On an Example of Jacobson

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In Vol. III of Nathan Jacobson's celebrated book [**2**], there appears the following exercise on p. 49:

Let $\mathbb{F}_p$ be the field with p elements, and $P = \mathbb{F}_p(x, y)$ where x, y are indeterminates. Let E be the subfield $\mathbb{F}_p(x^p - x,\ y^p - x)$. Show that $[P : E] = p^2$, that P/E is not separable, and that P/E contains no purely inseparable element.

Now, it is seen immediately that Jacobson's example is really a nonexample. Surprisingly, none of the other standard graduate texts seem to give an example, although one can be found in [**1**, Ex. 17, Ch. V]. Here is another:

Example. Let $P = \mathbb{F}_p(x, y)$ and let E be the subfield $\mathbb{F}_p(x^p - x, y^p x)$. Then

(i) $[P : E] = p^2$,
(ii) P/E is not separable, and
(iii) P/E contains no purely inseparable element over E except those contained in E.

Recall that an element x in an algebraic closure $\overline{K}$ of a field K is *separable* if its minimal polynomial $f(T)$ in $K[T]$ has all roots (in $\overline{K}$) simple.
It is said to be *purely inseparable* over K if it is fixed by all K-automorphisms of $\overline{K}$. More generally, an algebraic extension L of K is said to be *purely inseparable* if the only elements of L that are separable over K are the elements of K itself. Any algebraic extension L of K is built in two stages: $K \subset L_{sep} \subset L$, where L_{sep} is separable over K, and L is purely inseparable over L_{sep}.
One has as a consequence of this definition:

Let $x \in \overline{K}$, and let $f(T)$ be its minimal polynomial over K. Then, the following statements are equivalent:

(i) x is not separable over K;
(ii) The derivative $f'(T)$ is the zero polynomial; and
(iii) K is of characteristic $p > 0$, and $f(T) \in K[T^p]$.

Under any of these equivalent hypotheses, if n is the smallest integer such that $x^{p^n} \in K$, then the minimal polynomial over K is $f(T) = T^{p^n} - x^{p^n}$.

We return to our example now.

$P = \mathbb{F}_p(x, y) \supset E = \mathbb{F}_p(a, b)$ where $a = x^p - x$, $b = y^p x$. Then, over E, y satisfies the polynomial $g(T) = T^{p^2} + rT^{p(p-1)} - s$, where $r = b/a$ *and* $s = b^p/a$. Also, $P = E(y)$.

We note:

(a) x is separable over E and y is inseparable over E.
The separability of x follows from the preceding remarks by looking at the polynomial $T^p - T - (x^p - x)$. This is a polynomial over E satisfied by x. In fact, the Artin-Schreier Theorem [**3**, Ch. 8] shows that this polynomial is irreducible and is the minimal polynomial of x over E. However, we do not need this fact for the proof.
The inseparability of y is a consequence of the observation that $T^p - y^p$ is the minimal polynomial of y over the field $E(x)$.

(b) y is not purely inseparable over E.
As $y^p \notin \in E$, one has also $y^{p^2} \notin \in E$; otherwise, from $g(y) = 0$ one concludes $y^{p(p-1)} \in E$, which would lead to the erroneous conclusion $y^p \in E$.

(c) x is not a p-th power in P.
This is easy to check by a simple comparison of like powers of x in view of the algebraic independence of x and y over $\mathbb{F}_p$.

Suppose $t \in P \backslash E$ is purely inseparable over E. Then $t^p \in E$ (because if $t^{p^n} \in E$ for some $n \geq 2$, then since the degree of $P = E(y)$ over E is at most p^2, $n \leq 2$. But, if $n \neq 1$, then P would be purely inseparable, a contradiction).

Look at $P \supset E(t) \supset E$. Now, the minimal polynomial of t over E is $T^p - t^p$, and $[E(t): E] = p$. Note that $\alpha^p \in E$ for all $\alpha \in E(t)$.

Let $[P : E(t)] = l$, say. If the minimal polynomial of y over $E(t)$ is $f(T) = \sum_{i=0}^{l} a_i T^i$, then y satisfies the polynomial $f(T)^p = \sum_{i-0}^{l} a_i^p T^{ip}$. As this is the minimal polynomial of y over E, $f(T)^p$ divides $g(T)$. If $f(T)^p \sum u_i T^i = T^{p^2} + rT^{p(p-1)} - s$, one gets $u_i = 0$ if $i \not\equiv 0 \bmod p$. Renaming $u_{i/p}$ as v_i, the equation $\sum_{i=0}^{l} a_i^p T^{ip} \sum_{i=0}^{p-l} v_i T^{ip} = T^{p^2} + rT^{p(p-1)} - s$ gives inductively that $v_i = sb_i^p$ for some $b_i \in P$. Therefore, comparing the coefficients of $T^{p(p-1)}$ on both sides, we see $r = sv^p$ for some $v \in P$. This means that x is a p-th power in P, which is a contradiction.

Therefore, P has no purely inseparable elements outside of E.

Remarks. As a consequence of the proof, it is clear that $T^{p^2} + rT^{p(p-1)} - s$ is the minimal polynomial of y over E. The extension P of E is built up in two steps $P \supset E(x) \supset E$ with P purely inseparable of degree p over $E(x)$ and $E(x)$ separable of degree p over E.

REFERENCES

1. N. Bourbaki, *Algebre*, Actualites Scientifiques et Industrielles 1102, Hermann, Paris, 1950.
2. N. Jacobson, *Lectures in abstract algebra*, Vol. III, Van Nostrand, 1964.
3. S. Lang, *Algebra*, Addison-Wesley Publishing Company, Mass., 1965.