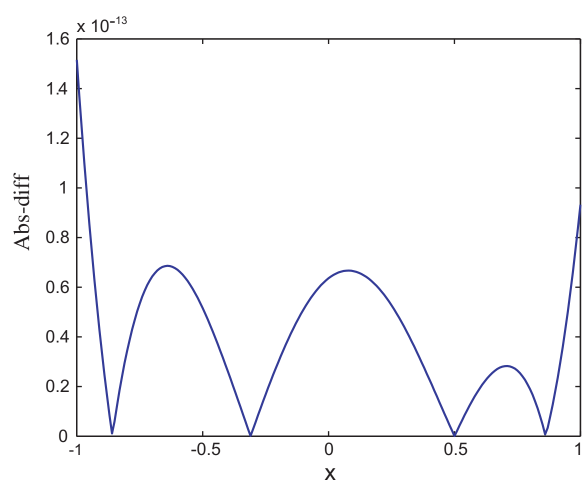

Fig. 2. Absolute difference between exact and approximate solutions of Eq. (3.8).

where $\Psi(x)$ is a well-behaved unknown function of $x \in [-1, 1]$. The representation (3.16) is chosen due to the known end point behaviors of $\phi(x)$. Now $\Psi(x)$ is approximated in terms of Bernstein polynomials in the form

$$\Psi(x) = \sum_{i=0}^{n} a_i B_{i,n}(x), \quad -1 \leqslant x \leqslant 1. \tag{3.17}$$

Then Eq. (3.12) produces the relation

$$\sum_{i=0}^{n} a_i A_i(x) = f(x), \quad -1 \leqslant x \leqslant 1, \tag{3.18}$$

where

$$A_i(x) = \frac{1}{2^n}\binom{n}{i}\sum_{k=0}^{n} d_k^{i,n}\left[-\pi(k+1)x^k + \sum_{m=0}^{k-2}\frac{1+(-1)^m}{4}\frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{m+4}{2})}(k-m-1)x^{k-m-2}\right], \tag{3.19}$$

the summation inside the square bracket being understood to be absent for $k < 2$.

The unknown constants $a_i (i = 0, 1, \ldots, n)$ can be found by a collocation method as has been done by Mandal and Bera [7] who used an expansion for $\Psi(x)$ in terms of simple polynomials instead of the Bernstein polynomials. However, here we follow the method described in Section 2 above. Multiplying both sides of (3.18) by $B_{j,n}(x) (j = 0, 1, \ldots, n)$ we obtain

$$\sum_{i=0}^{n} a_i c_{ij} = f_j, \quad j = 0, 1, \ldots, n, \tag{3.20}$$

where now

$$\begin{aligned} c_{ij} &= \int_{-1}^{1} A_i(x) B_{j,n}(x)\mathrm{d}x = \frac{1}{2^{2n}}\binom{n}{i}\binom{n}{j}\sum_{k=0}^{n}\sum_{r=0}^{n} d_k^{i,n} d_r^{j,n} \\ &\times\left[-\pi(k+1)\frac{1+(-1)^{k+r}}{k+r+1} + \sum_{m=0}^{k-2}\frac{1+(-1)^{k+r-m}}{k+r-m-1}\frac{1+(-1)^m}{4}\frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{m+4}{2})}(k-m-1)\right] \end{aligned} \tag{3.21}$$

and

$$f_j = \int_{-1}^{1} f(x) B_{j,n}(x)\mathrm{d}x, \quad j = 0, 1, \ldots, n. \tag{3.22}$$

We note that when $f(x) = 1$,

$$f_j = \frac{1}{2^n}\binom{n}{j}\sum_{k=0}^{n}\frac{1+(-1)^k}{1+k} d_k^{j,n}, \quad j = 0, 1, \ldots, n. \tag{3.23}$$

The constants $d_k^{i,n}$ appearing in (3.21) and (3.23) are defined in (3.6).

In our numerical computation here, $f(x)$ is chosen to be 1, and n to be 3. The constants $a_i (i = 0, 1, 2, 3)$ are calculated by solving the linear system (3.20) for $n = 3$ and $f_j (j = 0, 1, 2, 3)$ given by (3.23). Thus, the function $\Psi(x)$ is found approximately and hence, by using the relation (3.16), $\phi(x)$ is obtained approximately. A comparison between this approximate solution and the exact solution given by (3.15) is presented in Table 3 for $x = 0, \pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8$. It is seen that the approximate and the exact values are same and they coincide. The absolute difference between exact and approximate solutions is plotted in Fig. 3. It is found from this figure that the accuracy here is of the order of 10^{-17}.

Table 3
Approximate and exact solutions of Eq. (3.12)

x	0	±0.2	±0.4	±0.6	±0.8
$\phi(x)$ (approx)	−0.318310	−0.311879	−0.291736	−0.254648	−0. 190986
$\phi(x)$ (exact)	−0.318310	−0.311879	−0.291736	−0.254648	−0. 190986

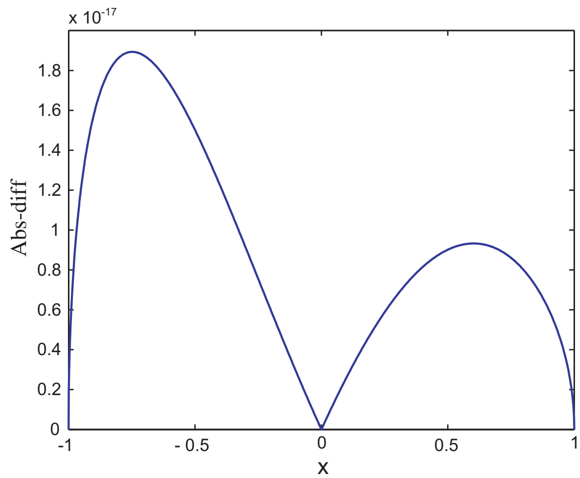

Fig. 3. Absolute difference between exact and approximate solutions of Eq. (3.12).

Example 4. Here, we consider a hypersingular integral equation of second kind as given by

$$\phi(x) - \frac{\alpha}{\pi}(1-x^2)^{\frac{1}{2}} \fint_{-1}^{1} \frac{\phi(t)}{(t-x)^2}\,\mathrm{d}t = f(x), \quad -1 \leqslant x \leqslant 1 \tag{3.24}$$

with end conditions $\phi(\pm 1) = 0$. This is a generalization of the elliptic wing case of Prandtl's equation. Expressing $\phi(x)$ in the form (3.16) above, we find that $\Psi(x)$ satisfies

$$\Psi(x) - \frac{\alpha}{\pi} \fint_{-1}^{1} (1-t^2)^{\frac{1}{2}} \frac{\Psi(t)}{(t-x)^2}\,\mathrm{d}t = F(x), \quad -1 \leqslant x \leqslant 1, \tag{3.25}$$

where

$$F(x) = \frac{f(x)}{(1-x^2)^{\frac{1}{2}}}. \tag{3.26}$$

If $\Psi(x)$ is represented in terms of Bernstein polynomials in the form (3.17), and the following the same procedure as in Example 3, we find that, in place of (3.2) we obtain the linear system

$$\sum_{i=0}^{n} a_i d_{ij} = F_j, \quad j = 0, 1, \ldots, n, \tag{3.27}$$

where

$$d_{ij} = \frac{1}{2^{2n}} \binom{n}{i} \binom{n}{j} \sum_{k=0}^{2n} \frac{1+(-1)^k}{1+k} d_k^{i+j,n} - \frac{1}{2} c_{ij}, \tag{3.28}$$

where c_{ij}s being the same as given in (3.21), and

$$F_j = \int_{-1}^{1} F(x) B_{j,n}(x)\,\mathrm{d}x. \tag{3.29}$$

Once the linear system is solved, the approximate solution is obtained.

For the special case when $\alpha = \frac{\pi}{2\beta}(\beta > 0)$, and $f(x) = \frac{2\pi k}{\beta}(1-x^2)^{\frac{1}{2}}$, Eq. (3.24) reduces to the Prandtl's equation and has the exact solution given by (cf. [8])

$$\phi(x) = \frac{4k}{1+\frac{2}{\pi}\beta}(1-x^2)^{\frac{1}{2}}. \tag{3.30}$$

Table 4
Approximate and exact solutions of Eq. (3.24)

x	0	±0.2	±0.4	±0.6	±0.8
$\phi(x)$ (approx)	2.444061	2.394682	2.400197	1.955250	1. 466437
$\phi(x)$ (exact)	2.444060	2.394680	2.400180	1.955248	1. 466436

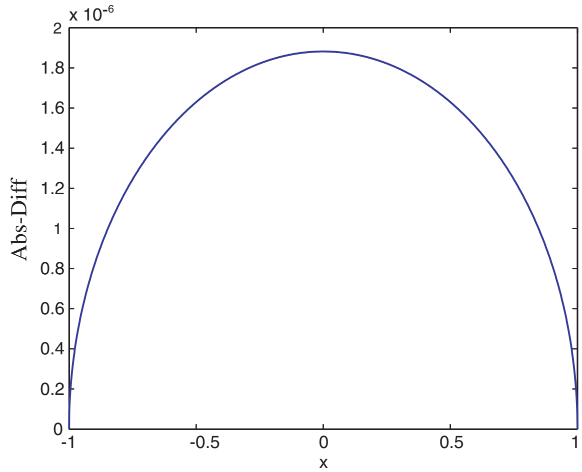

Fig. 4. Absolute difference between exact and approximate solutions of Eq. (3.24).

In this case, for $\beta = k = 1$,

$$F_j = \frac{\pi}{2^{j-1}} \binom{n}{i} \sum_{k=0}^{n} d_k^{j,n} \frac{1 + (-1)^k}{1 + k}. \tag{3.31}$$

Choosing $n = 3$, the coefficients a_0, a_1, a_2, a_3 are found. In Table 4, approximate and exact values of $\phi(x)$ at $x = 0, \pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8$ for $\beta = k = 1$ are given. The approximate and exact values almost coincide. Also in Fig. 4, a plot of the absolute difference between the approximate and exact solutions (for $\beta = k = 1$) is given. This figure shows that the error is of the order of 10^{-6}.

4. Error analysis

4.1. Fredholm integral equation

For the Fredholm integral equation, written in the operator form,

$$((I - K)\phi)(x) = f(x), \quad -1 \leqslant x \leqslant 1, \tag{4.1}$$

where I is an identity operator and $(K\phi)(x)$ denotes the integral $\int_{-1}^{1} K(x,t)\phi(t)\mathrm{d}t$.

The Bernstein polynomials are not orthogonal. However, these can be expressed in terms of some orthogonal polynomials, such as the Chebychev polynomials $U_n(x)$ of second kind (cf. [9]). It can be shown that

$$B_{i,n}(x) = \frac{1}{2^n} \binom{n}{i} \sum_{s=0}^{n} d_s^{i,n} \frac{1}{2^s} \sum_{m=0}^{[\frac{s}{2}]} \left\{ \binom{s}{m} - \binom{s}{m+1} \right\} U_{s-2m}(x). \tag{4.2}$$

Thus, an approximation $p_n(x)$ of the function $\phi(x)$ in terms of the Bernstein polynomials in the form

$$\phi(x) \simeq p_n(x) = \sum_{i=0}^{n} a_i B_{i,n}(x) \tag{4.3}$$

is eventually expressed as

$$p_n(x) = \sum_{j=0}^{n} b_j U_j(x), \tag{4.4}$$

where $b_j(j = 0, 1, \ldots, n)$ can be expressed in terms of $a_i(i = 0, 1, \ldots, n)$ and vice-versa. If $u_j(x) = \sqrt{\frac{2}{\pi}} U_j(x)$, then $u_j(x)(j = 0, 1, \ldots, n)$ form an orthonormal polynomial basis in $[-1, 1]$ with respect to the weight function $w(x) = (1 - x^2)^{\frac{1}{2}}$. Thus, (4.4) can be further expressed as

$$p_n(x) = \sum_{j=0}^{n} c_j u_j(x) \text{ with } c_j = \sqrt{\frac{\pi}{2}} b_j. \tag{4.5}$$

It is proved in Golberg and Chen [10, p. 178] that if $K(x, t) \in C^r$ and $f \in C^r (r > 0)$, then

$$\|\phi - p_n\|_w < c_0 n^{-r}, \quad r > 0$$

where $\|l\|_w \equiv \int_{-1}^{1} \{l(x)\}^2 w(x) \mathrm{d}x$ and c_0 is some constant. Thus, the convergence is very fast if r is large. In our two examples on Fredholm integral equations, both K and f are C^∞-functions, and as such, the method converges rapidly. This is also reflected in the numerical computations.

4.2. Hypersingular integral equation

The simple hypersingular integral equation (3.12) has the representation in the operator form

$$(H\Psi)(x) = f_1(x), \quad -1 \leqslant x \leqslant 1 \tag{4.6}$$

where H is the operator defined by

$$(H\Psi)(x) = \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}x} \left[\int_{-1}^{1} \frac{(1 - t^2)^{\frac{1}{2}}}{t - x} \Psi(t) \mathrm{d}t \right], \quad -1 \leq x \leq 1, \tag{4.7}$$

the integral within the square bracket being in the sense of Cauchy principal value, and

$$f_1(x) = \frac{1}{\pi} f(x). \tag{4.8}$$

Since

$$(HU_n)(x) = -(n + 1)U_n(x), \quad n \geqslant 0,$$

where H can be extended as a bounded linear operator (cf. [10, p. 306]) from $L_1(w)$ to $L(w)$, where $L_1(w)$ is the space of functions square integrable with respect to the weight function $w(x) = (1 - x^2)^{\frac{1}{2}}$ in $[-1, 1]$, and $L_1(w)$ is the subspace of functions $u \in L(w)$ satisfying

$$\|u\|_1^2 = \sum_{k=0}^{\infty} (k + 1) \langle u, u_k \rangle_w^2 < \infty, \tag{4.9}$$

where

$$\langle u, u_k \rangle_w = \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} u(x) u_k(x) \mathrm{d}x. \tag{4.10}$$

Now the function $\Psi(x)$ satisfying Eq. (4.6) is approximated in terms of the Bernstein polynomials $B_{i,n}(x)$ in the form

$$\Psi(x) = p_n(x),$$

where $p_n(x)$ is the same as in (4.3). In terms of the orthonormal Chebychev polynomials $u_j(x)$, $p_n(x)$ can be expressed in the form (4.5). If $f_1 \in C^r[-1, 1], r > 0$, then it follows that (cf. [10, p. 306])

$$\|\Psi - p_n\|_1 < c_1 n^{-r},$$

where c_1 is a constant. Thus, as before, the convergence is quite fast if r is large. In our example, we have chosen f_1 to be a constant and thus $f_1 \in C^\infty[-1,1]$. Hence, convergence is very rapid and this has been reflected in the numerical computation.

The second kind hypersingular integral equation (3.24) or rather (3.25) can be written in the operator form

$$((I-\alpha H)\Psi)(x) = F(x), \quad -1 \leqslant x \leqslant 1. \tag{4.11}$$

The proof of the convergence of the series representation of $\Psi(x)$in terms of Bernstein polynomials follows almost immediately by the same arguments. The details are omitted.

5. Conclusion

A simple method of approximating unknown function in terms of truncated series involving Bernstein polynomials is proposed here for solving several classes of integral equations. The method is illustrated by simple examples for which the exact solutions of the integral equations are available in the literature. The approximate solutions are compared with exact solutions numerically as well as by plotting the absolute difference between the approximate and exact solutions. Excellent agreement is seen to have been achieved between the exact and approximate solutions computed numerically by choosing a few terms for the truncated series. Also an error analysis is presented for a general Fredholm integral equation of the second kind and the two types of hypersingular integral equations. The method employed here can be probably extended to obtain approximate numerical solutions of integral equations arising in various areas of mathematical physics.

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References

[1] M.I. Bhatti, P. Bracken, Solutions of differential equations in a Bernstein polynomial basis, J. Comput. Appl. Math., 2007, doi:10.1016/j.cam2006.05.002.
[2] D.D. Bhatta, M.I. Bhatti, Numerical solution of KdV equation using modified Bernstein polynomials, Appl. Math. Comput. 174 (2006) 1255–1268.
[3] R.P. Kanwal, Linear Integral Equations, Academic Press, Orlando, 1973, pp. 12.
[4] A.D. Polyanin, A.V. Manzhirov, Handbook of Integral Equations, CRC Press, Boca Raton, 1988, pp. 257.
[5] P.A. Martin, Exact solution of a simple hypersingular integral equation, J. Integr. Equat. Appl. 4 (1992) 197–204.
[6] A. Chakrabarti, B.N. Mandal, Derivation of the solution of a simple hypersingular integral equation, Int. J. Math. Educ. Sci. Technol. 29 (1998) 47–53.
[7] B.N. Mandal, G.H. Bera, Approximate solution for a class of hypersingular integral equations, Appl. Math. Lett. 19 (2006) 1286–1290.
[8] A. Chakrabarti, B.N. Mandal, U. Basu, S. Banerjea, Solution of a class of hypersingular integral equations of the second kind, ZAMM 77 (1997) 319–320.
[9] M.A. Snyder, Chebychev Methods in Numerical Approximation, Prentice-Hall, Englewood Clifs, NJ, 1996.
[10] M.A. Golberg, C.S. Chen, Discrete Projection Methods for Integral Equations, Computational Mechanics Publications, Southampton, 1997, pp. 178 and 306.