QUALITY ENGINEERING®
Vol. 16, No. 4, pp. 577–584, 2004

Estimation of Life with Piecewise Linear-Quadratic Hazard Rate

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ABSTRACT

Bladder is an important accessory in the curing operation for tire manufacturing. Failure of bladder, during its use, results in scrap of a large amount of tire under curing. This leads to heavy financial loss to the company. In this article, we deal with estimation of life and determination of replacement plan for bladder. The life of bladder has been modeled using a piecewise linear-quadratic hazard function. A computational procedure is proposed for estimation of life. Finally, a replacement plan for bladder is derived.

Key Words: Replacement plan; Linear-quadratic hazard rate function; Wear-out point estimation; Bootstrap.

INTRODUCTION

In many manufacturing processes, the output quality largely depends on a few critical machinery parts. Such parts are required to be replaced periodically on completion of their useful life. For example, grinding tools play a major role in maintaining proper dimension and surface finish of components that are processed during grinding. When the tool wears out, it must be replaced immediately. Otherwise, the process will generate undesirable output. Therefore, it is quite important to determine a replacement plan for such parts in order to maintain high quality of output. A replacement plan is drawn using the information on its life. This leads us to the problem of life estimation. It is also required for a reliability improvement program. In this article, we deal with such a part, called *bladder*, to estimate its life for the derivation of replacement plan.

Bladder is a shaper medium for the manufacture of tires. It is used to shape the green tire onto the tire mould during the curing process as follows. Hot water/steam at specified temperature and pressure is circulated through bladder. Bladder transfers heat to the surrounding green tire, similar to an exothermic process. It is therefore essential that a bladder possess good physical properties and high impermeability.

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A bladder is expected to perform its intended function for a certain number of curing cycles. The life of a bladder is designated by the number of such cycles it has performed. Failure of bladder during the curing process results in scrap of the whole tire that is put for curing. On the other hand, an early replacement of bladder leads to under-utilization (and hence additional bladder cost). In the absence of proper knowledge on the life of bladder, the operators carry out inspection (in between cycles) for possible defect(s) in order to arrive at the decision for its replacement. This has resulted in a substantial scrap of tire due to bladder failure during curing. A suitable replacement plan will thus help in (1) reducing inspection cost through lesser inspection, (2) bringing down scrap of tire during curing process, and (3) arriving at a decision on inventory of bladder. Further, as these bladders are produced in-house (using a single facility), reliability assessment of the same is also important.

A plausible approach to the development of a replacement plan is based on the estimate of the wear-out point of the corresponding hazard rate function. Wear-out point of the hazard rate function is defined as a *change-point*. Generally, a change-point refers to the time point when hazard rate pattern changes abruptly. The literature on estimation of change-point is quite rich. For reference, see Gombay (2000), Loader (1991), Nguyen et al. (1984), Rukhin (1997). Motivation for this problem can be found in various applications. For instance, Mathews and Farewell (1982) studied the effectiveness of a new therapy in terms of reduction in relapse rate. Basu et al. (1988) (hereafter referred to as BGJ) deal with change-point estimation in the context of optimal burn-in strategy in reliability engineering. Gürler and Yenigün (2002) proposed a method for detection of AIDS (acquired immune deficiency syndrome) in HIV-infected patients. A large majority of these articles consider piecewise linear hazard rate function (with or without a jump at the change-point). The solution procedures are developed using likelihood method or bayesian approach. A common feature of these articles is the assumption that hazard rate is well known. On the contrary, a practical problem generally calls for modeling and estimation of hazard rate.

This article is organized as follows. We discuss the basic data on bladder life and study the pattern of empirical hazard rate function. Lifetime is modeled using a piecewise linear-quadratic hazard function. We propose a computational procedure for the estimation of associated parameters and for derivation of replacement plan of bladder. We also discuss the results of maximum likelihood method and the procedure of BGJ and contrast them with that of the proposed method.

DATA COLLECTION AND PRELIMINARY ANALYSIS

It is customary to maintain a life-history card for every bladder. This contains information on its life (i.e., the cumulative number of completed cycles), in addition to other particulars such as dates of mounting and failure, failure mode, etc. Failures have been largely attributed to design or manufacturing deficiency. Also, local controls in usage such as handling, pressure, and temperature are technically expected to have bearing on bladder life. A defective (failed) bladder is not repairable.

Depending upon the extent of usage (number of cycles) per day, a bladder lasts only a couple of days. The basic data on observed life correspond to 257 bladders that failed during three months in the recent past. Therefore, these constitute a random sample without censoring. Among these failures, 251 bladders correspond to leakage failure mode. The remaining six failures are due to bladder cut, weak bladder, or bladder impression. Let t_i be the life (in cycles) of bladder i. That is, we have the observations t_i, $i = 1, \ldots, 257$. The first step in life data analysis is to estimate the hazard rate (failure rate) that leads us to guess fairly well about the underlying distribution of life. See Lawless (1982), Meeker and Escobar (1998), Nelson (1982) for reference.

The hazard rate for bladder is estimated in Appendix A, and is plotted in Fig. 1. We observe that the values of six t_i's corresponding to the *non-leakage* failure modes are spread over the entire range of remaining t_i's, and hazard rate function remains almost unaltered even when they are dropped. Therefore, we decide to retain them for all subsequent computation and discussion.

It is evident that (empirical) hazard rate is more or less constant up to a certain time point and then increases very rapidly. This time point is referred to as wear-out point or change-point. On comparison of Fig. 1 with a typical *bath tub curve* that generally describes failure rate over the whole life-span of a product, we conclude that

- The *infant mortality* phase is absent,
- The period indicating constant failure rate is *useful life*, and

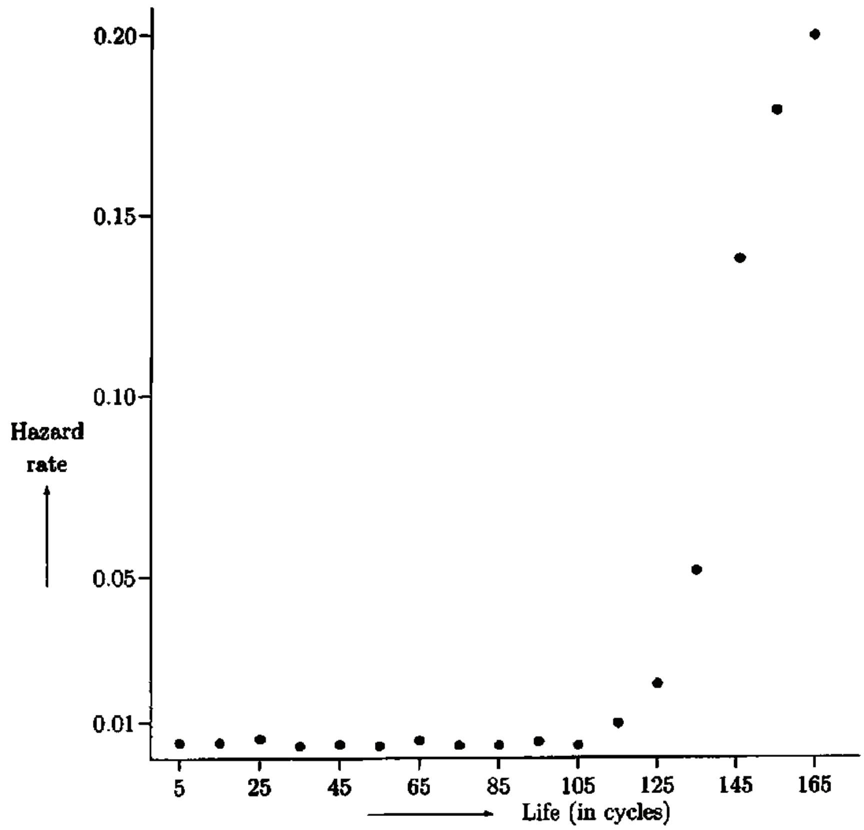

Figure 1. Hazard rate plot.

- The last phase describes *wear-out* period starting from the change-point (or wear-out point).

ESTIMATION OF LIFE

In this section, we try to identify a suitable distribution that describes the life of bladder. We refer to the hazard rate plot of the last section and the conclusions drawn therein. It may be observed that standard probability distributions, like Weibull distribution, log-normal distribution, or extreme-value distribution are not well suited to describe this life pattern. Therefore, we look for a function that closely resembles the observed hazard rate function.

We consider a piecewise linear-quadratic hazard rate function $h(t)$ given by Eq. (1), where t denotes the life and λ is the constant failure rate up to the change-point τ. Beyond τ, $h(t)$ is quadratic in form.

$$h(t) = \begin{cases} \lambda & \text{if } t < \tau \\ \lambda + \beta_1(t-\tau) + \beta_2(t-\tau)^2 & \text{if } t \geq \tau \end{cases} \qquad (1)$$

Hence, it is expected to represent our data well. However, we note that unless both β_1 and β_2 are positive, the corresponding distribution function need not be nondecreasing. In fact, it is found that the least-square estimates are $\hat{\beta}_1 < 0$ and $\hat{\beta}_2 > 0$. (The estimation procedure is discussed later). Although the function given by Eq. (1) represents our empirical hazard rate extremely well, the corresponding distribution function of life is not well defined due to the signs of $\hat{\beta}_1$ and $\hat{\beta}_2$. Therefore, we resort to the following modified form of hazard rate function:

$$h(t) = \begin{cases} \lambda & \text{if } t < \tau \\ \lambda + \beta(t-\tau)^2 & \text{if } t \geq \tau. \end{cases} \qquad (2)$$

The corresponding density function $f(t)$ and distribution function $F(t)$ are obtained by using the following relationship (Kapur and Lamberson, 1977):

$$f(t) = h(t)\exp\left[-\int_0^t h(x)\,dx\right] \tag{3}$$

and

$$1 - F(t) = \frac{f(t)}{h(t)}. \tag{4}$$

The explicit form of $f(t)$ and $F(t)$ are

$$f(t) = \begin{cases} \lambda\exp(-\lambda t) & \text{if } t < \tau \\ [\lambda + \beta(t-\tau)^2] & \\ \quad \exp[-\{\lambda t + \beta(t-\tau)^3/3\}] & \text{if } t \geq \tau, \end{cases} \tag{5}$$

$$F(t) = \begin{cases} 1 - \exp(-\lambda t) & \text{if } t < \tau \\ 1 - \exp[-\{\lambda t + \beta(t-\tau)^3/3\}] & \text{if } t \geq \tau. \end{cases} \tag{6}$$

The graphical forms of $f(t)$ and $F(t)$ are shown in Appendix B, along with the moments of the distribution.

In order to estimate the parameters involved, we fit the hazard rate function given by Eq. (2) to our data by method of least squares. Note that besides the usual parameters λ and β, the change-point τ is also required to be estimated. In this context, we propose the following procedure.

1. Suppose τ_0 be a trial value of τ. Divide the set of all 257 bladders into two groups, namely, $G_1 = \{\text{bladder } i: t_i < \tau_0\}$, $G_2 = \{\text{bladder } i: t_i \geq \tau_0\}$.
2. Obtain LSE (least-square estimate) of λ, say $\hat{\lambda}$, from G_1, and let ESS(G_1) be the corresponding error sum of squares. Use this estimate of λ in G_2 and obtain $\hat{\beta}$, the LSE of β. Let ESS(G_2) be the error sum of squares in G_2.
3. Denote the estimate of τ by $\hat{\tau}$. Select $\hat{\tau} = \tau_0$ when the sum ESS(G_1)+ESS(G_2) is minimum for $\tau = \tau_0$. Further, the corresponding $\hat{\lambda}$ and $\hat{\beta}$ are taken as the estimates of λ and β, respectively.

This yields the estimates as

$\hat{\tau} = 112$ cycles,

$\hat{\lambda} = 0.002522$, and

$\hat{\beta} = 0.000127$.

Table 1. Test for goodness of fit.

Class interval of life (cycles)	Observed frequency	Expected frequency	Estimated χ^2
0–10	7	6.400	0.056
10–20	7	6.241	0.092
20–30	9	6.086	1.396
30–40	4	5.934	0.630
40–50	5	5.786	0.107
50–60	4	5.642	0.478
60–70	7	5.502	0.408
70–80	4	5.365	0.347
80–90	4	5.231	0.290
90–100	6	5.101	0.159
100–110	4	4.974	0.191
110–120	15	8.921	4.142
120–130	31	41.165	2.510
130–140	60	73.366	2.435
140–150	73	54.035	6.656
Above 150	17	17.251	0.004
Total	257	257.000	19.900

At 5% level of significance, tabulated value of $\chi^2_{12} = 21.026$.

The mean (μ) and standard deviation (σ) of life are then estimated as $\hat{\mu} = 116.20$ cycles and $\hat{\sigma} = 40.20$ cycles. (See Appendix B for the formulae). The corresponding observed values in the sample are 116.94 cycles and 41.63 cycles, respectively.

Subsequently, the adequacy of $f(t)$, given in Eq. (5), is evaluated with a χ^2-test (see Table 1). We conclude that $f(t)$ describes the data quite satisfactorily. The graph for the associated reliability function $R(t)$ is given in Fig. 2.

REPLACEMENT POLICY

It is evident that a safe replacement policy is to withdraw the bladder when it has reached the wear-out point τ. However, the true value of τ is unknown; we have merely an estimate of it. In such case, we depend on its confidence interval. This is derived using the *bootstrap methodology* (Efron and Tibshirani, 1993). One thousand bootstrap samples are drawn from our basic data set, and change-point is estimated (following the procedure presented earlier) for each of these samples. The percentiles of $\hat{\tau}$, thus obtained, are presented in Table 2.

It is to be noted that in order to decide on the replacement time of bladder, we are interested in the lowest possible value of τ. That is, a lower confidence

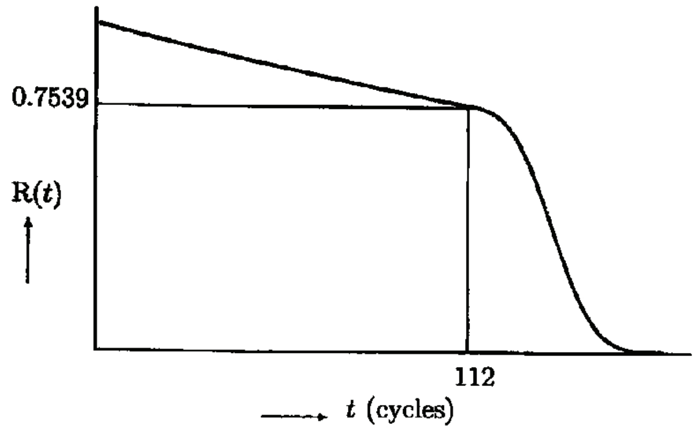

Figure 2. Reliability function.

Table 2. Percentiles of $\hat{\tau}$.

Probability	0.05	0.10	0.50	0.90	0.95
Value of $\hat{\tau}$	94	101	113	123	125

interval for τ is of interest. A 95% lower confidence interval for τ is given by

$$\tau \geq 94 \text{ cycles},$$

that is, a bladder should be replaced when it has completed 94 cycles.

DISCUSSION

The method of maximum likelihood can also be applied for estimation of the parameters. It may be observed that the corresponding likelihood function is not differentiable; therefore, we obtain the estimates numerically.

As we have mentioned earlier, most researchers deal with piecewise linear hazard rate functions. However, the procedure of BGJ (Basu et al., 1988) is quite relevant here for the estimation of change-point, but it does not address the problem of life estimation. Minor trivial modifications are made to this procedure to suit the present problem. See Appendix C for a brief description. The estimate of τ by BGJ is obtained as $\tilde{\tau} = 110$ cycles, and the 95% lower confidence interval for τ (estimated through bootstrap method using 1000 bootstrap samples) is given by: $\tau \geq 97$ cycles. Hence, we find that there is almost no discrepancy in replacement plan obtained from these two methods.

Based on the findings, namely, (1) high failure rate during useful life and (2) low wear-out point, the concerned management promptly initiated action to thoroughly review the design and manufacturing aspects of bladder for improvement. The management has also taken note of the fact that about 21% of the bladders will fail prior to the proposed replacement time of 94 cycles. After analyzing the associated cost components, it has been decided to implement this replacement time of bladder with immediate effect.

CONCLUSION

We have studied the problem of life estimation and replacement policy for bladder. It is observed that the hazard rate is a piecewise linear-quadratic function. We have proposed a procedure for estimation of life in this situation. Subsequently, the replacement plan for bladder is derived using the confidence interval of the wear-out point.

As in the present case of bladder, there can be many situations where life distribution may not be easily visualized as one of the standard probability distributions. Under such circumstances, we rely heavily on the underlying hazard function. The advantage of the proposed procedure over existing ones is that it makes direct use of the empirical hazard function in order to identify the distribution of life. This estimation procedure is primarily based on the least square method and is applicable to general piecewise hazard function. However, utmost care must be taken in estimation of proper distribution function of life while dealing with quadratic hazard function.

We treat the proposed procedure as a new alternative for estimation of life. We intend to carry out a detailed study to evaluate its performance in comparison with the established methods for both complete as well as censored data.

APPENDIX A

Estimation of Hazard Rate

Hazard rate during an interval = (number of failures during the interval) ÷ (Average number of bladders exposed to the risk of failure at the midpoint of the interval) × (length of the interval) (Srinath, 1975). The calculations are shown below (Table 3).

APPENDIX B

Graphs of $f(t)$ and $F(t)$

Depending upon the values of λ and β, both $f(t)$ as well as $F(t)$ take different forms. They are displayed in Figs. 3 and 4, respectively. Given λ, a larger value of β increases the peak of $f(t)$. Whereas, the value of λ (for fixed β) decides its shape. The value of τ mainly gives the time-point after which $f(t)$ starts increasing (see cases b and c of Fig. 3).

Table 3. Estimation of hazard rate.

Class interval of life (cycles)	Number of failures	Number survivors	Hazard rate
0–10	7	250	0.00276
10–20	7	243	0.00284
20–30	9	234	0.00377
30–40	4	230	0.00172
40–50	5	225	0.00220
50–60	4	221	0.00179
60–70	7	214	0.00322
70–80	4	210	0.00189
80–90	4	206	0.00192
90–100	6	200	0.00296
100–110	4	196	0.00202
110–120	15	181	0.00796
120–130	31	150	0.01873
130–140	60	90	0.05000
140–150	73	17	0.13645
150–160	16	1	0.17778
160–170	1	0	0.20000

The mean (μ) of life can be computed as follows:

$$\mu = \begin{cases} \dfrac{1}{\lambda}[1-e^{-\lambda\tau}] + \dfrac{e^{-\lambda\tau}}{\sqrt[3]{9\beta}} \displaystyle\sum_{j=0}^{\infty}\left(-\lambda\sqrt[3]{\frac{3}{\beta}}\right)^{j} \times \dfrac{\Gamma((j+1)/3)}{j!} & \text{if } \lambda\sqrt[3]{\dfrac{3}{\beta}} \le 1 \\ \dfrac{1}{\lambda}[1-e^{-\lambda\tau}] + \dfrac{e^{-\lambda\tau}}{\lambda} \displaystyle\sum_{j=0}^{\infty}\left(-\frac{\beta}{3\lambda^3}\right)^{j} \times \dfrac{(3j)!}{j!} & \text{if } \lambda\sqrt[3]{\dfrac{3}{\beta}} > 1 \end{cases}$$

and, it may be observed that $\mu > (1-e^{-\lambda\tau})/\lambda$. In general, the rth raw moment of life is given by

$$\mu_r' = \begin{cases} \dfrac{r!}{\lambda^r} - \tau^r e^{-\lambda\tau} \displaystyle\sum_{k=1}^{r} \frac{1}{(\lambda\tau)^k}\frac{r!}{(r-k)!} + re^{-\lambda\tau}\sum_{k=0}^{r-1}\binom{r-1}{k}\frac{\tau^{r-k-1}}{\sqrt[3]{3^{2-k}\beta^{k+1}}} \times \sum_{j=0}^{\infty}\left(-\lambda\sqrt[3]{\frac{3}{\beta}}\right)^{j}\frac{\Gamma((k+j+1)/3)}{j!} & \text{if } \lambda\sqrt[3]{\dfrac{3}{\beta}} \le 1 \\ \dfrac{r!}{\lambda^r} - \tau^r e^{-\lambda\tau} \displaystyle\sum_{k=1}^{r} \frac{1}{(\lambda\tau)^k}\frac{r!}{(r-k)!} + re^{-\lambda\tau}\sum_{k=0}^{r-1}\binom{r-1}{k}\frac{\tau^{r-k-1}}{\lambda^{k+1}} \times \sum_{j=0}^{\infty}\left(-\frac{\beta}{3\lambda^3}\right)^{j}\frac{(3j+k)!}{j!} & \text{if } \lambda\sqrt[3]{\dfrac{3}{\beta}} > 1. \end{cases}$$

APPENDIX C

BGJ Procedure

Suppose that $\hat{F}(t)$ is the empirical distribution function of life, and $y(t) = -\log[1-\hat{F}(t)]$. Let $p_0 > 0$ be very small and p_1 be a good lower bound for $F(\tau)$ (<1), where $F(t)$ is the true distribution function of life.

Then, the estimate of λ is given by

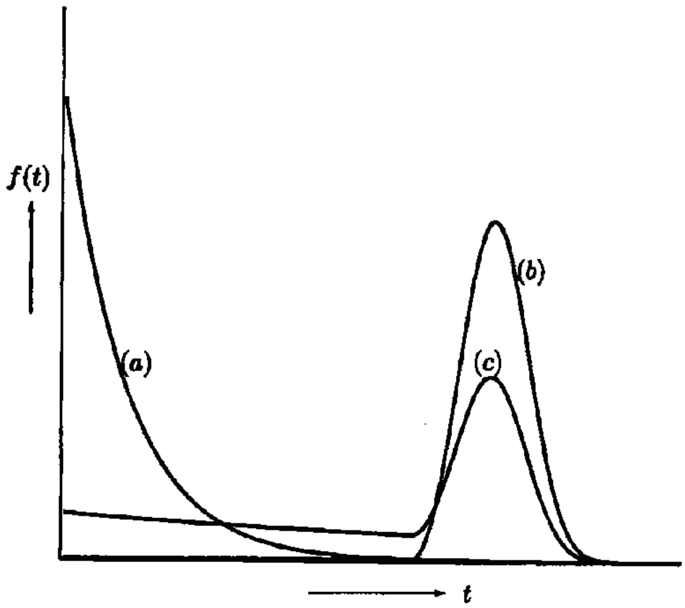

Figure 3. Graphs of $f(t)$ for $\tau = 112$ and (a) $\lambda = 0.05$, $\beta = 0.0001$ (i.e., $\lambda \gg \beta$), (b) $\lambda = 0.0001$, $\beta = 0.0001$ (i.e., λ very small), (c) $\lambda = 0.005$, $\beta = 0.0001$.

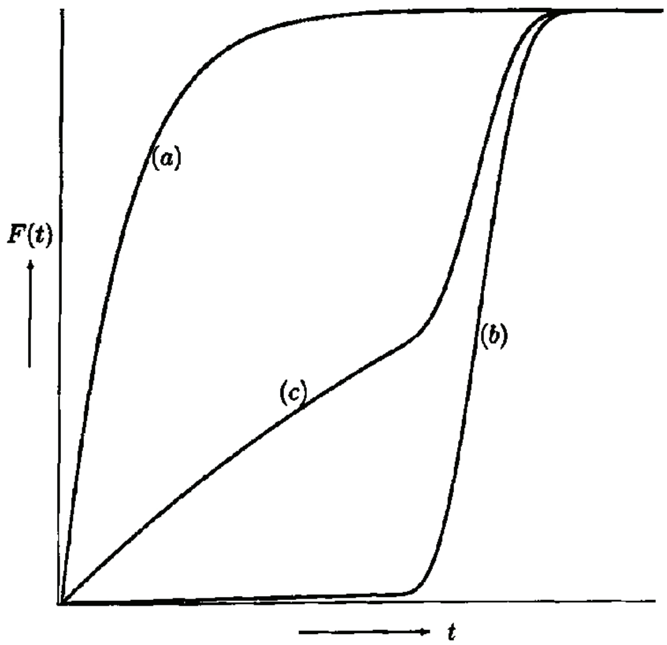

Figure 4. Graphs of $F(t)$ for $\tau = 112$ and (a) $\lambda = 0.05$, $\beta = 0.0001$ (i.e., $\lambda \gg \beta$), (b) $\lambda = 0.0001$, $\beta = 0.0001$ (i.e., λ very small), (c) $\lambda = 0.005$, $\beta = 0.0001$.

$$\bar{\lambda} = \frac{\sum t_{[i]} y(t_{[i]})/(k+1) - (\sum t_{[i]}/(k+1))(\sum y(t_{[i]})/(k+1))}{\sum t_{[i]}^2/(k+1) - (\sum t_{[i]}/(k+1))^2}, \quad (7)$$

where $t_{[i]}$ is the ith ordered life, $k = [np_1] - [np_0]$, n is the total number of observations, and all the summations range over $i = [np_0] + 1$ to $[np_1]$.

Then the BGJ estimator for τ is estimated as

$$\bar{\tau} = \sup\left\{ t : \frac{y(t+\Delta t) - y(t)}{\Delta t} < \bar{\lambda} \right\}. \quad (8)$$

$\Delta t > 0$ and $\bar{\lambda}$ is obtained from Eq. (7).

In order to derive the estimates, we have $n = 257$, $\hat{F}(t_{[i]}) = (i - 0.3)/(n + 0.4)$ (average ranks are used for tie cases), and set $p_0 = 0.004$, $p_1 = 0.20$, $\Delta t = 1$. Using Eqs. (7) and (8), we get

$\hat{\lambda} = 0.002445$ per cycle, and

$\hat{\tau} = 110$ cycles.

ACKNOWLEDGMENT

The author thanks Prof. Necip Doganaksoy for many helpful comments and suggestions that have improved substantially both the content as well as the presentation of this article.

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REFERENCES

Basu, A. P., Ghosh, J. K., Joshi, S. N. (1988). On estimating change point in a failure rate. In: Gupta, S. S., Berger, J. O., eds. *Statistical Decision Theory and Related Topics IV*. Vol. 2. Springer-Verlag, 239–252.

Efron, B., Tibshirani, R. J. (1993). *An Introduction to the Bootstrap*. Chapman & Hall.

Gombay, E. (2000). Sequential change-point detection with likelihood ratios. *Statistics & Probability Letters* 49:195–204.

Gürler, Ü., Yenigün, C. D. *Hazard Change-point Models with Truncated Data*. Technical Report 2002–05, Department of Industrial Engineering, Bilkent University, Ankara, Turkey, 2002.

Kapur, K. C., Lamberson, L. R. (1977). *Reliability in Engineering Design*. John Wiley & Sons.

Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data*. John Wiley & Sons.

Loader, C. R. (1991). Inference for a hazard rate change-point. *Biometrika* 78:749–757.

Matthews, D. E., Farewell, V. T. (1982). On testing for a constant hazard against a change-point alternative. *Biometrics* 38:463–468.

Meeker, W. Q., Escobar, L. A. (1998). *Statistical Methods for Reliability Data*. John Wiley & Sons.

Nelson, W. (1982). *Applied Life Data Analysis*. John Wiley & Sons.

Nguyen, H. T., Rogers, G. S., Walker, E. A. (1984). Estimation in change-point hazard rate models. *Biometrika* 71:299–304.

Rukhin, A. L. (1997). Change-point estimation under asymmetric loss. *Statistics & Decisions* 15:141–163.

Srinath, L. S. (1975). *Concepts in Reliability*. Affiliated East-West Press.