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On gravity-capillary waves due to interface disturbance

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The generation of gravity-capillary waves at the interface between two superposed fluids due to interface disturbance is considered, assuming linear theory. Fourier and Laplace transform techniques are employed in the mathematical analysis and the form of the interface depression is obtained as an infinite integral involving oscillatory functions when the disturbance is concentrated at the origin. The method of stationary phase is then employed to evaluate this infinite integral asymptotically. The asymptotic form of the interface depression is presented graphically and compared with the non-capillary case. It is observed that the interface capillarity has some significant effect on the wave motion.

1. Introduction

The two-dimensional Cauchy–Poisson problem concerning the generation of water waves due to local disturbance of the free surface is well studied in the literature within the framework of linearized theory of water waves. The problem was studied by Lamb [1] and Stoker [2] by the use of Fourier integral transform technique, and the free surface elevation was evaluated asymptotically by employing the method of stationary phase when the disturbance is confined to the immediate neighbourhood of the origin. The problem of an axially-symmetric initial surface disturbance in water of uniform finite depth was considered by Kranzer and Keller [3]. They compared theory with experimental results. They also gave a brief account of various earlier works related to this problem. The initial disturbance problem over an arbitrary region of the free surface was considered by Wen [4] who also obtained the free surface depression by using the method of stationary phase. There has been considerable interest in the study of the problem of generation of surface waves in water covered by an inertial surface (IS) composed of a thin but uniform distribution of non-interacting floating particles. Mandal [5] considered the two-dimensional unsteady motion in a deep ocean covered by an IS due to initial disturbances at the IS. The corresponding problem for an ocean of uniform finite depth was considered by Mandal and Ghosh [6]. Again, Mandal and Mukherjee [7] studied three-dimensional unsteady motion in a deep ocean covered by an IS due to a prescribed axisymmetric initial disturbance at the IS, while the corresponding problem for an ocean of uniform finite depth was considered by Mandal and Ghosh [8]. Also, Mandal and Ghosh [9] considered the problem of generation of water waves in an ocean of uniform finite depth covered by an IS due to an arbitrary periodic pressure distribution on the IS as well as an initial displacement of the IS.

In this paper we consider the problem of generation of gravity-capillary waves due to initial interface disturbance at the interface between two superposed fluids wherein the upper fluid extends infinitely upwards and the lower fluid extends infinitely downwards. In the mathematical analysis of the problem we use a new potential function defined in the lower fluid region which is a linear combination of the velocity potential for the lower fluid and another potential defined in the lower fluid region by reflection of the upper fluid about the common interface. This new potential function satisfies an initial value problem. Fourier and Laplace transform techniques are employed to solve the problem and the interface depression is obtained in terms of double integrals involving the initial interface depression. Assuming the initial interface depression to be concentrated at the origin, the interface depression is obtained in terms of an integral involving oscillatory functions. An asymptotic form of the interface depression is obtained by employing the method of stationary phase. Known results for free surface gravity waves are recovered in the absence of upper fluid as well as capillarity.

The asymptotic form of the interface depression is presented graphically. The figures exhibit variations of the interface depression at a fixed point x for different time t, and at a fixed time t for different x. In the absence of the upper fluid, the gravity wave profile has features which are qualitatively similar to those for the profile given in Lamb's [1] book. It is also observed from the figures that the presence of capillarity has some significant effect on the wave motion, while the presence of the upper fluid is not of much significance if the density of the upper fluid is very small.

2. Formulation of the problem

We consider two-dimensional motion at the interface between two inviscid, incompressible and homogeneous superposed fluids wherein the upper fluid extends infinitely upwards and the lower fluid extends infinitely downwards, in the presence of capillarity at the interface. The fluid motion is generated due to an initial disturbance in the form of an initial interface depression. A rectangular Cartesian coordinate system is chosen in which the origin is taken at the interface, y-axis vertically downwards in the lower fluid so that $y = 0$ is the mean position of the interface. As the motion starts from rest, it is irrotational and can be described by the velocity potentials $\phi_1(x, y, t)$ and $\phi_2(x, y, t)$ in the lower and upper fluids, respectively. They satisfy

$$\left.\begin{aligned} \nabla^2\phi_1 &= 0 \qquad \text{in } y \geq 0 \\ \nabla^2\phi_2 &= 0 \qquad \text{in } y \leq 0 \end{aligned}\right\} t \geq 0 \tag{2.1}$$

together with the linearized interface condition

$$\left.\begin{aligned} &\phi_{1y} = \phi_{2y} = \eta_t \\ &g(1-s)\eta - (\phi_{1t} - s\phi_{2t}) = \frac{T}{\rho_1}\eta_{xx} \end{aligned}\right\} \text{on } y = 0, t > 0 \tag{2.2}$$

where $\eta(x, t)$ is the interface depression, g is gravity, $s = \rho_2/\rho_1$ $(0 \leq s < 1)$, ρ_1 and

ρ_2 are the densities of the lower and upper fluids respectively and T is capillarity at the interface, the condition of no motion at infinite depth and height

$$\left.\begin{array}{ll}\nabla\phi_1 \to 0 & \text{as } y \to \infty \\ \nabla\phi_2 \to 0 & \text{as } y \to -\infty\end{array}\right\} \tag{2.3}$$

the initial conditions

$$\phi_1 = \phi_2 = 0 \quad \text{at } t = 0 \text{ on } y = 0 \tag{2.4}$$

$$\eta(x,t) = f(x) \quad \text{at } t = 0 \tag{2.5}$$

where $f(x)$ is the initial interface depression. We suppose that ϕ_1, ϕ_2, η are defined in the sense of generalized functions.

3. Method of solution

Let us define a new potential function $\phi(x,y,t)$ in $y \geq 0$ as

$$\phi(x,y,t) = \phi_1(x,y,t) - s\phi_2(x,-y,t) \tag{3.1}$$

From relations (2.1) to (2.5) we find that ϕ satisfies

$$\left.\begin{array}{l}\nabla^2\phi = 0 \text{ in } y \geq 0, t \geq 0 \\ \left.\begin{array}{l}\phi_y = (1+s)\eta_t \\ (1-s)\eta = \dfrac{1}{g}\phi_t + M\eta_{xx}\end{array}\right\} \quad \text{on } y = 0, t > 0 \\ \nabla\phi \to 0 \quad \text{as } y \to \infty, \\ \phi = 0 \quad \text{at } t = 0 \text{ on } y = 0 \\ \eta(x,t) = f(x) \quad \text{at } t = 0\end{array}\right\} \tag{3.2}$$

where $M = T/\rho_1 g$.

Let $\Phi(\xi,y,t)$ denote the Fourier transform of $\phi(x,y,t)$ denoted by

$$\Phi(\xi,y,t) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \phi(x,y,t)\exp(\mathrm{i}\xi x)\,\mathrm{d}x$$

By application of the Fourier transform, equations (3.2) reduce to

$$\left.\begin{array}{l}\Phi_{yy} - \xi^2\Phi = 0 \quad \text{in } y \geq 0, t \geq 0 \\ \left.\begin{array}{l}\Phi_y = (1+s)\rho_t \\ (1-s)\rho = \dfrac{1}{g}\Phi_t - M\xi^2\rho\end{array}\right\} \quad \text{on } y = 0, t > 0 \\ \Phi, \Phi_y \to 0 \quad \text{as } y \to \infty, \\ \Phi = 0 \quad \text{at } t = 0 \quad \text{on } y = 0 \\ \rho(\xi,t) = F(\xi) \quad \text{at } t = 0\end{array}\right\} \tag{3.3}$$

where $\rho(\xi,t)$ and $F(\xi)$ denote the Fourier transform of $\eta(x,t)$ and $f(x)$, respectively.

By the application of the Laplace transform in time, equations (3.3) reduce to

$$\left.\begin{aligned}&\bar{\Phi}_{yy} - \xi^2\bar{\Phi} = 0 \quad \text{in } y \geq 0\\ &\bar{\Phi}_y = -(1+s)F(\xi) + p(1+s)\bar{\rho}(\xi,p)\\ &\bar{\rho}(\xi,p) = \frac{p\bar{\Phi}}{g(1-s+M\xi^2)}\end{aligned}\right\} \quad \text{on } y = 0,$$

$$\bar{\Phi}, \bar{\Phi}_y \to 0 \quad \text{as } y \to \infty$$

where the bar denotes the Laplace transform in time.

Then we find

$$\bar{\Phi}(\xi, y, p) = \frac{g(1-s+M\xi^2)F(\xi)}{p^2+\omega^2}\exp\left(\{-|\xi|y\}\right) \tag{3.4}$$

$$\bar{\rho}(\xi,p) = \frac{pF(\xi)}{p^2+\omega^2} \tag{3.5}$$

where

$$\omega^2 = \frac{g|\xi|}{1+s}(1-s+M\xi^2) \tag{3.6}$$

If the initial depression of the interface is concentrated at the origin, then $f(x) = \delta(x)$, so that

$$F(\xi) = \frac{1}{(2\pi)^{1/2}}$$

and hence from equation (3.5), by Laplace inversion we find

$$\rho(\xi, t) = \frac{1}{(2\pi)^{1/2}}\cos\omega t$$

The Fourier inversion then produces

$$\begin{aligned}\eta(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\omega t \exp(-\mathrm{i}\xi x)\,\mathrm{d}\xi\\ &= \frac{1}{\pi}\int_0^{\infty}\cos\alpha t\cos\xi x\,\mathrm{d}\xi\end{aligned} \tag{3.7}$$

where

$$\alpha^2(\xi) = \frac{g\xi(1-s+M\xi^2)}{1+s} \tag{3.8}$$

For $s = 0$ and $M = 0$, equation (3.7) reduces to the result given in Stoker [2].

The velocity of interface wave propagation is given by

$$c^2 = \frac{\alpha^2(\xi)}{\xi^2} = \frac{1-s}{1+s}\frac{g}{\xi} + \frac{T}{\rho_1(1+s)}\xi \tag{3.9}$$

From this the following interesting observation may be made.

For a very small value of ξ, the first term of equation (3.9) is large compared to the second so that

$$c^2 \approx \frac{1-s}{1+s}\frac{g}{\xi} \tag{3.10}$$

This implies that the ensuing motion mainly consists of gravity waves only since the same result is also obtained for $T = 0$. On the other hand, when ξ is sufficiently large, the second term dominates so that

$$c^2 \approx \frac{T}{\rho_1(1+s)}\xi \tag{3.11}$$

This implies that gravity is mostly irrelevant, and the ensuing motion is mainly due to capillarity. These facts will be more evident in the following section.

4. Asymptotic form of the interface depression

To obtain the asymptotic form of the interface depression we apply the method of stationary phase to equation (3.7). Now, equation (3.7) can be written as

$$\eta(x,t) = \frac{1}{4\pi}\int_0^\infty \Big[\exp\Big\{it\Big(\alpha + \frac{\xi x}{t}\Big)\Big\} + \exp\Big\{-it\Big(\alpha + \frac{\xi x}{t}\Big)\Big\}$$

$$+ \exp\Big\{it\Big(\alpha - \frac{\xi x}{t}\Big)\Big\} + \exp\Big\{-it\Big(\alpha - \frac{\xi x}{t}\Big)\Big\}\Big]\, d\xi \tag{4.1}$$

We see that the first two integrals of equation (4.1) have no stationary points. The stationary points for the third and fourth integrals are the real roots of

$$\alpha'(\xi) = x/t \tag{4.2}$$

where $\alpha'(\xi)$ is given by

$$\alpha'(\xi) = \frac{1}{2}\left(\frac{g}{1+s}\right)^{1/2} \frac{1-s+3M\xi^2}{\{\xi(1-s+M\xi^2)\}^{1/2}} \tag{4.3}$$

It may be noted that $\alpha'(\xi)$ is positive and $\alpha'(\xi) \to \infty$ as $\xi \to 0, \infty$ and $\alpha'(\xi)$ has only one minimum value at $\xi = \xi_0$, say, where

$$\xi_0^2 = \frac{1-s}{M}\frac{3-2\sqrt{3}}{3} \tag{4.4}$$

It then follows that equation (4.2) has two real solutions, $\xi_1, \xi_2 (> \xi_1)$, say, only when $x/t > \alpha'(\xi_0)$ with $\alpha''(\xi_1)$ negative and $\alpha''(\xi_2)$ positive.

By the method of stationary phase applied to the third and fourth integrals of equation (4.1) we find

$$\eta(x,t) \approx \sum_{j=1}^{2}\{2\pi t|\alpha''(\xi_j)|\}^{-1/2}\cos\left[\alpha(\xi_j)t - \xi_j x + \frac{\pi}{4}\operatorname{sgn}\{\alpha''(\xi_j)\}\right] \tag{4.5}$$

This approximation is valid only when the ratio

$$\frac{\alpha'''(\xi)}{\{t|\alpha''(\xi)|^3\}^{1/2}}$$

is small enough for both roots, which is seen to be satisfied here. If we put $s = 0$ and $M = 0$, there is only one root of equation (4.2) given by

$$\xi = \frac{gt^2}{4x^2}$$

so that equation (4.5) reduces to the asymptotic form of the free surface depression given previously [1,2].

Again, it may be noted that when x/t is large compared to the minimum value of $\alpha'(\xi_0)$ then ξ_1 will be very small and ξ_2 will be very large. In this case, ξ_1 will be a gravity wave solution $(T = 0)$ given by

$$\xi_1 = \frac{1-s}{1+s}\frac{gt^2}{4x^2} \tag{4.6}$$

with the behaviour of the interface as

$$\eta(x,t) \approx \frac{1}{\pi^{1/2}x}\left(\frac{1-s}{1+s}\frac{gt^2}{4x}\right)^{1/2}\cos\left(\frac{1-s}{1+s}\frac{gt^2}{4x} - \frac{\pi}{4}\right) \tag{4.7}$$

while ξ_2 will be a capillary wave solution $(g = 0)$ given by

$$\xi_2 = \frac{(1+s)\rho_1}{T}\frac{4x^2}{9t^2} \tag{4.8}$$

with the behaviour of the interface given by

$$\eta(x,t) \approx \frac{2}{3}\left(\frac{\rho_1(1+s)}{\pi T}\right)^{1/2}\frac{x^{1/2}}{t}\cos\left[-\frac{(1+s)\rho_1}{T}\frac{4x^3}{27t^2} + \frac{\pi}{4}\right] \tag{4.9}$$

The approximation in equation (4.7) holds when $\frac{1}{2}gt^2$ is large compared to x and the approximation in equation (4.9) holds when $8\rho_1 x^3/T$ is large compared to t^2.

The asymptotic form given by solution (4.7) corresponds to pure gravity waves while solution (4.9) corresponds to pure capillary waves. This is consistent with the observation made from the wave velocities given in equations (3.10) and (3.11).

At a fixed position (i.e. for a given x) it is seen from solution (4.7) that a pure gravity wave has an envelope that increases with time and an effective period (which can be estimated by looking at zero crossings of the curves) that decreases with time. On the other hand, from solution (4.9) it is seen that a pure capillary wave has an envelope that decreases with time and an effective period that increases with time. The relative contribution of the capillary component to the gravity is

$$(1+s)\left\{\frac{\rho_1}{Tg(1-s)}\right\}^{1/2}\frac{4x^2}{3t^2} \tag{4.10}$$

These facts become apparent when we study the qualitative features of the interface waves (or suface waves for $s = 0$) depicted in Figures 1 and 2.

5. Discussion

The asymptotic form of the interface depression $\eta(x,t)$ is plotted graphically in Figures 1 and 2. Figure 1 shows the variation of $\eta(x,t)$ against t between $t = 1{\cdot}0$ s and 2·0 s for $x = 40$ cm. Figure 2 shows the variation of $\eta(x,t)$ against x between $x = 200$ cm and 250 cm for $t = 10$ s. For an air–water model, the value of s is 0·0013 ([1], p. 576) and $T = 74$ cgs units ([1], p. 455) so that $M = 0{\cdot}075$ cgs units, $s = 0$ and $T = 0$ corresponds to the free surface behaviour. In each figure, three curves are drawn corresponding to $s = 0, M = 0$ (curve (I)); $s = 0{\cdot}0013, M = 0$ (curve (II)) and $s = 0{\cdot}0013, M = 0{\cdot}075$ (curve (III)).

As x/t varies from 40 to 20 in Figure 1, x/t is fairly large and as such the roots ξ_1 and ξ_2 of equation (4.2) are separated significantly so that ξ_2 dominates ξ_1. The curves (I) and (II) correspond to gravity waves only at the free surface and interface, respectively. They almost coincide, which is expected since $s(= 0{\cdot}0013)$ is very small. From these curves it is observed that for gravity waves, the wave amplitude increases while wave period decreases with time. This feature of curves

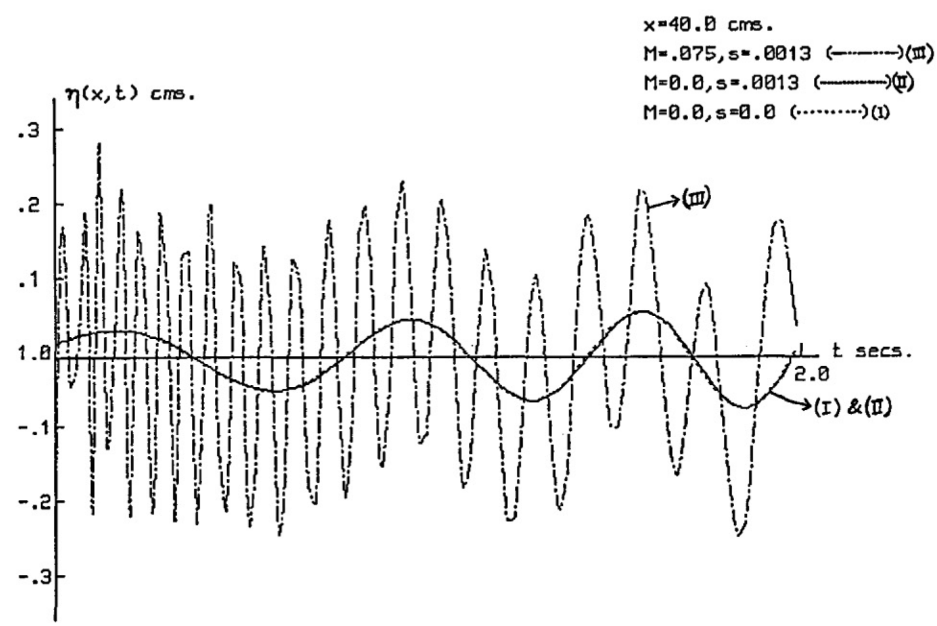

Figure 1.

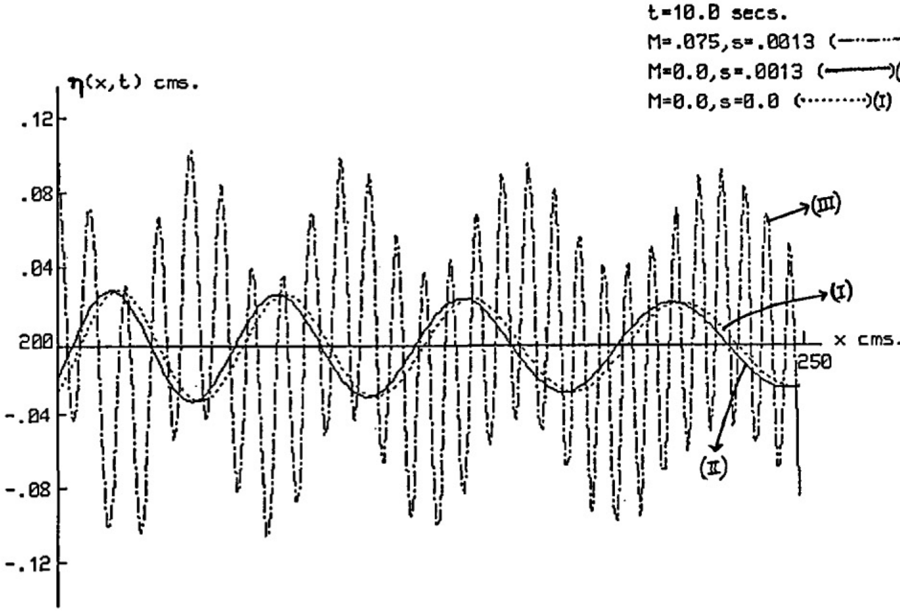

Figure 2.

(I) and (II) follows the pattern of the curve given in [1]. The curve (III) in Figure 1 shows the asymptotic form of the interface in the presence of capillarity. From this curve, it is observed that the wave amplitude falls while the period increases with time. This is due to the fact that that the capillary root dominates in the stationary phase computation. Also, the capillary wave periods are much smaller than gravity wave periods.

In Figure 2 a similar comparison can be made. Here x lies between 200 cm and 250 cm, and t is taken as 10 s. As the spatial position (x) becomes larger, the separation between pure gravity and pure capillary waves becomes larger so that the capillary solution becomes more dominant. This is consistent with the behaviour seen in Figure 2. It is observed that for capillary waves (curve (III)), the amplitude increases and period decreases with x while for gravity waves (curves (I) and (II)) the amplitude decreases and period increases with x. This feature of the curves (I) and (II) follows the same qualitative pattern as the curve given in [1].

Finally, it is observed that the presence of capillarity has some significant effect on the wave motion, while the presence of the upper fluid is not of much significance if the density of the upper fluid is very small.

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