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CLARKSON INEQUALITIES WITH SEVERAL OPERATORS

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Abstract

Several inequalities for trace norms of sums of n operators with roots of unity coefficients are proved in this paper. When $n = 2$, these reduce to the classical Clarkson inequalities and their non-commutative analogues.

1. *Introduction*

The classical inequalities of Clarkson [9] for the Lebesgue spaces L_p, and their non-commutative analogues for the Schatten trace ideals C_p, play an important role in analysis, operator theory, and mathematical physics. They have been generalised in various directions. Among these are versions for more-general symmetric norms [4], and for the Haagerup L_p-spaces [10], as well as refinements [2]. In this paper, we obtain extensions of these (and related) inequalities in another direction, replacing pairs of operators by n-tuples.

Let A be a linear operator on a complex separable Hilbert space. If A is compact, we denote by $\{s_j(A)\}$ the sequence of decreasingly ordered singular values of A. For $0 < p < \infty$, let

$$\|A\|_p = \left[\sum (s_j(A))^p\right]^{1/p}. \tag{1}$$

For $1 \leqslant p < \infty$, this defines a norm on the class C_p consisting of operators A for which $\|A\|_p$ is finite; this is called the *Schatten p-norm*. By convention, $\|A\|_\infty = s_1(A)$ is the operator bound norm of A. These p-norms belong to a larger class of symmetric or unitarily invariant norms. Such a norm $|||\cdot|||$ is characterized by the equality

$$|||A||| = |||UAV|||, \tag{2}$$

for all A, and unitary U and V. The class of operators $C_{|||\cdot|||}$ on which such a norm is defined is an ideal in the algebra of bounded operators. This is called the *unitary ideal corresponding to* $|||\cdot|||$. When we use the symbol $\|A\|_p$ or $|||A|||$, it is implicit that the operator A belongs to the class C_p or $C_{|||\cdot|||}$, respectively; see [3] for the properties of these norms.

The case $0 < p < 1$ is less interesting. In this case, expression (1) defines a quasi-norm. In lieu of the triangle inequality, we have

$$\|A+B\|_p \leqslant 2^{1/p-1}(\|A\|_p + \|B\|_p), \quad \text{for } 0 < p < 1.$$

For $1 \leqslant p \leqslant \infty$, we denote by q the conjugate index defined by the relation $1/p + 1/q = 1$. The symbol $|A|$ stands for the positive operator $(A^*A)^{1/2}$.

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We prove the following four theorems. In each of the statements, $A_0, A_1, \ldots, A_{n-1}$ are linear operators and $\omega_0, \omega_1, \ldots, \omega_{n-1}$ are the nth roots of unity with $\omega_j = e^{2\pi ij/n}$, $0 \leqslant j \leqslant n-1$.

THEOREM 1. *For $2 \leqslant p \leqslant \infty$, we have*

$$n^{2/p} \sum_{j=0}^{n-1} \|A_j\|_p^2 \leqslant \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k A_j \right\|_p^2 \leqslant n^{2-2/p} \sum_{j=0}^{n-1} \|A_j\|_p^2. \tag{3}$$

For $0 < p \leqslant 2$, these two inequalities are reversed.

THEOREM 2. *For $2 \leqslant p < \infty$, we have*

$$n \sum_{j=0}^{n-1} \|A_j\|_p^p \leqslant \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k A_j \right\|_p^p \leqslant n^{p-1} \sum_{j=0}^{n-1} \|A_j\|_p^p. \tag{4}$$

For $0 < p \leqslant 2$, these two inequalities are reversed.

THEOREM 3. *For $2 \leqslant p < \infty$, we have*

$$n \left|\left|\left| \sum_{j=0}^{n-1} |A_j|^p \right|\right|\right| \leqslant \left|\left|\left| \sum_{k=0}^{n-1} \left| \sum_{j=0}^{n-1} \omega_j^k A_j \right|^p \right|\right|\right| \leqslant n^{p-1} \left|\left|\left| \sum_{j=0}^{n-1} |A_j|^p \right|\right|\right|, \tag{5}$$

for every unitarily invariant norm $|||\cdot|||$. For $0 < p \leqslant 2$, these two inequalities are reversed.

THEOREM 4. *For $2 \leqslant p < \infty$, we have*

$$n \left(\sum_{j=0}^{n-1} \|A_j\|_p^p \right)^{q/p} \leqslant \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k A_j \right\|_p^q. \tag{6}$$

For $1 < p \leqslant 2$, this inequality is reversed.

When $n = 2$, Theorem 1 gives, for any pair A, B, the inequalities

$$2^{2/p} \left(\|A\|_p^2 + \|B\|_p^2 \right) \leqslant \|A+B\|_p^2 + \|A-B\|_p^2 \leqslant 2^{2-2/p} \left(\|A\|_p^2 + \|B\|_p^2 \right), \tag{7}$$

for $2 \leqslant p \leqslant \infty$, and the reverse inequalities for $0 < p \leqslant 2$. Theorem 2 gives

$$2 \left(\|A\|_p^p + \|B\|_p^p \right) \leqslant \|A+B\|_p^p + \|A-B\|_p^p \leqslant 2^{p-1} \left(\|A\|_p^p + \|B\|_p^p \right), \tag{8}$$

for $2 \leqslant p < \infty$, and the reverse inequalities for $0 < p \leqslant 2$.

For $p = 2$, inequalities (7) and (8) both reduce to the *parallelogram law*

$$\|A+B\|_2^2 + \|A-B\|_2^2 = 2 \left(\|A\|_2^2 + \|B\|_2^2 \right). \tag{9}$$

The special norm $\|\cdot\|_2$ arises from an inner product $\langle A, B \rangle = \operatorname{tr} A^* B$, and must satisfy this law. The generalisation given in Theorem 1 can be obtained easily in this case.

The inequalities (8) are one half of the celebrated Clarkson inequalities. A recent generalisation, due to Hirzallah and Kittaneh [11], states that

$$2|||\,|A|^p + |B|^p\,||| \leqslant |||\,|A+B|^p + |A-B|^p\,||| \leqslant 2^{p-1}|||\,|A|^p + |B|^p\,|||, \tag{10}$$

for $2 \leqslant p < \infty$; the two inequalities are reversed for $0 < p \leqslant 2$. The inequalities (8) follow from these by choosing for $|||\cdot|||$ the special norm $\|\cdot\|_1$. Theorem 3 includes the inequalities (10) as a special case.

When $n=2$, inequality (6) reduces to the inequality

$$2\left(\|A\|_p^p+\|B\|_p^p\right)^{q/p}\leqslant \|A+B\|_p^q+\|A-B\|_p^q, \tag{11}$$

for $2\leqslant p<\infty$, and to the reverse inequality for $1<p\leqslant 2$. These constitute the other half of the Clarkson inequalities. They are much harder to prove, and are stronger, than the inequalities (8).

A simple proof and a generalisation of the inequalities (8) were given by Bhatia and Holbrook in [**4**]. Some of their ideas were developed further in our paper [**5**]. In Section 2 we give a proof of Theorems 1 and 2 using these results. In Section 3 we discuss some extensions of these results as given in [**4**]. In Section 4, we outline a proof of Theorem 3, as well as proofs of some more general theorems. We follow the approach taken in [**11**]. This was based on results of Ando and Zhan [**1**], and we show how these can be generalised to n-tuples. The harder Clarkson inequalities (11) are usually proved by complex interpolation methods. In Section 5, we show how one such proof (as given by Fack and Kosaki [**10**]) can be modified to give Theorem 4. Section 6 contains further generalisations of Theorems 1–4, where the ω_j^k in the inequalities are replaced by more general coefficients.

Sharper versions of (7), (8) and (11) have been proved by Ball, Carlen and Lieb [**2**], by the use of deeper arguments. Our results go in a different direction.

2. *Proofs of Theorems* 1 *and* 2

Proof of Theorem 1 *for* $p\geqslant 1$. Consider the $n\times n$ matrix

$$T=[T_{jk}],\qquad 0\leqslant j,\ k\leqslant n-1, \tag{12}$$

where the entries T_{jk} are operators. In [**5**, Theorem 1] we showed that

$$\|T\|_p^2\leqslant\sum_{j,k}\|T_{jk}\|_p^2,\quad \text{for } 2\leqslant p\leqslant\infty. \tag{13}$$

Now, given n operators $A_0,\dots,A_{n-1}$, let T be the block circulant matrix

$$T=\operatorname{circ}(A_0,\dots,A_{n-1}). \tag{14}$$

This is the $n\times n$ matrix whose first row has entries $A_0,\dots,A_{n-1}$, and the other rows are obtained by successive cyclic permutations of these entries. Let

$$F_n=\frac{1}{\sqrt{n}}\begin{bmatrix}\omega_0^0 & \omega_1^0 & \cdots & \omega_{n-1}^0\\ \omega_0^1 & \omega_1^1 & \cdots & \omega_{n-1}^1\\ \vdots & \vdots & \ddots & \vdots\\ \omega_0^{n-1} & \omega_1^{n-1} & \cdots & \omega_{n-1}^{n-1}\end{bmatrix}$$

be the finite Fourier transform matrix of size n. Let $W=F_n\otimes I$. This is the block matrix whose (j,k) entry is $\omega_k^j I/\sqrt{n}$. It is easy to see that if T is the block circulant matrix in (14), then $X=W^*TW$ is a block-diagonal matrix, and the kth entry on its diagonal is the operator

$$X_{kk}=\sum_{j=0}^{n-1}\omega_j^k A_j. \tag{15}$$

Now note that

$$\|T\|_p = \|X\|_p = \left(\sum_{k=0}^{n-1} \|X_{kk}\|_p^p\right)^{1/p}. \tag{16}$$

Using (13)–(16), we obtain

$$\left[\sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k A_j \right\|_p^p\right]^{2/p} \leqslant n \sum_{j=0}^{n-1} \|A_j\|_p^2, \tag{17}$$

for $2 \leqslant p < \infty$. For these values of p, the function $f(x) = x^{2/p}$ is concave on the positive half-line. Hence

$$n^{2/p-1}(x_0^{2/p} + \ldots + x_{n-1}^{2/p}) \leqslant (x_0 + \ldots + x_{n-1})^{2/p}. \tag{18}$$

Using this, we obtain from (17) the inequality

$$n^{2/p-1} \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k A_j \right\|_p^2 \leqslant n \sum_{j=0}^{n-1} \|A_j\|_p^2, \tag{19}$$

for $2 \leqslant p \leqslant \infty$. This is the second inequality in (3).

The first inequality in (3) can be obtained from this by a change of variables. Let

$$B_k = \sum_{j=0}^{n-1} \omega_j^k A_j, \quad \text{for } 0 \leqslant k \leqslant n-1. \tag{20}$$

Replace the n-tuple $(A_0, \ldots, A_{n-1})$ in the inequality just proved by $(B_0, \ldots, B_{n-1})$. Note that the n-tuple whose kth entry is $\sum_j \omega_j^k B_j$ is the same as the n-tuple $(nA_0, nA_1, \ldots, nA_{n-1})$ up to a permutation. This leads to the first inequality in (3).

When $1 \leqslant p \leqslant 2$, inequality (13) is reversed; see [**5**, Theorem 1]. So inequality (17) is reversed. The function $f(x) = x^{2/p}$ is convex in this case, and inequality (18) is reversed. As a result, both the inequalities in (3) are reversed. This completes the proof of Theorem 1 for $1 \leqslant p \leqslant \infty$. The case $0 < p < 1$ is discussed in Section 3. □

Proof of Theorem 2 for $p \geqslant 1$. The proof of Theorem 2 runs parallel to that of Theorem 1. For T as in (12) we have, from [**5**, Theorem 2],

$$\sum_{j,k} \|T_{jk}\|_p^p \leqslant \|T\|_p^p, \quad \text{for } 2 \leqslant p < \infty, \tag{21}$$

and the inequality is reversed for $0 < p \leqslant 2$. Start with this instead of (13), and follow the steps of the proof of Theorem 1. One obtains Theorem 2 for $1 \leqslant p < \infty$. The case $0 < p < 1$ is discussed in Section 3. □

The inequalities of Theorems 1 and 2 are sharp. For $0 \leqslant j \leqslant n-1$, let A_j be the diagonal matrix with its (j,j) entry equal to 1, and all its other entries equal to 0. In this case, the first inequalities in (3) and (4), respectively, are equalities. On the other hand, if we choose $A_j = (\omega_0^j, \omega_1^j, \ldots, \omega_{n-1}^j)$ for $0 \leqslant j \leqslant n-1$, we see that the other two inequalities are equalities in this case.

A simple consequence of inequality (7) is the following result, proved in [**6**]. Let T be any operator, and let $T = A + iB$ be its Cartesian decomposition with A and B Hermitian. Then, for $2 \leqslant p \leqslant \infty$,

$$2^{2/p-1}\left(\|A\|_p^2 + \|B\|_p^2\right) \leqslant \|T\|_p^2 \leqslant 2^{1-2/p}\left(\|A\|_p^2 + \|B\|_p^2\right), \tag{22}$$

and the inequalities are reversed for $0 < p \leqslant 2$. Note that in this case we have, from (8),

$$\|A\|_p^p + \|B\|_p^p \leqslant \|T\|_p^p \leqslant 2^{p-2}\left(\|A\|_p^p + \|B\|_p^p\right) \tag{23}$$

for $2 \leqslant p < \infty$, and the reverse inequalities for $0 < p \leqslant 2$. The inequalities (22) can be derived from (23) by a simple convexity argument. More subtle norm inequalities for the Cartesian decomposition may be found in [7] and [8].

3. *Extensions and remarks*

We have proved Theorems 1 and 2 for $p \geqslant 1$, using results given in [5]. There are other connections between [4], [5] and the present paper. We point out some of them.

1. Let T be the block matrix (12), and let U_j be the block-diagonal operator

$$U_j = \operatorname{diag}\left(\omega_0^j I, \ldots, \omega_{n-1}^j I\right), \qquad 0 \leqslant j \leqslant n-1.$$

Let $A_j = U_j^* T U_j$. The second inequality in (3) then gives

$$n^{4/p-2} \sum_{j,k} \|T_{jk}\|_p^2 \leqslant \|T\|_p^2, \quad \text{for } 2 \leqslant p \leqslant \infty.$$

This is the inequality complementary to (13), proved in [5] by other arguments.

2. A unitarily invariant norm $|||\cdot|||$ is called a *Q-norm* if there exists another unitarily invariant norm $|||\cdot|||^\wedge$ such that $|||A|||^2 = |||A^*A|||^\wedge$. The Schatten p-norms for $p \geqslant 2$ are Q-norms, since $\|A\|_p^2 = \|A^*A\|_{p/2}$. The crucial observation in [4] is a reinterpretation of the Clarkson inequalities (8) in such a way that a generalisation to Q-norms and their duals becomes possible. The next remarks concern similar generalisations of Theorems 1 and 2.

3. The following useful identity can be easily verified.

$$\frac{1}{n}\sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1}\omega_j^k A_j\right)^*\left(\sum_{j=0}^{n-1}\omega_j^k A_j\right) = \sum_{j=0}^{n-1} A_j^* A_j. \tag{24}$$

For $n = 2$, this reduces to

$$\frac{(A+B)^*(A+B) + (A-B)^*(A-B)}{2} = A^*A + B^*B. \tag{25}$$

4. We use the notation $A_0 \oplus \ldots \oplus A_{n-1}$, or $\oplus A_j$, for the block-diagonal operator with operators A_j as its diagonal entries.

For positive operators A_j, $0 \leqslant j \leqslant n-1$, we have the inequality

$$|||A_0 \oplus \ldots \oplus A_{n-1}||| \leqslant \left|\left|\left|\left(\sum_{j=0}^{n-1} A_j\right) \oplus 0 \oplus \ldots \oplus 0\right|\right|\right|, \tag{26}$$

for all unitarily invariant norms [5, Lemma 4]. For the p-norms, this gives (for positive operators)

$$\sum_{j=0}^{n-1} \|A_j\|_p^p \leqslant \left\|\sum_{j=0}^{n-1} A_j\right\|_p^p, \qquad 1 \leqslant p < \infty. \tag{27}$$

For $n = 2$, this is a starting point of a proof of the Clarkson inequalities (8), and its generalisation as in (26) led to stronger versions in [4].

To bring out the relevance of Q-norms, we give a different proof of Theorem 1, based on the identity (24) and the inequality (27).

Another proof of Theorem 1 for all $p > 0$. Let $A_0, \dots, A_{n-1}$ be any operators, and let B_k be the sum defined in (20). Then, for $2 \leqslant p < \infty$, we have

$$\begin{aligned}\sum_{k=0}^{n-1} \|B_k\|_p^2 &= \sum_{k=0}^{n-1} \|B_k^* B_k\|_{p/2} \\ &\geqslant \left\| \sum_{k=0}^{n-1} B_k^* B_k \right\|_{p/2} \\ &= n \left\| \sum_{j=0}^{n-1} A_j^* A_j \right\|_{p/2} \\ &\geqslant n \left[\sum_{j=0}^{n-1} \|A_j^* A_j\|_{p/2}^{p/2} \right]^{2/p} \\ &= n \left[\sum_{j=0}^{n-1} \left(\|A_j\|_p^2\right)^{p/2} \right]^{2/p} \\ &\geqslant n \left[n^{1-p/2} \left(\sum_{j=0}^{n-1} \|A_j\|_p^2 \right)^{p/2} \right]^{2/p} \\ &= n^{2/p} \sum_{j=0}^{n-1} \|A_j\|_p^2.\end{aligned}$$

In this chain of reasoning, the second equality follows from the identity (24), while the three inequalities are derived from the triangle inequality, (27) and (18), respectively. This proves the first inequality in (3). For $0 < p \leqslant 2$, the three inequalities used in the derivation above are reversed. It has been noted [**6**, Lemma 1] that for positive operators A_j and $0 < p \leqslant 1$,

$$\sum \|A_j\|_p \leqslant \left\| \sum A_j \right\|_p,$$

and also that the inequality (27) is reversed in this case; see [**6**, p. 111] or [**12**, p. 20]. The inequality (18) is reversed too in this case.

This proves Theorem 1 for all $p > 0$. □

5. Let us now recast Theorem 2 in the mould of [4]. Taking pth roots, we rewrite the first inequality in (4) as

$$n^{1/p} \left\| \bigoplus_{j=0}^{n-1} A_j \right\|_p \leqslant \left\| \bigoplus_{k=0}^{n-1} B_k \right\|_p, \qquad 2 \leqslant p < \infty,$$

where B_k is as in (20), and then as

$$\left\| \bigoplus_{n \text{ copies}} \left[\bigoplus_{j=0}^{n-1} A_j \right] \right\|_p \leqslant \left\| \bigoplus_{k=0}^{n-1} B_k \right\|_p, \qquad 2 \leqslant p < \infty. \tag{28}$$

(For brevity, we use the notation $\oplus_{n \text{ copies}} X$ to mean the n-fold direct sum $X \oplus X \oplus \ldots \oplus X$.) In the same way, the second inequality in (4) can be rewritten as

$$n^{1/p}\left\|\bigoplus_{k=0}^{n-1} B_k\right\|_p \leqslant n\left\|\bigoplus_{j=0}^{n-1} A_j\right\|_p, \qquad 2 \leqslant p < \infty,$$

and then as

$$\left\|\bigoplus_{n \text{ copies}}\left[\bigoplus_{k=0}^{n-1} B_k\right]\right\|_p \leqslant n\left\|\bigoplus_{j=0}^{n-1} A_j\right\|_p, \qquad 2 \leqslant p < \infty. \tag{29}$$

In this form, the inequalities (28) and (29) shed some of their dependence on p, compared to the (equivalent) inequalities (4). What is left of p can be removed too. The inequalities (28) and (29) are true for all Q-norms. For the duals of Q-norms, they are reversed. This can be proved using the ideas expounded in [4] and in this paper. We do not give the details here.

6. The case $0 < p < 1$ of Theorem 2 is proved on the same lines as those used in Remark 4 above.

7. It is tempting to attempt a generalisation of Theorem 1 on the same lines as for Theorem 2 in Remark 5. Let us start with the special case, $n = 2$. The first inequality in (7) can be rewritten as

$$\|A \oplus A\|_p^2 + \|B \oplus B\|_p^2 \leqslant \|A + B\|_p^2 + \|A - B\|_p^2, \quad \text{for } 2 \leqslant p \leqslant \infty. \tag{30}$$

This is the same as saying that

$$\|A^*A \oplus A^*A\|_p + \|B^*B \oplus B^*B\|_p \leqslant \|(A+B)^*(A+B)\|_p + \|(A-B)^*(A-B)\|_p, \quad \text{for } 1 \leqslant p \leqslant \infty. \tag{31}$$

To ask whether the inequality (30) might be true for all Q-norms is to ask whether (31) might be true for all unitarily invariant norms; that is, whether we have

$$|||A^*A \oplus A^*A||| + |||B^*B \oplus B^*B||| \leqslant |||(A+B)^*(A+B) \oplus 0||| + |||(A-B)^*(A-B) \oplus 0||| \tag{32}$$

for all unitarily invariant norms. The answer is: 'no'.

On 8×8 matrices, consider the norm

$$|||A||| = \left[(s_1(A) + s_2(A))^2 + (s_3(A) + s_4(A))^2\right]^{1/2}.$$

Let $A = \operatorname{diag}(1, 1, 0, 0)$ and $B = \operatorname{diag}(0, 0, 2^{1/4}, 0)$. The direct sums involved in (32) are then 8×8 matrices. Each of the two norms on the left-hand side of (32) is equal to $2\sqrt{2}$, while each of the two norms on the right-hand side is equal to $(4+2\sqrt{2})^{1/2}$. Thus the putative inequality (32) is not always valid.

8. Ball, Carlen and Lieb [2] have proved the following inequalities for $1 \leqslant p \leqslant 2$:

$$\|A\|_p^2 + (p-1)\|B\|_p^2 \leqslant \frac{1}{2}(\|A+B\|_p^2 + \|A-B\|_p^2);$$

$$\|A\|_p^2 + (p-1)\|B\|_p^2 \leqslant \frac{1}{2^{2/p}}(\|A+B\|_p^p + \|A-B\|_p^p)^{2/p}.$$

Compare the first of these with one of the inequalities in (7):

$$2^{1-2/p}(\|A\|_p^2+\|B\|_p^2)\leqslant\frac{1}{2}(\|A+B\|_p^2+\|A-B\|_p^2),$$

and compare the second with the inequality obtained by following some of the steps of Remark 4:

$$\|A\|_p^2+\|B\|_p^2\leqslant\frac{1}{2}(\|A+B\|_p^p+\|A-B\|_p^p)^{2/p}$$

4. *Proof of Theorem 3, and generalisations*

This section has to be read along with the paper of Ando and Zhan [**1**] and that of Hirzallah and Kittaneh [**11**]. We indicate how the results obtained there for $n=2$ can be proved for $n>2$.

Recall that a non-negative function f on $[0,\infty)$ is said to be *operator monotone* if $f(A)\geqslant f(B)$ whenever A and B are positive operators with $A\geqslant B$. The function $f(t)=t^p$ is operator monotone for $0<p\leqslant 1$. Thus for $1\leqslant p<\infty$, the inverse function of $f(t)=t^p$ is operator monotone; see [**3**, Chapter V].

THEOREM 5 (Generalised Ando–Zhan theorem). *Let $A_0,\dots,A_{n-1}$ be positive operators. Then for every unitarily invariant norm, the following statements hold.*

(i) *For every non-negative operator monotone function f on $[0,\infty)$,*

$$\left|\left|\left|\sum_{j=0}^{n-1}f(A_j)\right|\right|\right|\geqslant\left|\left|\left|f\left(\sum_{j=0}^{n-1}A_j\right)\right|\right|\right|. \tag{33}$$

(ii) *This inequality is reversed if f is a non-negative increasing function on $[0,\infty)$ such that $f(0)=0$, $f(\infty)=\infty$, and the inverse function of f is operator monotone.*

Ando and Zhan [**1**] have proved this for $n=2$. An analysis of their proof shows that all their arguments can be suitably modified when $n>2$. In particular, in their crucial Lemma 1 we can replace the sum $A+B$ by $\sum_j A_j$, and then check that the same proof works. Using this, we can prove the following theorem.

THEOREM 6. *Let $A_0,\dots,A_{n-1}$ be any operators. Then for every unitarily invariant norm, the following statements hold.*

(i) *For every increasing function f on $[0,\infty)$ such that $f(0)=0$, $f(\infty)=\infty$ and the inverse function of $g(t)=f(\sqrt{t})$ is operator monotone,*

$$\begin{aligned} n\left|\left|\left|\sum_{j=0}^{n-1}f(|A_j|)\right|\right|\right| &\leqslant \left|\left|\left|\sum_{k=0}^{n-1}f\left(\left|\sum_{j=0}^{n-1}\omega_j^k A_j\right|\right)\right|\right|\right| \\ &\leqslant \frac{1}{n}\left|\left|\left|\sum_{j=0}^{n-1}f(n|A_j|)\right|\right|\right|. \end{aligned} \tag{34}$$

(ii) *The two inequalities in (34) are reversed for every nonnegative function f on $[0,\infty)$ such that $h(t)=f(\sqrt{t})$ is operator monotone.*

The $n=2$ case of Theorem 6 has been proved by Hirzallah and Kittaneh [**11**]. Their arguments can be modified by replacing the Ando–Zhan theorem by its generalisation, pointed out above. Their Lemma 1 needs no change. At one stage, we need the identity

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|\sum_{j=0}^{n-1}\omega_j^k A_j\right|^2 = \sum_{j=0}^{n-1}|A_j|^2. \tag{35}$$

This is just the identity (24), and substitutes for its $n = 2$ version used in [**11**, p. 366, line 6]. We leave the rest of the details to the reader.

The two parts of Theorem 3 follow from the corresponding parts of Theorem 6 if we choose $f(t) = t^p$ with $p \geqslant 2$ and $0 < p \leqslant 2$, respectively.

REMARK. Note that [**1**, Corollaries 1–3] and [**11**, Corollaries 2 and 3] can also be generalised to n-tuples of operators in this manner.

5. *Proof of Theorem 4*

Imitating the standard complex interpolation proof of the $n = 2$ case, we give a proof of Theorem 4 for $1 < p \leqslant 2$. The ideas are the same as those used in [**10**]. At a crucial stage, we need a generalisation of the parallelogram law provided by Theorem 1.

LEMMA. *Let $A_0, \dots, A_{n-1}$ be operators in the Schatten p-class C_p for some $1 < p \leqslant 2$. Let B_k be the sum defined in* (20), *and let Y_k, $0 \leqslant k \leqslant n-1$, be operators in the dual class C_q. Then*

$$\left|\operatorname{tr}\sum_{k=0}^{n-1} Y_k B_k\right| \leqslant n^{1/q}\left(\sum_{j=0}^{n-1}\|A_j\|_p^p\right)^{1/p}\left(\sum_{k=0}^{n-1}\|Y_k\|_q^p\right)^{1/p}. \tag{36}$$

Proof. Let $A_j = |A_j|W_j$ and $Y_k = V_k|Y_k|$ be right and left polar decompositions of A_j and Y_k, respectively. Here, W_j and Y_k are partial isometries.

We have $1/2 \leqslant 1/p < 1$. For the complex variable $z = x+iy$ with $1/2 \leqslant x \leqslant 1$, let

$$\begin{aligned} A_j(z) &= |A_j|^{pz}W_j;\\ Y_k(z) &= \|Y_k\|_q^{pz-q(1-z)}V_k|Y_k|^{q(1-z)}. \end{aligned}$$

Note that $A_j(1/p) = A_j$ and $Y_k(1/p) = Y_k$. Let

$$f(z) = \operatorname{tr}\sum_{k=0}^{n-1} Y_k(z)B_k(z).$$

The left-hand side of (36) is $|f(1/p)|$. We can estimate this if we have bounds for $|f(z)|$ at $x = 1/2$ and $x = 1$. If $x = 1$, we have

$$|\operatorname{tr} Y_k(z)A_j(z)| = \|Y_k\|_q^p\,|\operatorname{tr} V_k|Y_k|^{-iqy}\,|A_j|^{p(1+iy)}W_j|.$$

Using the information that for any operator T,

$$|\operatorname{tr} T| \leqslant \|T\|_1, \quad \text{and} \quad |||XTZ||| \leqslant \|X\|\,|||T|||\,\|Z\|$$

for any three operators X, T and Z and unitarily invariant norm $|||\cdot|||$, we see that

$$|\operatorname{tr} Y_k(z)A_j(z)| \leqslant \|Y_k\|_q^p\|A_j\|_p^p, \quad \text{for all } 0 \leqslant j,\ k \leqslant n-1.$$

Hence

$$|f(z)| = \left|\operatorname{tr}\sum_{k=0}^{n-1} Y_k(z)B_k(z)\right| \leqslant \left(\sum_{k=0}^{n-1}\|Y_k\|_q^p\right)\left(\sum_{j=0}^{n-1}\|A_j\|_p^p\right), \tag{37}$$

when $x = 1$.

When $x = 1/2$, the operators $A_j(z)$ and $Y_k(z)$ are in C_2 and

$$\begin{aligned}|f(z)| &\leqslant \sum_{k=0}^{n-1} |\operatorname{tr} Y_k(z)B_k(z)| \\ &\leqslant \sum_{k=0}^{n-1} \|Y_k(z)\|_2\|B_k(z)\|_2 \\ &\leqslant \left(\sum_{k=0}^{n-1}\|Y_k(z)\|_2^2\right)^{1/2}\left(\sum_{k=0}^{n-1}\|B_k(z)\|_2^2\right)^{1/2} \\ &= n^{1/2}\left(\sum_{k=0}^{n-1}\|Y_k(z)\|_2^2\right)^{1/2}\left(\sum_{j=0}^{n-1}\|A_j(z)\|_2^2\right)^{1/2}\end{aligned}$$

The equality at the last step is a consequence of Theorem 1, specialised to the case $p = 2$. Note that when $x = 1/2$, we have $\|A_j(z)\|_2^2 = \|A_j\|_p^p$, and $\|Y_k(z)\|_2^2 = \|Y_k\|_q^p$. Hence

$$|f(z)| \leqslant n^{1/2}\left(\sum_{k=0}^{n-1}\|Y_k\|_q^p\right)^{1/2}\left(\sum_{j=0}^{n-1}\|A_j\|_p^p\right)^{1/2}, \tag{38}$$

when $x = 1/2$. If M_1 is the right-hand side of (37) and M_2 that of (38), then by the three-line theorem we have, for $1/2 \leqslant 1/p < 1$,

$$|f(1/p)| \leqslant M_1^{2(1/p-1/2)} M_2^{2(1-1/p)}$$

This gives (36). □

Now, to prove Theorem 4, let $B_k = U_k|B_k|$ be a polar decomposition, and let

$$Y_k = \|B_k\|_p^{q-p}|B_k|^{p-1}U_k^\star.$$

It is easy to see that

$$\operatorname{tr} Y_k B_k = \|B_k\|_p^q = \|Y_k\|_q^p.$$

So we get, from (36):

$$\sum_{k=0}^{n-1}\|B_k\|_p^q \leqslant n^{1/q}\left(\sum_{j=0}^{n-1}\|A_j\|_p^p\right)^{1/p}\left(\sum_{k=0}^{n-1}\|B_k\|_p^q\right)^{1/p}$$

This is the same as saying that

$$\sum_{k=0}^{n-1}\|B_k\|_p^q \leqslant n\left(\sum_{j=0}^{n-1}\|A_j\|_p^p\right)^{q/p}, \qquad 1 < p \leqslant 2. \tag{39}$$

This proves Theorem 4 for $1 < p \leqslant 2$. The reverse inequality for $2 \leqslant p < \infty$ can be obtained from this by a duality argument. □

By a change of variables, a pair of complementary inequalities can be obtained as in Theorems 1–3. As pointed out earlier [2, 4], the inequalities of Theorem 2 follow from those of Theorem 4 by simple convexity arguments. Theorem 1 too can be derived from Theorem 4 by such arguments. For example, for $2 \leqslant p < \infty$, we have from (6):

$$\left(\sum_{j=0}^{n-1} \|A_j\|_p^p\right)^{1/p} \leqslant \left(\frac{1}{n}\sum_{k=0}^{n-1} \left\|\sum_{j=0}^{n-1} \omega_j^k A_j\right\|_p^q\right)^{1/q} \tag{40}$$

On the positive half-line, the function $f(x) = x^{2/q}$ is convex and the function $g(x) = x^{2/p}$ is concave. Using this information, we can obtain the first inequality in (3) from the inequality (40). The proof given in Section 2 is based on easier ideas.

6. *Further generalisations*

An anonymous referee has made the interesting observation that our proofs of Theorems 1 and 2 rely on one crucial property of the finite Fourier transform matrix F_n, namely that F_n^2 is a permutation matrix. More generally, let R_n be any $n \times n$ matrix such that

$$R_n^2 = zP, \tag{41}$$

where z is a complex number and P a permutation matrix. Then

$$R_n^* R_n = \alpha I_n, \quad \text{where } \alpha = |z|. \tag{42}$$

For $n = 2$, some examples of such matrices are:

$$\begin{bmatrix} r & \sqrt{2-r^2} \\ \sqrt{2-r^2} & -r \end{bmatrix}, \quad \begin{bmatrix} s & i\sqrt{s^2-2} \\ i\sqrt{s^2-2} & -s \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix},$$

where r and s are real numbers with $r^2 \leqslant 2$ and $s^2 \geqslant 2$.

Let $\boldsymbol{R} = R_n \otimes I$. Then $\boldsymbol{R}$ acts naturally on $n \times 1$ column vectors whose entries are operators $A_0, \ldots, A_{n-1}$. For a given operator $\boldsymbol{R}$ as above, let

$$\boldsymbol{A}' = \boldsymbol{R}\boldsymbol{A}. \tag{43}$$

With these notations, we have the following theorem.

THEOREM 7. *For $2 \leqslant \lambda \leqslant p < \infty$, we have*

$$n^{\lambda/p-\lambda/2}\alpha^{\lambda/2} \sum_{j=0}^{n-1} \|A_j\|_p^\lambda \leqslant \sum_{j=0}^{n-1} \|A_j'\|_p^\lambda \leqslant n^{\lambda/2-\lambda/p}\alpha^{\lambda/2} \sum_{j=0}^{n-1} \|A_j\|_p^\lambda. \tag{44}$$

Proof. The proof follows the steps in Remark 4 of Section 3. Replace the tuple $\boldsymbol{B}$ there by $\boldsymbol{A}'$. Note that

$$\sum_{j=0}^{n-1} A_j'^* A_j' = \alpha \sum_{j=0}^{n-1} A_j^* A_j, \tag{45}$$

and that for $2 \leqslant \lambda \leqslant p$, the function $f(x) = x^{\lambda/2}$ is convex and $g(x) = x^{\lambda/p}$ is concave. This leads to the first inequality in (44). The second can be obtained from this on replacing $\boldsymbol{A}$ by $\boldsymbol{A}'$ and noting that the tuple $\boldsymbol{A}''$ is a permutation of the tuple $\boldsymbol{A}$. We leave the details to the reader. □

Theorem 7 includes, as special cases, Theorems 1 and 2. If R_n is the matrix with ω_j^k as its (j,k) entry, then $\alpha = n$. Further, by choosing $\lambda = 2, p$, we obtain Theorems 1 and 2, respectively.

Similar generalisations of Theorems 3 and 4 may be obtained. Let us indicate this briefly. Replace the identity (35) by

$$\sum_{j=0}^{n-1} |A_j'|^2 = \alpha \sum_{j=0}^{n-1} |A_j|^2, \tag{46}$$

to obtain, instead of (34), the inequalities

$$n \left|\left|\left| \sum_{j=0}^{n-1} f\left(\sqrt{\frac{\alpha}{n}}\, |A_j|\right) \right|\right|\right| \leqslant \left|\left|\left| \sum_{j=0}^{n-1} f\left(|A_j'|\right) \right|\right|\right| \leqslant \frac{1}{n} \left|\left|\left| \sum_{j=0}^{n-1} f\left(\sqrt{n\alpha}\, |A_j|\right) \right|\right|\right|.$$

This leads to the inequalities

$$\begin{aligned} n^{1-p/2}\alpha^{p/2} \left|\left|\left| \sum_{j=0}^{n-1} |A_j|^p \right|\right|\right| &\leqslant \left|\left|\left| \sum |A_j'|^p \right|\right|\right| \\ &\leqslant n^{p/2-1}\alpha^{p/2} \left|\left|\left| \sum |A_j|^p \right|\right|\right|, \end{aligned} \tag{47}$$

for $2 \leqslant p \leqslant \infty$. The inequality (5) is included in this as a special case.

Let r_{ij} be the entries of the matrix R_n, and let $r = \max |r_{ij}|$. If we replace the tuple $\boldsymbol{B}$ in Section 5 by the tuple $\boldsymbol{RA}$, we obtain modified versions of the inequalities (37) and (38). For the first of these, we pick up an extra factor r on the right-hand side; for the second, we need to replace the factor $n^{1/2}$ by $\alpha^{1/2}$. The interpolation argument then shows that

$$\left| \operatorname{tr} \sum_{k=0}^{n-1} Y_k A_k' \right| \leqslant r^{2(1/p-1/2)} \alpha^{1/q} \left(\sum_{j=0}^{n-1} \|A_j\|_p^p \right)^{1/p} \left(\sum_{k=0}^{n-1} \|Y_k\|_q^p \right)^{1/p}.$$

This replaces the inequality (36). Imitating the rest of the argument in the proof of Theorem 4, one obtains at the end

$$\sum_{j=0}^{n-1} \|A_j'\|_p^q \leqslant r^{q-2} \alpha \left(\sum_{j=0}^{n-1} \|A_j\|_p^p \right)^{q/p}, \tag{48}$$

for $1 < p \leqslant 2$. Once again, when R is the matrix with entries ω_j^k, then $\alpha = n$, $r = 1$, and the inequality (48) reduces to (39).

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