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GENERALIZED LYAPUNOV EQUATIONS AND POSITIVE DEFINITE FUNCTIONS*

RAJENDRA BHATIA† AND DRISS DRISSI‡

Abstract. We establish the positive definiteness of some functions and of some matrices that arise as solutions of generalized Lyapunov equations.

Key words. positive definite matrix, positive definite function, Fourier transform, Bochner's theorem, Lyapunov matrix equation, operator means

AMS subject classifications. 42A82, 47A62, 15A24, 15A48

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1. Introduction.

Let A be a positive definite matrix and consider the following matrix equations:

(L1) $$AX + XA = B.$$

(L2) $$A^2X + XA^2 + tAXA = B.$$

(L3) $$A^3X + XA^3 + t(A^2XA + AXA^2) = B.$$

(L4) $$A^4X + XA^4 + t(A^3XA + AXA^3) + 6A^2XA^2 = B.$$

(L5) $$A^4X + XA^4 + 4(A^3XA + AXA^3) + tA^2XA^2 = B.$$

Equation (L1) is known as the *Lyapunov equation* and has been studied extensively.

We may choose an orthonormal basis for the underlying space in which A is diagonal, $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then, in component form, (L1) may be written as

$$(\lambda_i + \lambda_j)x_{ij} = b_{ij}$$

and solved as

(1) $$x_{ij} = \frac{b_{ij}}{\lambda_i + \lambda_j}.$$

The matrix C with entries

(2) $$c_{ij} = \frac{1}{\lambda_i + \lambda_j}$$

is called the *Cauchy matrix*. It is positive semidefinite. One way of seeing this is by writing

(3) $$\frac{1}{\lambda_i + \lambda_j} = \int_0^\infty e^{-t(\lambda_i+\lambda_j)}dt$$

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