Complexity analysis, uncertainty management and fuzzy dynamical systems

A cybernetic approach

D. Dutta Majumder and Kausik Kumar Majumdar
*Electronics and Communication Sciences Unit,
Indian Statistical Institute, Calcutta, India*

Abstract In this paper, we present a brief study on various paradigms to tackle complexity or in other words manage uncertainty in the context of understanding science, society and nature. Fuzzy real numbers, fuzzy logic, possibility theory, probability theory, Dempster-Shafer theory, artificial neural nets, neuro-fuzzy, fractals and multifractals, etc. are some of the paradigms to help us to understand complex systems. We present a very detailed discussion on the mathematical theory of fuzzy dynamical system (FDS), which is the most fundamental theory from the point of view of evolution of any fuzzy system. We have made considerable extension of FDS in this paper, which has great practical value in studying some of the very complex systems in society and nature. The theories of fuzzy controllers, fuzzy pattern recognition and fuzzy computer vision are but some of the most prominent subclasses of FDS. We enunciate the concept of fuzzy differential inclusion (not equation) and fuzzy attractor. We attempt to present this theoretical framework to give an interpretation of cyclogenesis in atmospheric cybernetics as a case study. We also have presented a Dempster-Shafer's evidence theoretic analysis and a classical probability theoretic analysis (from general system theoretic outlook) of carcinogenesis as other interesting case studies of bio-cybernetics.

Introduction

Any mathematical model of a physical phenomenon consists of a number of variables and parameters, each of which is measurable. But unfortunately, any measurement is prone to error. The more we want to make a measurement error-free the more time is needed to accomplish that measurement. So achieving accuracy is a time-complex (apart from other complexities) task. We can reduce this complexity by allowing and accommodating uncertainties in our measurements. Fuzzy sets as enunciated by Lotfi Zadeh are the basis of fuzzy uncertainties. Apart from fuzzy uncertainties other uncertainties are also there. The more the uncertainties involved the more complex is the system (Dutta Majumder, 1993). To model these complex systems, uncertain systems are born, where complexity is reduced by accommodating uncertainty. In this paper, we have discussed various uncertainties and their role in modeling complex general and cybernetic systems.

Progress of sciences and all other components of civilization can be traced to certain types of challenges and responses. The subject of cybernetics emerged

through the interaction of traditional sciences when scientists faced a set of problems concerned with communication, control and computation in machines and living tissues. The foundation of cybernetics was laid and its basic principles formulated over centuries by the work of mathematicians, physicists, physicians and engineers. Though the decisive factor in the emergence of cybernetics (Wiener, 1949) was the swift development of electronic automation and especially the appearance of the high-speed computers which opened up boundless vistas in data processing, simulation and the modeling of control systems (Dutta Majumder, 1975, 1979).

A problem of fundamental importance in nature, man, society and machines, in developing a general theory of physical systems is the system causalty (Dutta Majumder, 1979). From this comes the notion of general dynamical system. By a dynamical system we mean a system which changes with respect to time. Like Euclidean geometry a general dynamical system too has a few undefined basic concepts. They are precisely three in number, namely *time*, *state* (*phase*) *space* and *time evolution law* (Katok and Hasselblatt, 1995). Although time is generally regarded as an undefined concept, Majumdar (2000a) has recently proposed a definition of time from a mathematical analytical stand point, which is different from the notion of time as presented by Coveney (1988). When we intend to mathematically model a physical phenomenon, we identify a set of attributes (e.g. mass, length and time in physics). Then we "quantify" each attribute, i.e. construct a real valued function over the domain of that attribute (this way we obtain an unit for an attribute, e.g. second for time, etc.). Collection of all such independent (i.e. one's value does not depend on any other) attributes make the state space. If there are n independent attributes the dimension of the state space is n. Let this n-dimensional state space be denoted by X_n. The time evolution law is a continuous function $f : X_n \times T \rightarrow X_n$, where T is the space of time. Additionally f also satisfies the following two conditions.

(1) $f(x, 0) = x$,   for any $x \in X_n$.

(2) $f(x, s + t) = f(f(x, s), t)$ for $s, t \in T$.

A dynamical system is formally denoted by (X_n, f) or just by f, when there is no confusion about the state space X_n.

The concept of state has long played an important role in the physical sciences. It was towards the turn of this century that the concept of state was given a more precise formulation by H. Poincare and later by the subsequent works of Birkoff (1977), Kalman (1962), Markov (1931), Nemytskii *et al* (1963). The concept of state space is a great unifying tool in the theory of general dynamical systems and in the theory of cybernetic systems (Dutta Majumder, 1979).

For any physical system (X_n, f) to determine the system it is essential to be able to measure each input state and each output state. In other words, we must be able to measure each point in the state space X_n. But by Heisenberg's uncertainty principle, for any physical dynamical system, we have

$$\Delta x \Delta t \geq h/2\pi, \tag{1.1}$$

where Δx is the error in measuring $x \in X_1$ (assuming the state space to be one-dimensional for simplicity), Δt is the error in measuring t, h Planck's constant. According to equation (1.1), if we want to make $\Delta x \to 0$, we automatically get $\Delta t \to \infty$. That is, if we employ a computer (or an automaton more fundamentally) to "exactly" determine the location of a point in X_1 it will take infinite time. A more realistic interpretation of this fact is that, "The more we wish our computer to be accurate in locating a state in the state space the more time it will take." Or in other words, "The more is the demand for accuracy from the computer (in determining a value) the more time complex the job will be for the machine." So the inherent (and insurmountable) uncertainty in determining a state in the state space gives rise to complexity in a cybernetic or a general dynamical system.

Heisenberg's uncertainty principle gives the ultimate limit of uncertainty or indeterminacy in a dynamical system. But in our day-to-day real life we even need not go that far to find an uncertainty. Suppose a die is thrown and you are asked to guess the top face. Your uncertainty about the outcome is attributed to *randomness*. The best way to approach this question might be to describe the status of the die in terms of probability distribution on the six faces. Uncertainty that arises due to chance is called *probabilistic uncertainty* (PU) (Pal and Bezdek, 1994).

To make the situation more complex, suppose an artificial vision system analyzes a digital image of the top face. Based on the evidence gathered, the system might suggest that the top face is either a 5 or 6, but cannot be more specific. This kind of uncertainty arises from limitations (for example, sensor resolution) of the evidence gathering system. Uncertainty in the second situation reflects ambiguity in specifying the *exact* solution, and is called *non-specificity* by Yager. Pal and Bezdek (1994) have preferred to use the alternate term *resolutional uncertainty* (RU). If we are certain that the top face is either 5 or 6, this case involves only nonspecificity. More generally, the vision system might also supply a certainty factor with its information. For example, the system might suggest that the top face is either a 5 or 6 with belief of 0.8. In this case, uncertainty due to chance is also present because the top face can take any value, so the system contains both PU and RU (Pal and Bezdek, 1994).

Finally, you are asked to interpret the top face of the die as, say, *high* (or *low*). Here a third type of uncertainty appears due to linguistic imprecision or vagueness. This is called *fuzzy uncertainty* (FU). FU differs from PU and RU because it deals with situations where set boundaries are not sharply defined.

PUand RU are not due to ambiguity about set boundaries, but rather, about the belongingness of elements or events to crisp sets (Pal and Bezdek, 1994).

Uncertainties in information theory have been categorized differently by Klir (1991). There he talks about probabilistic and possibilistic uncertainties, both of which are special cases of Dempster-Shafer theory of evidence (Shafer, 1976). In the next section, we shall describe various forms of uncertainties.

Different uncertainties

Webster's New Twentieth Century Dictionary gives the following six clusters of meaning for the term uncertainty:

(1) not certainly known; questionable; problemetical;

(2) vague; not definite or determined;

(3) doubtful; not having certain knowledge; not sure;

(4) ambiguous;

(5) not steady or constant; varying;

(6) liable to change; not reliable or dependable.

In Indian languages, particularly in Sanskrit-based languages, there are other higher (spiritual) levels of uncertainties, the approximate meaning of which are mysterious and unknowable (Dutta Majumder, 1993; Klir and Folger, 1988). We shall not deal with them here.

In our view, these six or seven or eight levels of uncertainties are linked with different levels of human cognition (according to the Buddhist philosophy there are eight levels of cognition). But these six semantic clusters indicated above emerge out with two distinct groups or forms. The first group of three is under the form *vagueness* and the second group of three is under the form *ambiguity*. Some of the concepts connected with vagueness are fuzziness, haziness, cloudiness, unclearness, indistinctiveness and sharplessness. Some of the concepts connected with ambiguity are nonspecificity, one-to-many relations, variety, generality, divergence and diversity (Figure 1).

Fuzzy sets and fuzzy measures reflect two fundamentally different types of uncertainties, namely:

(1) vagueness, and

(2) ambiguity

 • Type 1 ambiguity. Nonspecificity in evidence (measures of nonspecificity).

 • Type 2 ambiguity. Dissonance of evidence (measures of dissonance).

 • Type 3 ambiguity. Confusion in evidence (measures of confusion).

It can be seen that the concept of fuzzy set provides a good mathematical framework for dealing with the concepts connected with vagueness.
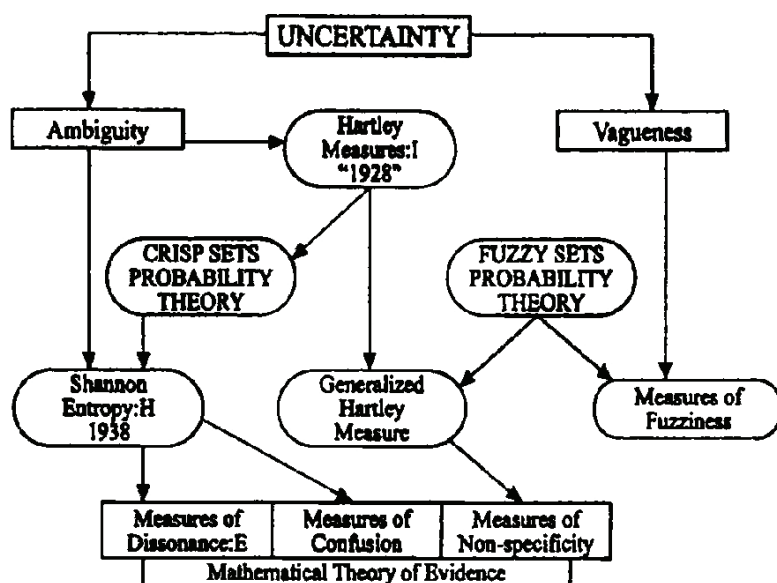
Figure 1.
Summary of seven
uncertainty measures

For dealing with the clusters of concepts connected with ambiguity the concept of fuzzy measures provides a general mathematical framework.

A fuzzy measure is defined by a function

$$g : P(X) \rightarrow [0,1],$$

which assigns to each crisp subset of X a number in the unit interval $[0,1]$. It should be noted that the domain of the function g is the power set $P(X)$ of crisp set X and not $F(X)$ the collection of all the fuzzy subsets of X. When a number is assigned to a subset $A \in P(X)$, $g(A)$ represents the degree of available evidence or our belief that a given element of X (a $priori$ nonlocated in any subset of X) belongs to the subset A (Figure 2).
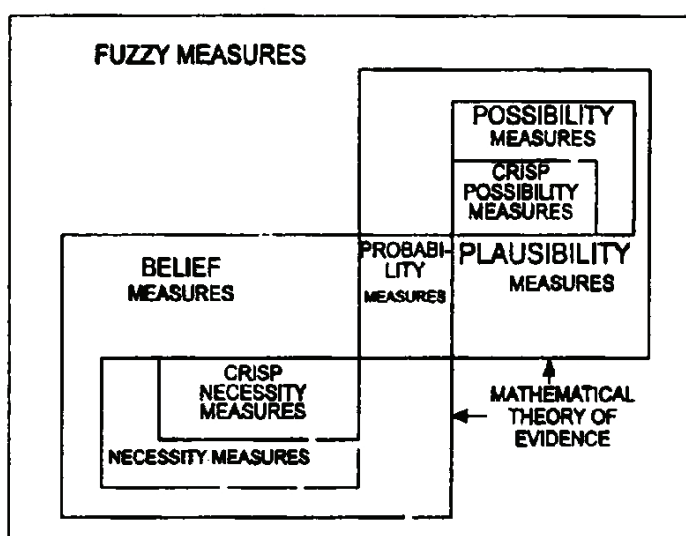

Figure 2.
Inclusion relationship
among the main classes
of fuzzy measures

Some approaches to tackle complexity

Complex systems are much less understood and not even well defined mathematically. They are in the frontier between simple and chaotic systems: a complex system is a dynamical system depending on many parameters, in constant evolution and distances along trajectories increase (decrease) polynomially and not exponentially. One considers that the brain's neural network is one such system (Palis, 2002). Some salient features of complex systems are:

(1) it is a dynamical system in constant evolution formed by a great number of units;

(2) some characteristics of the system are randomly distributed; and

(3) the system may have several attractors (Palis, 2002).

Our view of complexity is slightly different from that of Palis (2002). A chaotic is also very much a complex system. Therefore, the trajectories may vary exponentially also. Not all uncertainties in a complex system are essentially statistical in nature. There may be other uncertainties too as we are going to describe below. The possible notions of applications of complex systems are not yet conclusive, like the case of the brain, origin of life, evolution of tumour (Majumdar and Dutta Majumder, 2004a), economics, etc. Some of the methods of dealing with complexity, in general dynamical and cybernetic systems, used to model scientific, social and natural phenomena are described below (Majumdar, 2001).

(1) *Probability theory* is the oldest branch of science dealing with complexity due to uncertainty in a system. Like fuzzy measure probability is also a measure. But probability measure is much more restricted compared to fuzzy measure. Each probability measure is to satisfy a set of three axioms due to Kolmogrov. In a dynamical system for an x if $f(x, t)$ cannot be determined exactly for any $t > 0$, instead $f(x, t)$ takes a value y with certain probability, we call the dynamical system a stochastic process. Stochastic processes are certainly more complex than the ordinary deterministic dynamical systems.

(2) *Fuzzy set theory* unlike probability theory is based on nonstandard or many-valued logic. A dynamical system whose state space is a fuzzy set is known as a fuzzy dynamical system (FDS). In a fuzzy state space each point is a fuzzy point, i.e. each point is a fuzzy subset. In a dynamical system $f(x, t)$ we can neither specify the initial value x, nor can we specify the final value $f(x, t)$, moreover we cannot measure the inherent uncertainties in determining x and $f(x, t)$ by probability measure but by fuzzy measure only. In that case, the dynamical system is a FDS. Clearly an FDS is even more complex than a probabilistic dynamical system or a stochastic process.

(3) *Chaos theory* is also known as *applied nonlinear dynamics*. According to Devaney (1989), a dynamical system f is said to be chaotic if:

- f has sensitive dependence on initial conditions. That is, for any $\in > 0$, $\exists \delta > 0$, such that, $|x - y| < \in$, there exists t for which $|f(x, t) - f(y, t)| > \delta$;

- f is topologically transitive. That is, for any two open sets u, v in the phase space, such that, $u \cap v = \emptyset$, there exists t for which $f(u, t) \cap v \neq \emptyset$

- the periodic points of f are dense in the phase space.

For a given x, we cannot, in general, assign any probability or fuzzy measure to $f(x, t)$. That is, the uncertainty involved in determining $f(x, t)$ is neither fuzzy nor probabilistic in nature.

(4) *Dempstar-Schafer's theory of evidence* is a unifying approach for fuzzy set theory and probability theory. Both possibility or fuzzy membership measure and probability measure follow from this theory. A brief outline of the theory is as follows.

Let X be an universal set. $P(X)$ be the power set of X. Then, the *Dempster-Shafer theory* (DST) (Shafer, 1976) is based upon a function

$$m : P(X) \rightarrow [0, 1],\qquad(3.1)$$

such that,

$$m(\emptyset) = 0 \text{ and } \sum_{A \subseteq X} m(A) = 1.\qquad(3.2)$$

where $\emptyset$ is the null set. The function m is called *basic probability assignment* and the quantity $m(A)$ is called A's *basic probability number*, and it is understood to be the measure of the *belief* that is committed exactly to A. But $m(A)$ is not the total belief committed to A. To calculate the total belief, $Bel(A)$, committed to A

$$Bel(A) = \sum_{B \subseteq A} m(B).\qquad(3.3)$$

A function $Bel : P(X) \rightarrow [0, 1]$ is called a *belief function* over X if it is given by equation (3.3) for some probability assignment $m : P(X) \rightarrow [0, 1]$. A subset A of X is called a *focal element* of a belief function Bel over X if $m(A) > 0$. The pair (F, m), where F denotes the set of all focal elements of m, is called the *body of evidence*. According to DST, belief measure is less than or equal to the probability measure, and probability measure is a special case of belief measure.

Another fundamentally important measure of DST is the *plausibility measure*, denoted by Pl and given by $Pl(A) = 1 - Bel(A')$, where A' is the complement of A. We have, $Bel(A) \leq Pl(A)$ for all $A \subseteq X$. According to DST plausibility measure is greater than or equal to the probability measure. When all focal elements are *singletons*, $Bel(A) = Pl(A)$ for all $A \subseteq X$ and we have $Bel(A) = Pl(A) =$ probability measure of A.

When all focal elements and hence, also the body of evidence, are nested, i.e. ordered by set inclusion, then the belief function is said to be *consonant*. In this particular case, the plausibility measure is called *possibility measure*, denoted by Pos, and the belief measure is called *necessity measure*, denoted by Nec. Pos is described by a *possibility distribution function* $r : X \rightarrow [0, 1]$ via the formula

$$Pos(A) = \max_{X \in A} r(x), \qquad (3.4)$$

for all $A \subseteq X$. Also for all $A \subseteq X$, $Nec(A) = 1 - Pos(A')$.

Let A be a fuzzy subset of X. $\mu_A : X \rightarrow [0, 1]$ is a membership function denoting the *grade of membership* (degree of belongingness) of any point $x \in X$ to A. If r_A is the possibility distribution function over A, then Zadeh (1978) has proposed that,

$$r_A(x) = \mu_A(x), \qquad (3.5)$$

for all $x \in X$. Also for all $B \in P(X)$

$$Pos_A(B) = \max_{x \in B} r_A(x). \qquad (3.6)$$

In this interpretation of possibility theory, focal elements correspond to distinct α-*cuts* of the fuzzy set A. Zadeh (1978) has taken equation (3.5) to define r_A over A in terms of μ_A. But we propose to just reverse this approach here to obtain a definition of μ_A in terms of r_A. Yen (1990) has generalized the DST to fuzzy sets. He has suggested methods of *linear programming problems* to calculate plausibility measure. So by that method $Pos(N(x))$ may be calculated, where $N(x)$ is an infinitesimal neighborhood of x. This way by substituting $N(x)$ for A in equation (3.4) we have $r_{N(x)}(x) = \mu_{N(x)}(x)$ for any x. If $\mu_{N(x)}(y) = 0$ for $y \notin N(x)$ then,

$$\mu_A(x) = \sum_{A = \cup N(x), N(x) \cap N(y) = \varnothing \text{ for } x \neq y} \mu_{N(x)}(x). \qquad (3.7)$$

This method of calculation of μ_A will work in most (but not in all) practical situations.

Some other soft-computing approaches obtained, including fusion methods (the methods obtained by combining more than one method out of fundamenta

methods, described in (1), (2), (3) and (4) above, to tackle complexity) to tackle complexity in science, society and nature are

- Genetic algorithm (nonlinear dynamical system combined with evolutionary optimization). The direction of evolution (bio-cybernetic in nature) is modeled either by statistical or possibilistic inference;

- artificial neural network (nonlinear dynamical system modeled after neuronal bio-cybernetics);

- neuro-fuzzy (fusion of artificial neural network and fuzzy set theory);

- fuzzy genetic algorithm (fusion of fuzzy set theory and genetic algorithm); and

- neuro genetic algorithm (fusion of neural network and genetic algorithm).

Concerning all the above uncertainties, three major areas of investigation for engineering applications are:

(1) design of control systems for diverse applications;

(2) pattern recognition and machine learning systems design;

(3) uncertainty management in (1) and (2) above and many other applications in science, society and nature.

In a recent paper, Yajnik (1999) has identified and separated regularity and irregularity, which remain intertwined in a complex system. Regularity means existence of periodic trajectories or existence of periodic part in an arbitrary trajectory. On the other hand, irregularity means the existence of nonperiodic trajectories or existence of nonperiodicity in an arbitrary trajectory. This means that a complex system can be decomposed into two mutually disjoint systems, one regular (accurate prediction is possible in this part) and the other is irregular (prediction will fail exponentially in this part). It points out to an aspect in the analysis of a complex system, namely the optimum analysis of a complex system will be one in which the maximal regular part is identified, the complement of which is the minimal irregular part. The most efficient prediction making will then be possible for the system. There may be a potentially remarkable application of Yajnik's (1999) method in the dynamics of evolution of tumour in human tissues. Yajnik's (1999) method can be applied to identify maximal regular and minimal irregular parts of the dynamics of tumour evolution. Once this becomes possible we shall be able to accurately predict which therapy will have how much effect within certain range.

Various dynamical systems

(1) *A general dynamical system* consists of a continuous function $f: X \times T \rightarrow X$, where X is the phase or state space and T is the space of time, such that, $f(x, 0) = x$ for all $x \in X$ and $f(x, s + t) = f(f(x, s), t)$. When f is obtained as a solution of a set of differential equations,

the dynamical system is called a differential dynamical system. The dynamical system is linear or nonlinear depending on whether f is linear or nonlinear.

(2) *Cybernetics* is the systemic study of automatic activities of animal and, in particular, human bodies and their systems of control on the one hand and the study of machines particularly for mechanization of thought process and its mathematical counterpart, namely mathematical logic (standard or nonstandard) on the other. So the mathematical model of a cybernetic system is any dynamical system f such that, f is common to a given biological and a given mechanical system. In the application of cybernetics, we derive f in biology and implement it in engineering.

(3) *Fuzzy dynamical system* is a dynamical system whose phase space is the collection of fuzzy subsets of some given set. An FDS is often understood as a fuzzy set of collection of crisp dynamical systems representing some uncertain or imprecise system. We do not know which crisp dynamical system exactly models that imprecise system. Instead each crisp member of the fuzzy set has certain possibility or membership grade to represent the uncertain system. We shall discuss FDSs in detail in the next section.

(4) *Chaotic dynamical system* is a nonlinear dynamical system f, such that,

- f has sensitive dependence on initial conditions (for any $\in > 0$, $\exists \delta > 0$, such that, $|x - y| < \in$ there exists t for which $|f(x,t) - f(y,t)| > \delta$);

- f is topologically transitive (for any two open sets u, v in the phase space, such that, $u \cap v = \varnothing$, there exists t for which $f(u,t) \cap v \neq \varnothing$);

- The periodic points of f are dense in the phase space.

(5) *Attractor* is the asymptotic phase space of a dynamical system. That is if X is the initial phase space (i.e. the dynamical system may start from any point of X) then a subset A of X is called the attractor of the dynamical system f iff

$$\lim_{t \to \infty} f(X, t) = A.$$

When f is an FDS, A is called a fuzzy attractor. Of course, a fuzzy attractor is a fuzzy subset of the state space.

In practice, a chaotic dynamical system f is characterized by:

(1) sensitive dependence on initial conditions (e.g. by measuring Liapunov exponents); and

(2) by the presence of a strange attractor, i.e. a fractal attracting set.

By a fractal attracting set we mean an attracting set (attractor) whose Hausdorff-Besicovitch dimension strictly exceeds the topological dimension.

Fuzzy dynamical systems

When a scientific, social or natural phenomenon is studied not all of its aspects are studied simultaneously. The aspect which is identified for study is modeled for analysis and understanding. To model a scientific, social or natural phenomenon the identifiable attributes are quantified often by real numbers, e.g. mass in physics, price in economics, etc. After that we determine relations among these quantified attributes in the form of a set of mathematical equations (often differential or difference equations). If the system is changing with respect to time then the solutions of this set of equations are time dependent functions. The set of these time dependent functions is the mathematical dynamical system modeling the corresponding scientific, social or natural phenomenon. Each quantifiable attribute must be measurable. This means locating a particular value in the set of real numbers. The more complex the system the more is the difficulty to measure a value of an attribute. Nevertheless, each and every practical system is endowed with uncertainties. Heisenberg's uncertainty principle is for microlevel in physical systems. But something like some form of uncertainty principle is inherent in micro and middle levels of biological systems and at all levels of natural and social systems. According to Zadeh, "Fuzziness comes from description of complex systems." The more the complexity of a system the greater is its uncertainty due to fuzziness. That is, each quantity we want to measure becomes fuzzy valued instead of precise valued. Thus, the set of crisp numbers is replaced by set of fuzzy numbers. When more than one attribute is involved, we obtain multidimensional fuzzy numbers, i.e. fuzzy vectors (Majumdar, 2000b, 2002a). Equations become fuzzy equations and differential equations become fuzzy differential equations (FDEs).

The potential application to system theory has from the beginning significantly motivated and influenced the direction of development of the theory of fuzzy sets. Fuzzy systems were first discussed in 1965 by Zadeh in his expository paper (Zadeh, 1965). The first systematic treatment of abstract FDSs was by Nazaroff (1973), who fuzzified Halkin's crisp topological polysystems to obtain fuzzy topological polysystems. These were further investigated by Warren (1976). They, however, suffer the shortcoming of not explicitly exhibiting the time dependence of the systems. Time dependent fuzzy sets were considered by Lientz (1972), but were not, strictly speaking, FDSs as they did not admit variations in initial conditions. Kloeden (1982) defined and developed the notion of FDSs on line of the classical notion of mathematical dynamical systems developed since the time of Poincare in the 1880s.

In a parallel development almost at the same time, FDS theory was developed from a more system theoretic and less topological outlook.

In a system, where some deterministic dynamical characteristics are unknown or deliberately ignored as well as uncertainties attached to their mathematical model, probabilistic approach cannot be used. This observation led Chang and Zadeh (1972) to the concept of fuzzy system. De Glas (1983) considered FDS (or fuzzy dynamical system as he himself liked to call it F as the fuzzy set of all possible crisp dynamical systems representing an uncertain dynamical system. He even considered α-cuts of this fuzzy set to signify the system behavior with the degree of possibility α or more. De Glas (1984) was the first to consider stability and attractor of an FDS.

Before defining an FDS, let us define a *multivalued* or *generalized semi-dynamical system* as a prerequisite. From this point onwards, we shall call it *generalized dynamical system* (GDS) only.

Basic definition of fuzzy dynamical systems

Definition 5.1. GDS is defined axiomatically in terms of an attainability set mapping $F : X \times T \to C((C, h)$ be the metric space of all nonempty compact subsets of X with the Hausdorff metric h), where for each $(x, t) \in X \times T$ the attainability set $F(x, t)$ is the set of all points in X attainable at time $t \in T$ (unless or otherwise mentioned T will always mean the set of time in this chapter) from the initial point x, satisfying following four generalizations of axioms (1)-(4):

(1) F is defined for all $(x, t) \in X \times T$;

(2) $F(x, 0) = \{x\}$ for all $x \in X$;

(3) $F(x, s + t) = F(F(x, s), t) = \cup \{F(y, t); y \in F(x, s)\}$ for all $x \in X$ and $s, t \in T$;

(4) F is jointly continuous in (x, t).

Definition 5.2. A *trajectory* of a GDS F is defined to be a single-valued mapping $\phi : [t_0, t_1] \to X$ for which

$$\phi(t) \in F(\phi(s), t - s), \tag{5.1}$$

for all $t_0 \leq s \leq t \leq t_1$.

The existence of trajectories and attainability by trajectories of all points in the attainability sets follows from axioms (1)-(4).

Before we extend Kloeden's (1982) definition of FDS we should do a little ground work. Let D be a metric on $X \times I (I = [0, 1])$ defined by

$$D((x_1, r_1), (x_2, r_2)) = \max\{d(x_1, x_2), |r_1 - r_2|\}, \quad x_1, x_2 \in X \text{ and } r_1, r_2 \in I.$$

Then $(X \times I, D)$ is also a complete, locally compact metric space (usually R^n for some n). A fuzzy subset A is defined by its membership function $\mu_A : X \to I$.

Definition 5.3. The *support* of fuzzy subset μ_A, denoted by supp μ_A, such that

$$\text{supp } \mu_A = \text{closure}\{x \in X | \mu_A(x) > 0\}.$$

Definiton 5.4. The *endograph* of a fuzzy subset μ_A, denoted by end μ_A, is the subset

$$\text{end } \mu_A = \{(x,r) \in X \times I | \mu_A(x) \geq r\},$$

of $X \times I$ and the *supported endograph*, denoted by send μ_A, the subset

$$\text{send } \mu_A = \text{end } \mu_A \cap \text{supp } \mu_A \times I.$$

μ_A is a compact fuzzy subset of X if and only if μ_A is an upper semi-continuous function on X and supp μ_A is compact subset of X or equivalently, if and only if send μ_A is a compact subset of $X \times I$. Let $\mathcal{J}$ denote the collection of all nonempty compact fuzzy subsets of X, and let δ be the metric on $\mathcal{J}$ defined by

$$\delta(\mu_A, \mu_B) = H(\text{send } \mu_A, \text{ send } \mu_B),$$

where H is the Hausdorff metric for nonempty compact subsets of $X \times I$.
$(\mathcal{J}, \delta)$ is not a complete metric space (Kloeden, 1982). The metric space (C, h) is embedded in the metric space $(\mathcal{J}, \delta)$ under the mapping $i : C \to \mathcal{J}$ defined by $i(A) = \chi_A$, the characteristic function of A. Each individual point of X is a compact subset of X. So $\{x\} \subset C$, for all $x \in X$. So $\{x\} \subset \mathcal{J}$ for $x \in X$.

Now, we are in a position to present an extended version of Kloeden's definition of FDS.

Definition 5.5. FDS $(\mathcal{J}, \sigma)$ on a state space $\mathcal{J}$ is defined axiomatically in terms of a fuzzy attainability set mapping (FAM) (the so-called time-evolution law) $\sigma : \mathcal{J} \times T \to \mathcal{J}$ satisfying the following four axioms:

(1) $\sigma(x, t)$ is defined for all $(x, t) \in \mathcal{J} \times T$;

(2) $\sigma(x, 0) = \mu_x$ for all $x \in \mathcal{J}$;

(3) $\sigma(x, s + t) = \sigma(\sigma(x, s), t)$ for all $x \in \mathcal{J}$ and $s, t \in T$;

(4) σ is jointly continuous in (x, t).

Since each individual point of X is also a member of $\mathcal{J}$, an FDS σ can start from X as well. In fact, in Kloeden's original definition of an FDS σ always starts from X.

Definition 5.6. $\sigma(x, t)$ is called the *fuzzy attainability set* or *fuzzy reachable set* of $(\mathcal{J}, \sigma)$ at time t starting from the fuzzy point (which is a nonempty compact fuzzy subset of X) $x \in \mathcal{J}$. The time-evolution law σ is also called *fuzzy attainability set mapping* (FAM).

Of course an attainability set is also a nonempty compact fuzzy subset of X and hence is an element of $\mathcal{J}$. Definition 5.5 is a slightly more generalized version of Kloeden's (1982) definition of FDSs in terms of FAMs. De Glas (1983) has defined an FDS as representable by a fuzzy subset of all the possible crisp time-evolution laws representing an uncertain crisp system. Here also

the time-evolution law representing a fuzzy system turns out to be an FAM (Theorem 4.1 of De Glas (1983)).

Axioms (1)-(4) of an FDS are fuzzified version of axioms (1)-(4) of a crisp GDS and because the metric space (C, h) is embedded in the metric space $(\mathscr{S}, \delta)$ by the identification mapping $i : C \to \mathscr{S}$ defined by $i(A) = \chi_A$ for each $A \in C$. Thus crisp dynamical systems can be considered as a subclass of FDSs.

Definition 5.7. A *fuzzy trajectory* of an FDS σ is a mapping $\psi : [t_0, t_1] \to \mathscr{S}$ for which $\psi(t)$ is a singleton fuzzy subset of X for each $t_0 \le t \le t_1$ and

$$\psi(t) \in \sigma(\psi(s), t - s), \qquad (5.2)$$

for all $t_0 \le s \le t \le t_1$.

Unlike a crisp trajectory a fuzzy trajectory need not be continuous (Kloeden, 1982). Also note that, for $t_0 \le t \le t_1$

$$\psi(t) = r(t)\chi_{\phi(t)}, \qquad (5.3)$$

where $\phi(t) = \operatorname{supp} \psi(t)$ and $r(t) = \psi(t)(\phi(t))$. If $r(t) \ge \alpha$ for all t then it is α-*trajectory* of De Glas (1983). Existence of fuzzy trajectories has been proved in theorem 5.2 of Kloeden (1982). The most important thing to note here is that the classical fuzzy trajectory is actually a "crisp" trajectory with each point on the trajectory having a membership value in $[0,1]$. Membership value of a particular point on the trajectory is determined by belongingness (membership) of that point to a nonempty compact fuzzy subset (the attainability set) of X i.e. a point of the state space $\mathscr{S}$.

FDSs have been defined in terms of *fuzzy reachable set mapping* (Hullermeier, 1997, 1999) also. Reachable set and attainability set are same. Another important notion of FDS is that, let S be the set of all possible crisp models of an imprecise dynamical system. By a fuzzy model of the system a suitable fuzzy subset of S is understood (De Glas, 1983). This clearly comes down to the notion of:

(1) identifying and accommodating the initial (fuzzy) imprecision by choosing a suitable fuzzy initial point in the phase space and

(2) then identifying and accommodating the propagation of imprecision in determining any subsequent state by a suitable FAM.

Obviously the description of FDSs in terms of FAMs is the most generalized and the most powerful notion so far. Also such description is an immediate generalization of the notion of crisp dynamical systems to the FDSs. Therefore, it is quite natural to expect that, many important notions of the crisp dynamical systems may be extended to the FDSs described in terms of FAMs. Now we want to extend the notion of dissipativeness to the FDSs to define fuzzy dissipative dynamical systems (Majumdar, 2003a).

Definition 5.8. $(\mathcal{f}, \sigma)$, where $\mathcal{f}$ is defined over X as above, is a *fuzzy dissipative dynamical system* if $\sigma(X, t_2) \subseteq \sigma(X, t_1)$ $(\mu_{\sigma(X,t_2)} \le \mu_{\sigma(X,t_1)})$ in fuzzy set terminology), where (X, d, μ) is a nonempty compact metric space, which is also a fuzzy set with membership function μ. $t_1, t_2 \in T$ such that, $t_1 < t_2$. X is the initial phase space of σ, i.e. at t_0 (the time at which the system starts) σ can start from any point of X. If $\sigma(X, t_2) \subset \sigma(X, t_1)$ holds strictly for $t_1 < t_2$ we call that the system is *strictly dissipative* or *monotonic dissipative*.Given the importance of dissipative dynamical systems in real life let us suggest here an easy-to-implement test for dissipativeness of an FDS (Majumdar, 2003a). If there exists a nonnegative $k \in R$ such that,

$$\lim_{t \to \infty} \delta(X, \sigma(X, t)) \le k, \qquad (5.4)$$

holds, then we call σ a dissipative FDS. Clearly, $X = \sigma(X, 0)$ and dissipativeness implies that $\sigma(X, t) \subseteq \sigma(X, 0)$ for $t \ge 0$, which means $\delta(\sigma(X, 0), \sigma(X, t)) \le k$ for some nonnegative $k \in R$ and all nonnegative $t \in T$. On the other hand, if equation (5.4) holds and $\sigma(X, t) \not\subseteq \sigma(X, 0)$ for some $t > 0$ then we can extend X in the following manner

$$\text{closure}\left(\bigcup_{t \ge 0} \sigma(X, t)\right) = X' \qquad (5.5)$$

Because of equation (5.4), X' is a bounded metric space. Being union of fuzzy subsets X' is also a fuzzy set. But X' is not compact in general. However, in most cases of interest, X is a compact subset of R^n for some n and in that case X' being closed bounded subset of R^n is compact (by generalized Heine-Borel theorem). We extend d to X' and get the space (X', d, μ). We just replace (X, d, μ) by (X', d, μ) and define $(\mathcal{f}', \delta)$ on (X', d, μ). Because of equation (5.4)

$$\lim_{t \to \infty} \delta(X', \sigma(X', t)) \le k \qquad (5.6)$$

holds, which implies σ is dissipative on X' or rather on $\mathcal{f}'$.

Now we are in a position to define fuzzy attractor (Majumdar, 2003a). De Glas (1984) was the first to define and discuss fuzzy attractor (or rather α-attractor). Cabrelli *et al* (1992) have used the term "attractor" only, instead of "fuzzy attractor". In a recent paper, Bassanezi *et al* (2000) have dealt (in a different way) attractors and stability of FDSs from a purely mathematical point of view. Here we have defined a fuzzy attractor in terms of an FAM σ of the FDS $(\mathcal{f}, \sigma)$.

Fuzzy attractor and stability of fuzzy dynamical systems

Definition 5.9. In a fuzzy dissipative dynamical system $\lim_{t \to \infty} \sigma(X, t) = A \subseteq X$, we define A as *fuzzy attractor* of σ. Of course A is a fuzzy subset of X.

If we introduce the notion of α-cuts into, it we will obtain $\lim_{t\to\infty}[\sigma(X,t)]^\alpha = [A]^\alpha$ for $0 \le \alpha \le 1$. $[A]^\alpha$ is the α-attractor defined by De Glas (1984).

Now, how to define the membership function μ_A for A? Note that the fuzzy trajectory given by $\psi(t)$ in expression (5.2) converges to A as $t \to \infty$. Also each point of A belongs to a fuzzy trajectory $\psi(t)$ for some t. From equation (5.3), it transpires that membership value of $\psi(t)$ is $r(t)$. So for each $x \in A$, there exists a $t \in T$ such that, $x = \psi(t)$, and

$$\mu_A(\psi(t)) = r(t) = \mu_A(x). \tag{5.7}$$

But through a given point $x \in A$ more than one trajectory may pass. In that case, to determine a unique value for $\mu_A(x)$ we are to take the fuzzy set union of $\mu_A(\psi_c(t))$ over all c, where $\psi_c(t)$ is a (fuzzy) trajectory passing through x.

$$\cup_c \mu_A(\psi_c(t)) = \sup_c \{r_c(t)\} = \mu_A(x). \tag{5.8}$$

Attractors have played an increasingly important role in thinking about (classical) dynamical systems since their introduction in the 1960s of the last century. Since attractor is actually the phase space of the underlying dynamical system as $t \to \infty$, we may consider attractor as a matured state of the phase space. Naturally, the dynamical system evolving within the attractor is rather matured compared to the initial state. To understand a dynamical system therefore the study of attractor, when one exists, is of considerable importance.

Definition 5.10. Let *fuzzy nonwandering set* of an FDS σ be denoted by $\Omega(\sigma)$, which is a fuzzy subset of X. Then, $x \in \Omega(\sigma)$ implies that $\sigma(x,t) \subseteq \Omega(\sigma)$ for all $t \in T$.

$\mu_{\Omega(\sigma)}: X \to [0,1]$ and for any $y \in \Omega(\sigma)$ $\mu_{\Omega(\sigma)}(y) > 0$. Now, following Ruelle and Tukens (Milnor, 1985) we can give the following.

Definition 5.11. A subset A of the fuzzy nonwandering set $\Omega(\sigma)$ is a *fuzzy attractor* if it has a neighborhood U such that,

$$\bigcap_{t>0} \sigma(U,t) = A. \tag{5.9}$$

Of course, $\sigma(U, t_2) \subseteq \sigma(U, t_1)$ for $t_1 < t_2$, i.e. σ is dissipative.

Note that the above definition of fuzzy attractor implies that all fuzzy trajectories sufficiently close to the fuzzy attractor A must converge to A. This is a standard stability condition of a dynamical system, and when this stability condition is satisfied we call the underlying FDS σ, *stable*. This form of stability is known as Liapunov stability. Let us state this more formally in the following.

Definition 5.12. A is a closed subset (in sense of fuzzy topology (Chang, 1968)) of X with $\sigma(A, t) = A(\mu_{\sigma(A,t)}) = \mu_A$ in fuzzy set terminology) for any $t \ge 0$ will be called *Liapunov stable* (also called *orbitally stable*) if A has arbitrarily small neighborhoods U with $\sigma(U,t) \subset U$ for all $t > 0$.

There is another important form of stability called asymptotical stability. Before we can define it for an FDS we need to develop some more concepts.

Definition 5.13. *Omega limit set* $\omega(x)$ of a point $x \in X$ is the collection of all accumulation points (y will be called an *accumulation point* of the sequence $\{\sigma(x,t)\}_{t\geq 0}$, if and only if there exists a fuzzy trajectory through x which converges to y for some $t \geq 0$) of the sequence $\{\sigma(x,t)\}_{t\geq 0}$.

Definition 5.14. The *realm of attraction* of an attractor A, denoted by $\rho(A)$, is the collection of all points $x \in X$ for which $\omega(x) \subseteq A$.

Obviously, for any (fuzzy) attractor A of a an FDS σ, $\sigma(A,t) = A$ for all $t \geq 0$.

Definition 5.15. A fuzzy attractor A of an FDS is called *asymptotically stable* if it is Liapunov stable and its realm of attraction $\rho(A)$ is an open set (in sense of fuzzy topology (Chang, 1968)).

In the asymptotically stable case if we choose U with closure (in sense of fuzzy topology) contained in $\rho(A)$, then it follows that A is equal to the intersection of the sequence of forward images $U \supset \sigma(U) \supset \sigma^2(U) \supset \sigma^3(U) \cdots\cdots$, where $\sigma^i(U) = \sigma(U,i)$. This discussion of stability of a fuzzy system is generalization of stability discussed by De Glas (1984) and Tathachar and Viswanath (1997).

Robustness of fuzzy dynamical systems

Let us next extend the concept of robustness to fuzzy attractors (Majumdar, 2003a). Again, we shall have to do some ground work.

Definition 5.16. The *likely limit set* $\Lambda = \Lambda(\sigma)$ of an FDS σ is the smallest closed subset (in sense of fuzzy topology (Chang, 1968) of X with the property that $\omega(x) \subseteq \Lambda$ for every point $x \in X$ (Remember (X, d) is a compact metric space.) outside a set of lebesgue measures zero. (Lebesgue measure can be defined on X, for (X, d) is compact and embeddable in R^n for some finite n.)

Clearly, the likely limit set Λ is the largest attractor of σ.

Definition 5.17. *Hausdorff distance* between two closed sets is the smallest number δ such that, each closed ball of radius δ centered at a point of either set necessarily contains a point of the other set.

Definition 5.18. The likely limit set $\Lambda(\sigma_1)$ of an FDS σ_1 is called *robust* if and only if the likely limit set $\Lambda(\sigma_2)$ of any other FDS σ_2 has Hausdorff distance with $\Lambda(\sigma_1)$ δ then σ_1 converges uniformly to σ_2 on X implies $\delta \rightarrow 0$.

Fuzzy chaotic dynamical systems

Our next target is to define fuzzy chaotic dynamical system. Crisp chaotic dynamical system has been defined by Devaney (1989). We shall make a straightforward extension of this definition to FDS σ (Majumdar, 2003a). But *fuzzy chaos* has already been defined by Buckley and Hayashi (1998) and Kloeden (1991). Kloeden has defined fuzzy chaos in terms of iterative maps on fuzzy sets on line of what Li and Yorke (1975) did for certain crisp cases. Li and Yorke's definition of chaos is only slightly different from that of Devaney.

Buckley and Hayashi's (1998) approach is also similar. A *chaotic fuzzy set A* is represented by μ_A and this μ_A is then taken as the limit of an iterated function, which is known to be chaotic under iterations, subjected to suitable conditions (e.g. the logistic function for suitable parameter values). Apart from Buckley and Hayashi (1998) and Kloeden (1991), fuzzy chaos has also been described by Teodorescu (1992). Kloeden (1991) has defined fuzzy chaos for discrete FDS. Next we shall extend Devaney's definition of chaos to the FDS.

Definition 5.19. Let σ be an FDS. A point $p \in X$ is called *periodic* of *period* $S(> 0)$ if and only if $\chi_p \in \sigma(p, S)$ and $\chi_p \notin \sigma(p, S')$ for any $0 < S' < S$.

Clearly p is a fuzzy singleton of X and hence a member of $\mathcal{F}$.

Definition 5.20. σ will be called a *fuzzy chaotic dynamical system* if and only if the following three conditions are satisfied.

(1) There exists $\in > 0$ such that, for any $x \in \mathcal{F}$ and any neighborhood $N(x)$ of x there exist $y \in N(x)$ and $t \geq 0$ such that, $\delta(\sigma(x, t), \sigma(y, t)) > \in$ (δ is Hausdorff distance). We say that, σ has *sensitive dependence on initial conditions*. As usual here x, y are nonempty compact fuzzy subsets of X, i.e. fuzzy points of X and hence individual elements of $\mathcal{F}$.

(2) For any two nonempty fuzzy open sets A and B in $(\mathcal{F}, \delta)$ such that, $\mu_A \wedge \mu_B = 0$, there exists $t > 0$, such that $\mu_{\sigma(A, t)} \wedge \mu_B \neq 0$. We call that σ is *topologically transitive*.

(3) Let P be the collection of periodic points of σ. $P \subseteq \mathcal{F}$. If P is dense in $\mathcal{F}$, i.e. equivalently, in each open subset of $(\mathcal{F}, \delta)$ there is at least one member of P, *density of periodic points* property is satisfied by σ.

Notice that if we take t in the set of all nonnegative integers instead of all nonnegative real numbers, we obtain the discrete chaotic FDSs of iterated functions on fuzzy sets. But this time the definition of fuzzy chaos is on line of Devaney rather than Li and Yorke as adopted by Kloeden (1991).

Buckley and Hayashi (1998) have iterated the *logistic function* for suitable parameter values to generate chaotic membership functions over the set of real numbers R to get the concomitant chaotic fuzzy numbers. But in reality *chaotic fuzzy sets* may be far more complicated.

In a crisp dynamical system an attractor plays a decisive role to determine whether the underlying dynamical system is chaotic. The presence of a *homoclinic point* in the attractor implies that the underlying dynamical system is behaving *chaotically*.

If two curves C_1 and C_2 intersect in R^n at a point B, then C_1 and C_2 will be called *transversal* to each other at B if and only if they cannot be pulled apart from each other at B by a small deformation of either of them or both, i.e. in other words, their intersection at B is stable (Figure 3).

Definition 5.21. In a crisp dynamical system p will be a *homoclinic point* if and only if two different trajectories T_1 and T_2 *transversally* intersect at p

such that, neighboring points of p on T_1 converge to p and neighboring points of p on T_2 diverge from p under the dynamical system.

Definition 5.22. In an FDS let two fuzzy trajectories T'_1 and T'_2 intersect transversally at p'. If membership values along T'_1 increase towards p' and membership values along T'_2 decrease towards p', then p' will be called a *fuzzy homoclinic point* of the underlying FDS.

Likewise, we define fuzzy homoclinic point of an FDS in terms of changing membership values (Majumdar, 2003a).

Horse shoe effect, described by Smale (1967), is another very important aspect of a multidimensional (crisp) chaotic dynamical system. Like fuzzy homoclinic point fuzzy horse shoe effect can also be defined in a similar manner.

This membership interpretation of p' is equivalent to crisp notions of converging towards (increasing membership values) and diverging from (decreasing membership values) p.

Definition 5.23. When there will be a fuzzy homoclinic point in a fuzzy attractor of an FDS, we call that attractor *fuzzy chaotic attractor.*

Definition 5.24. For all practical purposes we shall call a fuzzy dissipative dynamical system evolving in (X, d) chaotic if:

- the fuzzy attractor A is a fractal subset of X, and

- A contains at least one fuzzy homoclinic point.

According to the above criterion of chaotic FDS it is clear that the shape of the fuzzy attractor will usually be very complicated and the membership function defined on the attractor will take abruptly fluctuating values within any arbitrarily small neighborhood of almost every point (in sense of measure theory) of A. Ideally, the set of fuzzy homoclinic points will be *dense* in A in the relative metric topology of A as a subspace of (X, d).

Fuzzy Liapunov exponent
Liapunov exponent is perhaps the most important of all parameters of a dynamical system. Fuzzy system theorists have already defined and measured various forms of Liapunov exponent of a fuzzy system (Tathachar and Viswanath, 1997). Here, we would like to define the notion of Liapunov exponent for an FDS, which will be called fuzzy Liapunov exponent.

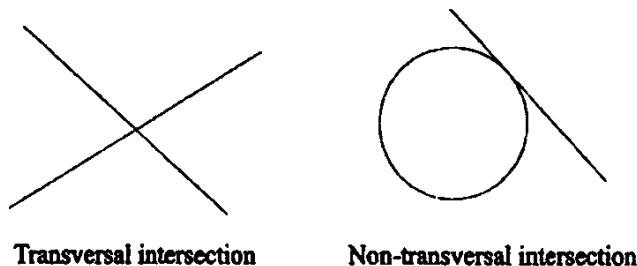Transversal intersection   Non-transversal intersection

Figure 3.
Transversal vis-à-vis
nontransversal
intersections

Note that we have been describing the FDS in terms of time series. In this type of description of a dynamical system the Liapunov exponents are relatively easy to measure. For an FDS $\sigma(x,t)$ (for simplicity we take here x one-dimensional, extension to higher dimension is straightforward) we define the Liapunov exponent λ as follows (Majumdar, 2003a).

Definition 5.25. For an FDS $(\mathscr{f}, \sigma)$ evolving on $(\mathscr{f}, \delta)$ the *fuzzy Liapunov exponent* λ is given by the equation

$$\lambda = \lim_{k \to \infty}(1/k)\sum_{k=1}^{\infty} \lim_{t_k \to t_{k-1}} \log \delta(\sigma(x,t_k), \sigma(x,t_{k-1}))/|t_k - t_{k-1}|. \qquad (5.10)$$

Generally, $\sigma(x,t)$ is a fuzzy subset of X (attainability set) for any given x and t. So fuzzy Liapunov exponent λ is in general a fuzzy number (because the Hausdorff distance between two fuzzy sets can never be determined with crisp precision), i.e. a crisp number equipped with a membership value. In a multidimensional fuzzy system if the largest Liapunov exponent tends to zero (i.e. membership value at 0 is 1 and membership values are uniformly zero outside a progressively smaller compact intervals of R containing 0) the system is said to be stable. Liapunov exponents are very crucial in analyzing local instability and predictability of a dynamical system.

Fuzzy metric entropy and fuzzy Liapunov time
Definition 5.26. In an FDS σ, evolving in $(\mathscr{f}, \delta)$, the *local metric entropy* (LME) is defined at any point x to be the sum (in the sense of fuzzy arithmetic) of all the fuzzy Liapunov exponents at x, where each fuzzy Liapunov exponent is such a fuzzy number that, no crisp member of that fuzzy number is nonpositive or in other words, this is a fuzzy number of the form $\mu_{[a,b]}, a > 0$, where $[a,b]$ is the support of $\mu_{[a,b]}$.

Definition 5.27. Sum of all the Liapunov exponents at x is the *divergence* of the phase space at x.

The greater the value of LME the more the system is chaotic and the less it is predictable. We can conclude that $T_\lambda \approx 1/\lambda$ (in sense of fuzzy arithmetic), where λ is the average value (calculated according to the arithmetic of fuzzy numbers) of the maximum Liapunov exponent over the elapsed time.

Definition 5.28. T_λ is called *Liapunov time*, i.e. the system is predictable almost up to a time T_λ.

Fuzzy differentiable dynamical systems

In the previous section, we extended some important dynamical system theoretic notions to the FDSs. In this section we shall be concentrating on the special case when the underlying fuzzy attainability set mapping of the FDS is a solution to a set of FDEs or fuzzy differential inclusions (FDIs), i.e. fuzzy differentiable dynamical systems (FDDSs).

Fuzzy differential equations

Definition 6.1. When one or more of the FDEs are time-dependent we call the system *nonautonomous FDDS*. Time-dependent expressions in a nonautonomous system are of the form

$$x'(t) = f(x, t), \quad x(0) = x_0, \tag{6.1}$$

where equation (6.1) is an FDE, x' is fuzzy derivative of the fuzzy valued function x.

Definition 6.2. When none of the FDEs is time-dependent we call the system *autonomous FDDS*. An FDE of an autonomous system takes the form

$$x'(t) = f(x), \quad x(t_0) = x_0. \tag{6.2}$$

FDDS was first defined by De Glas (1984). But his definition of "fuzzy derivative of a real valued function" is rather restrictive. Puri and Ralescu (1983) have presented two definitions of derivative of a fuzzy valued function whose domain of definition is an open subset of some normed (crisp) space. One of them is H-derivative or Hukuhara derivative and the other is a more generalized notion called canonical derivative. Since the H-derivative had great influence on latter works we are presenting the following definition.

Let U be an open subset of R. Let $S_0(R^n)$ be the collection of fuzzy subsets $(u : R^n \to [0, 1])$ of R^n satisfying the following properties:

(i) u is upper semicontinuous;

(ii) u is fuzzy convex, i.e. $u(\lambda x_1 + (1 - \lambda)x_2) \geq \min[u(x_1), u(x_2)]$ for $x_1, x_2 \in R^n, \lambda \in [0, 1]$;

(iii) closure($[u]^\alpha$) is compact for $\alpha \in (0, 1]$.

Definition 6.3. A fuzzy function $F : U \to S_0(R^n)$, which associates with each point $y \in U$ a fuzzy subset $F(y)$ of R^n with properties (i), (ii) and (iii) described above, is called *H-differentiable* at $y_0 \in U$ if there exists $DF(y_0) \in S_0(R^n)$ such that, the limits

$$\lim_{h \to 0+} \frac{F(y_0 + h) - F(y_0)}{h} \quad \text{and} \quad \lim_{h \to 0+} \frac{F(y_0) - F(y_0 - h)}{h} \tag{6.3}$$

both exist and are equal to $DF(y_0)$.

Kaleva (1987) was the first to define FDE in terms of H-derivative. At the same time, Seikkala (1987) also defined FDE in terms of a slightly generalized form of H-derivative. Their formulation of fuzzy initial value problem (FIVP) is as follows.

$$x'(t) = f(t, x(t)), \quad x(a) = x_0 \tag{6.4}$$

where $x(t)$ is a fuzzy valued function defined on T (whose range set is certain class of fuzzy subsets of R^n), $x'(t)$ is the H-derivative (or some generalized form of H-derivative) of $x(t)$, f is also a fuzzy valued function and x_0 is an initial

value of equation (6.4), which is a fuzzy point in the phase (solution) space. In their works and in almost all subsequent works $x(t)$ takes values only in the class of normal subsets of $S_0(R^n)$ denoted by E^n. This E^n is the phase (solution) space of the FDEs.

Recently Buckley and Feuring (2000a, b) have generalized equation (6.4) in terms of various definitions of derivatives of a fuzzy valued function.

$$\frac{dX}{dt} = f(t,X,K), \quad X(0) = C \tag{6.5}$$

assuming that we have adopted some definition for the derivative of the unknown fuzzy function $X(t)$. In equation (6.5), all capital letters denote fuzzy quantity. $X : T \rightarrow S(R^n)$, where $S(R^n)$ is the collection of fuzzy subsets of R^n. $K = (K_1, \ldots, K_n)$ is an n-dimensional fuzzy number and C is a fuzzy number. $X(0)$ is of course the initial value of X. Equation (6.5) is the Buckley-Feuring form or BF-form of FDEs. In their paper, Buckley and Feuring (2000a, b) have taken $K_i, i \in \{1, \ldots, n\}$ and C as triangular fuzzy numbers, but here for the sake of generality we prefer to keep them as arbitrary fuzzy numbers. To obtain a solution for equation (6.5) the entire range of crisp solution set of equation (6.5) is considered and then α-cuts over various fuzzy quantities responsible for generating that solution set is taken. Then the required solution set is constructed by using Zadeh's extension principle. Buckley and Feuring (2000a, b) have also defined fuzzy partial differential equations.

Solution of an FDE usually involves summation of two fuzzy numbers. But if we add two fuzzy quantities generally the diameter of the resultant fuzzy quantity is greater than any of the constituent fuzzy quantity. As a result diam $X(t) = \infty$ as $t \rightarrow \infty$, where $X(t)$ is the solution of an FDE. This render FDEs unsuitable for modeling. Also due to the same reason the classical dynamical systems' theoretic notions are not possible to extend to the FDEs.

Since the solution or phase space of equation (6.5) is supposed to represent the behavior of the system represented by equation (6.5), determining the degree of possibility of the solution set is very crucial to determine the system behavior. In this situation, if we only concentrate on determining the solution set of highest possibility, which obviously represents the best system behavior, we shall be able to reduce the computational complexity to a great extent. Taking the solution space with highest degree of possibility has been emphasized by Hullermeier (1997, 1999).

Fuzzy differential inclusions

FDI is a reformulation of equation (6.4) in a different form, where equality ($=$) is replaced by inclusion ($\in$) as formulated below:

$$x'(t) \in f(t,x(t)), \quad x(a) \in x_0 \tag{6.7}$$

Unlike in equation (6.4), in equation (6.7) $x(t)$ is a crisp trajectory and $f(t, x(t))$ is a fuzzy set of crisp functions (actually in equation (6.7) $x'(t) \in f(t, x(t))$ means $x'(t) \in [f(t, x(t))]^0$). The derivative of $x(t)$ is *not* the Hukuhara derivative but the classical derivative. Equation (6.7) in more general form in terms of arbitrary α-cuts becomes

$$x'(t) \in [f(t, x(t))]^\alpha, \quad x(a) \in x_0 \tag{6.8}$$

where $0 \le \alpha \le 1$. $x(a) \in x_0$ means $x(a) \in [x_0]^0$.

A classical FDE due to Kaleva (1987) or Seikkala (1987) proceeds from a fuzzification of the differential operator and considers the entire fuzzy flow (by the *fuzzy flow* of equation (6.4) we mean the FAM $x(t)$ as a solution of equation (6.4)) which describes the system behavior. As opposed to this the FDI proceeds from a generalization of differential inclusion relations and considers a fuzzy set of individual, crisp solutions, much the same way De Glas (1983) thought of an FDS. In the original formulation of FDI, equation (6.8) has been described as $x'(t) \in [f(t, x(t))]^\alpha$, $x(a) \in [x_0]^\alpha$. But in equation (6.8) we have taken x_0 (that is $[x_0]^0$) in place of $[x_0]^\alpha$. This has been done to keep the formulation of an FDI as general as possible. Our ultimate concern in equation (6.8) is to know the α-cut of the fuzzy set of functions $f(t, x(t))$. So initially we allow $x(a)$ to take values from whole of x_0. We may restrict $x(a)$ within some level subset of the support of x_0 only to determine $[f(t, x(t))]^\alpha$. So $x(a) \in [x_0]^\alpha$ is actually needed when we want to solve equation (6.8) (step 3 of Algorithm 6.1). At the formulation stage, it can be dispensed with to keep the formulation as general as possible, just as in equation (6.8). As long as the crisp initial value problem

$$x'(t) = f_0(t, x(t)), \quad x(a) = \lambda \in x_0,$$

where f_0 is a crisp member of the fuzzy set of crisp functions $f(t, x(t))$ and λ is a crisp member of x_0, has a unique solution, existence of unique solution of equation (6.8) as α-cuts of fuzzy set of crisp solutions is guaranteed. The α-cut of the fuzzy set of crisp solutions of equation (6.8) is the fuzzy flow representing the system behavior with possibility α or more.

Be it equations (6.4) or (6.8) the solution is more difficult compared to the classical crisp differential equations. So the quest for a qualitative rather than a quantitative solution to equation (6.4) or (6.8) is even more natural than the crisp case.

Numerical methods for solving FDIs like equation (6.8) have been developed by Hullermeier (1999), which is as follows. The admissible domain of t is γ to δ. We partition $[\gamma, \delta]$ in n equal subintervals, where the ith subinterval is denoted by I_i.

To obtain a solution of equation (6.8) we shall have to determine the fuzzy reachable set $x(t)$ for any arbitrary value of t. Like equation (17) of Hullermeier (1997), we can write

$$\frac{y_{i+1} - y_i}{\Delta t} \in f(t_i, y_i) \qquad (6.9)$$

where f is a fuzzy valued function with values in E^1. Note that a fuzzy set of crisp functions, when considered as a single function, maps a crisp or fuzzy set onto a fuzzy set and hence may be called a fuzzy valued function. E^n is the collection of normal fuzzy sets of $S_0(R^n)$. E^n is a metric space (E^n, D), where D is the Hausdorff metric on E^n. Based on equation (6.9) the following generalized difference equation scheme can be defined for fuzzy valued function of a crisp variable

$$Y(t_{i+1}) = \bigcup_{y \in Y(t_i)} \{y + \Delta t \cdot f(t_i, Y(t_i))\}, \quad Y(0) = x_0 \qquad (6.10)$$

Since the set $Y(t_i)$ may have very complicated structures, it is generally not possible to represent them exactly. So in addition to discretization of time, discretization of a class of subsets of R^n has also been considered. Let A be a class of subsets of R^n which can be represented by means of a certain data structure. Denote by $A(Y) \in A$ the approximation of a set $Y \in R^n$. The following approximation of equation (6.10) has been defined in Huellermeier (1997):

$$Z(t_{i+1}) = A\big(\cup z + \Delta t \cdot A(f(t_i, z))\big), \quad Z(0) = A(X) \qquad (6.11)$$

The implementation of solution of equation (6.8) is based on iteration of equation (6.11). The class A used in equation (6.11) for approximating sets was implemented as different classes of geometrical bodies, such as convex hulls or more general classes including nonconvex sets. All this have been very elaborately described by Hullermeier (1999). In Hullermeier (1999), some examples have also been presented with simulated results. However, the entire process is extremely computational intensive and complicated, which may be a stumbling block to the application of FDIs to model real situations despite its immense potentiality. Here we have been able to find out an easier method to solve FDIs. But unfortunately, this process will work only in one-dimensional case. It is not possible to extend it to solve the multidimensional FDIs. Nevertheless in one-dimensional cases our method will give better result than those obtained by the method of Hullermeier (1999).

A method for solving one-dimensional fuzzy differential inclusions

In one-dimensional formulation of an FDI in the form of equation (6.8) the graph of $x(t)$ is a (crisp) trajectory in R^2 and $f(t, x(t))$ is a fuzzy valued function (as a fuzzy set of crisp functions) defined on R^2 with values in E^2. $x(a) \in R$ is an initial (crisp) point (that is at $t = a$). x_0 is a fuzzy point in R. $x(a)$ varies over all crisp members of the fuzzy point x_0. x_0 is known as the fuzzy initial point. t is always a crisp quantity. Then $f(t, x(t))$ is fuzzy valued because $x(t)$ can

start (at $t = a$) from any member of the fuzzy point x_0 and some fuzzy valued parameters may also be present in $f(t, x(t))$. Let $f_0(t, x(t))$ be a crisp member of $f(t, x(t))$. We may call f_0 a representative member or *seed* of f. We shall attempt to obtain the fuzzy solution of equation (6.8) by "fuzzyfying" f_0. The algorithm is called *crystalline algorithm* (Majumdar, 2002d), where the entire fuzzy solution set of equation (6.8) is built around the seed solution just like formation of a crystal around a seed in a super-saturated liquid. The ordinary crisp numbers have also been considered as fuzzy numbers. Crisp numbers are only special fuzzy numbers. The set of all fuzzy real numbers is denoted by $\mathcal{R}$.

Algorithm 6.1. (Crystalline Algorithm):

START

Step 1: *Let* $f_0(t, x(t)) = \Phi(p_1, \ldots, p_k, t, x(t))$. *$x(t)$ starts from a crisp member λ of $[x_0]^0$ Solve (directly or numerically) $x'(t) = f_0(t, x(t))\ldots(A)$.*

Step 2: *Fix $\alpha \in [0, 1]$.*

Step 3: *Let $x(t) = \psi(q_1, \ldots, q_m, t)$ be the solution of (A) in step 1. Each $p_i = q_j$ for some $j (m > k)$. $q_m = \lambda \in [x_0]^0$. If f_0 is to vary over all members of f then each q_j must belong to a fuzzy number, that is, $q_j \in [a_j, b_j]$; $([a_j, b_j], \mu_{[a_j, b_j]}) \in \mathcal{R}$. But we are to make $x'(t) = f_0(t, x(t)) \in [f(t, x(t))]^\alpha$. This implies $q_j \in [\mu_{[a_j, b_j]}]^\alpha$ for all j and $x(a) = \lambda = q_m \in [x_0]^\alpha (x_0 = \mu_{[a_m, b_m]})$.*

Step 4: *Take $x(t) = \psi(a_1^\alpha, \ldots, a_m^\alpha, t)\ldots(B)$ and $x(t) = \psi(b_1^\alpha, \ldots, b_m^\alpha, t)\ldots(C)$, where $[\mu_{[a_j, b_j]}]^\alpha = [a_j^\alpha, b_j^\alpha]$. The region on R^2 bounded by (B) and (C) is the solution or phase space of equation (6.8).*

END

Solution of a very important one-dimensional FDI has been obtained by the above procedure in the next section. In step 1 standard numerical methods for solving ordinary crisp differential equations are to be employed when methods for direct solution are not available.

Justification: Now, let us justify Algorithm 6.1.

In practice in step 1 we just need to consider a crisp version of the expression $f(t, x(t))$. We accomplish this in two steps.

(1) We just choose a crisp member from each fuzzy parameter (constant fuzzy number) and substitute the respective fuzzy parameter by the chosen crisp number in the expression of $f(t, x(t))$.

(2) We treat any fuzzy variable as an ordinary crisp variable in the expression of $f(t, x(t))$.

This way we obtain $f_0(t, x(t)) = \Phi(p_1, \ldots, p_k, t, x(t))$. Then we solve (directly or numerically) the ordinary crisp differential equation $x'(t) = f_0(t, x(t)) = \Phi(p_1, \ldots, p_k, t, x(t))$. This way obtaining the seed solution as $x(t) = \psi(q_1, \ldots, q_m, t)$ is completed. $x(t_{i+1})$ satisfies the following equation up to the desired degree of approximation

$$x(t_{i+1}) = x + \Delta t f_0(t_i, x(t_i)), \quad x(a) = \lambda$$

from which we obtain

$$X(t_{i+1}) = \bigcup_{x \in X(t_i)} \{x + \Delta t f_0(t_i, X(t_i))\}, \quad X(a) = x_0 \qquad (6.12)$$

$X(t)$ is an approximation of the fuzzy attainability set obtained at time t by the fuzzy flow of solution of equation (6.8) starting at $t = a$. So by the way of equation (6.12) we can reach at an approximation of the fuzzy attainability set given by equation (6.10).

In step 2, we fix α, where $0 \le \alpha \le 1$. The more the value of α the higher is the possibility of the solution set of equation (6.8) to represent the actual system behavior. Often only the highest value of α is taken to represent the best system behavior.

In step 3, by allowing each crisp parameter value q_j to vary over the whole range of the corresponding fuzzy value we actually determine the whole range of the fuzzy valued function (as a fuzzy set of crisp functions) $f(t, x(t))$. In this algorithm we actually never try to approximate the fuzzy attainability set by $X(t)$ of equation (6.12) but in step 3 we directly try to determine the fuzzy attainability set (given by $Y(t)$ in equation (6.10)).

In step 4, we demarcate the α-level subset within the fuzzy attainability set by specifying the boundary of the α-level subset. In step 4, we achieve the same goal, which is set to be realized in Hullermeier (1997) by iteration of equation (6.11). Unfortunately, this technique of curving out the α-level subset out of the fuzzy attainability set will not be effective in multidimensional case. This is precisely where the algorithm fails in multidimensional case.

So far, the theory of FDEs or FDIs has remained confined to treating first-order and first-degree equations or inclusions only. The crystalline algorithm is also designed to solve first-order first-degree FDIs like equation (6.8). But in the following example we shall solve a simple second-order FDI with the help of this algorithm.

Example 6.1. (Majumdar, 2002d): Find the solution representing the best possible behavior of the system given by

$$x''(t) \in [kx]^{\alpha} \qquad (6.13)$$

$$k \in [K]^0 \qquad (6.14)$$

$$x'(0) \in [L]^{\alpha} \text{ for } x(t) \in [M]^{\alpha} \qquad (6.15)$$

$$x(0) \in [x_0]^0 \qquad (6.16)$$

where $x''(t) = dx'(t)/dt$, $x'(t) = dx(t)/dt$, K is a trapezoidal fuzzy number given by Figure 4, L is a triangular fuzzy number given by Figure 5, M is a triangular fuzzy number given by Figure 6 and x_0 is a triangular fuzzy

number given by Figure 7. $0 \leq \alpha \leq 1$. Also shown is the simulated state space of the system.

Solution. In search of a "seed" solution we shall have to fix a representative crisp expression in the fuzzy expression $[kx]^\alpha$ or $[kx(t)]^\alpha$ on the right of equation (6.13), surely $k \in [K]^\alpha$. Notice that we are to determine the solution representing the best system behavior only. In that case $\alpha = 1$. So $k \in [K]^1 = [0.6,1]$ (Figure 4). $x(t)$ is a crisp variable but $x(0)$ can take any crisp value from a fuzzy number. This contributes to the "fuzziness" of $x(t)$ (there may be other factors as well to contribute in this direction), that is $x(t)$ becomes a crisp member of a fuzzy set of crisp functions. So, in search of a seed solution it suffices to take

$$x''(t) = kx \qquad (6.17)$$

where $k \in [0.6,1]$ and x is an ordinary crisp variable. We shall proceed by solving equation (6.17) just like an ordinary crisp differential equation. Multiplying both sides of equation (6.17) by $2x'(t)$ and then integrating we obtain

$$(x')^2 = kx^2 + C \qquad (6.18)$$

where C is an integration constant. From equation (6.15), we obtain that $x'(0) \in [L]^\alpha$ for $x(t) \in [M]^\alpha$, which means $x'(0) \in [L]^1$ for $x(t) \in [M]^1$. Since L, M are traingular fuzzy numbers given by Figures 6 and 7, respectively, we have $x'(0) = 0$ for $x(t) = 1$. Hence from equation (6.18) we obtain $C = -k$. Or,

$$x'(t) = \sqrt{k}(x^2 - 1)^{1/2}. \qquad (6.19)$$

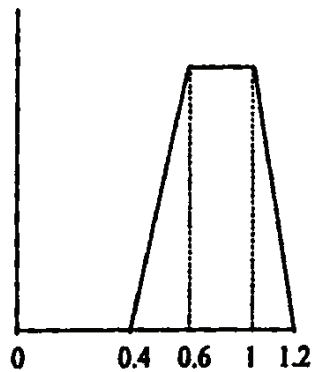Solving equation (6.19), we obtain
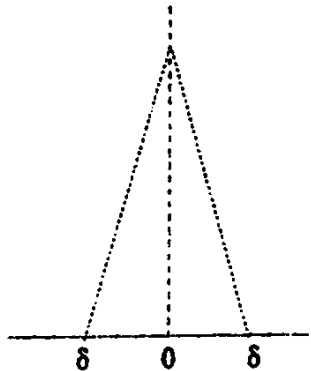

Figure 4.
Trapezoidal fuzzy
number K

Figure 5.
Triangular fuzzy
number x_0

$$\sin^{-1}x(t) = \sqrt{k}t + D, \qquad (6.20)$$

where D is an integration constant and we are considering the positive square root of k only. From equation (6.16), we obtain $x(0) \in [x_0]^0$. But we are interested only in $x(0) \in [x_0]^1$ or $x(0) = 0$ (Figure 5). So

$$x(t) = \sin(\sqrt{k}t), \qquad (6.21)$$

where $k \in [0.6, 1]$. The seed solution of equations (6.13)-(6.16) for any $k \in [0.6, 1]$. To determine the fuzzy flow corresponding to the best possible behavior of the system represented by equations (6.13)-(6.16) in the phase space we shall have to determine the boundary of the fuzzy flow in R^2. The boundary is represented by the graphs of $x(t) = \sin(0.7746t)$ (curve "1" of Figure 8) and $x(t) = \sin(t)$ (curve "2" of Figure 8).

Stability of a fuzzy system represented by a set of fuzzy differential inclusions
Of course the FDSs theoretic notions defined and discussed in the last section can be applied to the FDIs in a very straightforward manner. Let us describe the notion of stability in terms of critical points (not in terms of attractors) in case of a system of FDIs defined over a two-dimensional phase space. Let a system be described by

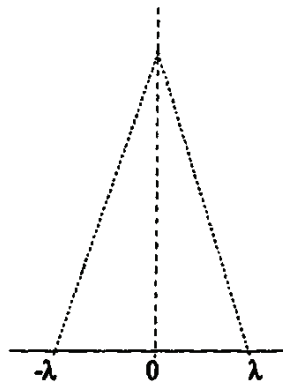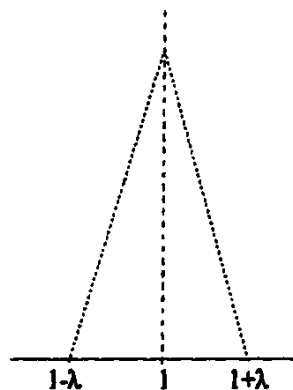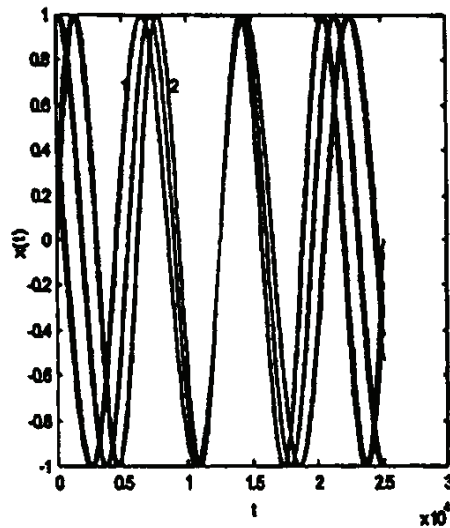**Figure 6.**
Triangular fuzzy
number L

Figure 7.
Triangular fuzzy
number M

Figure 8.
Fuzzy state space of the
system

Notes: Fuzzy state space of the system represented by (6.13), (6.14), (6.15) and (6.16) is enclosed by curves '1' (extreme left curve of the band) and '2' (extreme right curve of the band). Only the region representing the best system behaviour has been considered

$$x'(t) \in [F(x,y)]^\alpha, \qquad (6.22)$$

$$y'(t) \in [G(x,y)]^\alpha, \qquad (6.23)$$

where $F, G : R^2 \to E^1$. In order to solve this system we may write the equations as

$$\frac{dy}{dx} = \left[\frac{G(x,y)}{F(x,y)}\right]^\alpha \qquad (6.24)$$

The system represented by equation (6.24) will be stable on all points $(x,y) \in R^2$ such that, $0 \notin [G(x,y)]^\alpha$ and $0 \notin [F(x,y)]^\alpha$ for any $\alpha > 0$.

Definition 6.4. In the system described by equations (6.23) and (6.24) if $P_0 = (x_0, y_0)$ is a crisp point on the phase plane (i.e. two-dimensional phase space) such that, $0 \in [F(x_0, y_0)]^\alpha$ and $0 \in [G(x_0, y_0)]^\alpha$ for $\alpha > 0$ then we call P_0 a *critical point* of the system.

If S is the region of R^2 consisting all the critical points of the system represented by equations (6.23) and (6.24), then the system is stable at any point in the complement of S. For an implementation of this criterion let us consider a system represented by FDIs in Hullermeier (1999). Under certain assumptions dynamics of the corresponding economic system is described by the crisp differential equations

$$x'(t) = \alpha_x(\tanh(\kappa_x x + \sigma_x y) - x)\cosh(\kappa_x x + \sigma_x y), \qquad (6.25)$$

$$y'(t) = \alpha_y(\tanh(\kappa_y y + \sigma_y x) - y)\cosh(\kappa_y y + \sigma_y x). \qquad (6.26)$$

Here the state variables $-1 \leq x(t), y(t) \leq 1$ characterize the stages of business cycle. The parameters $\alpha_x, \alpha_y, \kappa_x, \kappa_y, \sigma_x, \sigma_y$, which have certain economic interpretation, not only influence the quantitative but also the qualitative behavior of the system.

But in an economic system precise values of the parameters $\alpha_x, \alpha_y, \kappa_x, \kappa_y, \sigma_x, \sigma_y$ may not be known. The uncertainty remains involved in equations (6.25) and (6.26) makes it more appropriate to formulate the system in terms of FDIs, which takes the form

$$x'(t) \in [\alpha_x(\tanh(\kappa_x x + \sigma_x y) - x)\cosh(\kappa_x x + \sigma_x y)]^\alpha \qquad (6.27)$$

$$y'(t) \in [\alpha_y(\tanh(\kappa_y y + \sigma_y x) - y)\cosh(\kappa_y y + \sigma_y x)]^\alpha \qquad (6.28)$$

where each of the parameters $\alpha_x, \alpha_y, \kappa_x, \kappa_y, \sigma_x, \sigma_y$ has been treated as a fuzzy number. It is easy to verify that the only critical point of the system is $(0, 0)$. Also 0 must not belong to the support of fuzzy numbers $\alpha_x, \alpha_y, \kappa_x, \kappa_y, \sigma_x, \sigma_y$ to make the system stable.

Let P_0 be an isolated critical point of the system represented by equations (6.22) and (6.23). If P_0 has a neighborhood $N_\delta(P_0)$ with diameter $\delta > 0$ such that, all trajectories starting from any point $P \in N_\delta(P_0)$ converges to P_0 then P_0 is called a Liapunov stable critical point of the system. This notion of Liapunov stability of the critical point P_0 follows directly from Definition 5.11, where P_0 is an *attracting critical point*. In the system given by equations (6.27) and (6.28) $(0, 0)$ is a Liapunov stable critical point.

Case studies

To demonstrate the efficacy of FDI relations in modeling very complex natural phenomena let us describe a case study here. We have in mind cyclogenesis, i.e. genesis of cyclones (Majumdar, 2002a, b). We shall apply Dempster-Shafer's theory of evidence in medical image fusion (Bhattacharya and Dutta Majumder, 2000; Dutta Majumder and Bhattacharya, 2000). We have also presented a statistical analysis (from general system theoretic point of view) of the development and regression of cancer in human body (Dutta Majumder and Roy, 2000). As a very interesting example of atmospheric cybernetics, following Majumdar (2002a, b), here we shall propose a FDS modeling of a climatic disturbance created by winds coming from different directions and colliding to give rise to a vortex under certain conditions. Under favorable conditions, this vortex may lead to the development of a cyclone (Case study I). Then, following Bhattacharya and Dutta Majumder (2000) and Dutta Majumder and Bhattacharya (2000) we shall present a novel bio-cybernetic fusion technique for multimodal medical images by applying Dempster-Shafer theory of evidence (Case study II). Finally, following Dutta Majumder and Roy (2000) we shall present a path breaking bio-cybernetic and general system

theoretic approach to induce spontaneous regression on malignant tumour by means of large fluctuating perturbations.

Complexity
analysis

1173

Case study I. Evolution of cyclonic weather – cyclogenesis
Through sustained research, particularly after the Second World War, the physics behind the genesis and development of a tropical storm has already been understood to a large extent. It is well known that despite the prevalence of favorable geographic and climatic conditions over a large part of the globe during the storm seasons the actual occurrence of a tropical storm is a relatively rare phenomenon. Even when a tropical storm is developed, about the half of all of them cannot reach hurricane strength (say of intensity T 3.5 or more in the Dvorak (1984) scale). The reason behind the relative rare occurrence of a strong tropical storm (called Hurricane in USA, Typhoon in China, Cyclone in India, etc.) is that, a sufficiently strong initial disturbing vortex is essential to give rise to an intense tropical storm (Emanuel, 1988). So not only the favorable geographic and climatic conditions are essential prerequisites but also a sufficiently strong initial disturbance is required. Following Ooyama (1969) there have been innumerable models of steady state cyclones. But there is almost no model for the initial strong disturbance instrumental behind the genesis of a cyclone. Here we shall present a simple but elegant model of this strong initial disturbance. Normally it could have been a very complex task. But due to the inherent efficiency of the FDIs to model complex systems it has become a rather simple task.

To present our idea in a concise form let us assume that not one but two initial disturbances are needed to give rise to a strong initial disturbing vortex. Each of the disturbance is in the form of linear wind jet propagating parallel to the ground. One of them is very strong (speed more than 40 km/h) acting in the cross-radial direction to the vortex to be created and the other is very weak (speed about 5 km/h) acting at the radial direction of the vortex.

With respect to a cylindrical coordinate system expressions for the radial, cross-radial and vertical components of the velocity are as follows.

$$dr/dt = \text{radial component;}$$

$$r(d\theta/dt) = \text{cross-radial component;}$$

$$dz/dt = \text{vertical component.}$$

We know that the shape of a cyclonic vortex, as observed through the satellite images and also through the RADARs, is log-spiral. But to have a log-spiral (or equiangular spiral) shaped vortex out of linear wind jets the ratio of radial and cross-radial components of the velocities must be a constant. Since the linear wind jets propagate parallel to the ground the vertical component of the velocity is zero. So the model of the initial disturbing vortex is given by

$$\frac{dr/dt}{r(d\theta/dt)} = m \quad \text{(a constant).} \tag{7.1}$$

But equation (7.1) in its present form is trivially inadequate. It can never model the real situation, because m being the ratio of two wind speeds cannot be a fixed value over an interval of time needed for the formation of the vortex. m is liable to fluctuate even within a very small interval of time. But surely m will not go beyond certain range. That is, the values of m will always lie within some interval of the real line. So we find that m takes values from a fuzzy real number M. m is not fixed. But M as a fuzzy number is fixed. M is a trapezoidal fuzzy number $\langle a, c, d, b \rangle$ as shown in Figure 9, which behaves as a (fuzzy) constant.

Let us explain why we have taken the fuzzy constant M as a trapezoidal fuzzy number. $M : R \rightarrow [0, 1]$ is of course a membership function. $M(m) = 0$ if $m \notin [a, b]$. This means that the created vortex for $m \notin [a, b]$ will not be a stable one. Vortex created for those m will collapse shortly due to the wind shear. As the value of m moves from a to c the possibility of the created vortex to become a cyclone increases under favorable climatic and geographical conditions. When $m \in [c, d]$ the possibility of the created vortex to mature into a tropical storm of hurricane strength is the highest under favorable climatic and geographical conditions (although $M(m) = 1$ for $m \in [c, d]$ this does not mean that a tropical storm will develop without fail for $m \in [c, d]$). Again as values of m progress from d to b the possibility of the created vortex to mature into an intense tropical storm even under favorable climatic and geographical conditions diminishes before finally becoming zero for $m \geq b$. Values of a, b, c and d will have to be determined experimentally. We have inferred the values of a, b, c and d for the northern hemisphere with the help of numerical simulation on synthetic data. These values are given in Table I.

So formulation of equation (7.1) as an FDE rather than a crisp differential equation is more realistic. But as we have already explained the limitations of FDEs in modeling and simulation, it is better to give a FDI relation formulation of equation (7.1), which will be the FDS modeling of the initial climatic disturbance leading to the genesis of a cyclone.
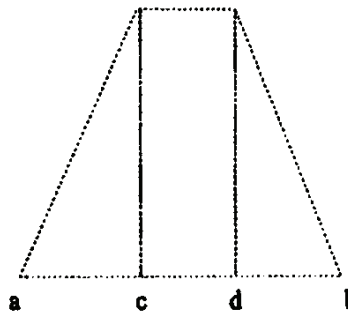

Figure 9.
Trapezoidal fuzzy
number M

$$r'(\theta) \in [mr(\theta)]^{\alpha}, \quad r(0) \in [r_0]^{\alpha}. \qquad (7.2)$$

Equation (7.2) is the FDI relation formulation of equation (7.1). r_0 is a triangular fuzzy number given in Figure 10.

Typical value of $r(0)$ is 1,000 km. But some inaccuracies always remain involved in its measurement. So we have taken r_0 as a triangular fuzzy number, which is the fuzzy set of all possible values of $r(0)$.

Now we are in a position to present the phase (solution) space of equation (7.2) with the help of the crystalline algorithm described in the last section, which is the simulated initial disturbing vortex. We shall opt only for those solutions which represent the best possible system behavior. That is, we shall consider the solution of equation (7.2) for $\alpha = 1$ only (Figure 11).

Case study II. Multimodal medical image fusion for cancerous cells

A more exciting case study is treating the very complex development of diseases in human body from a bio-cybernetic and general systems' theoretic point of view as proposed by Dutta Majumder and Bhattacharya (2000) and Dutta Majumder and Roy (2000). Here, the complexity is due to the extreme difficulty in determining the systems parameters and the exact relationships among them.

Dutta Majumder and Bhattacharya (2000) have made a very novel and interesting application of Dempster-Shafer theory of evidence to combine (fuse) three different kinds of images of brain tissues of an Alzheimer's patient.

The combination of information from multimodal imageries as a fused image would provide optimum information (Hall, 1998). The multimodal medical image fusion may provide integrated information to the medical practitioners

a	c	d	b	
−0.2	−0.1	−0.05	−0.001	

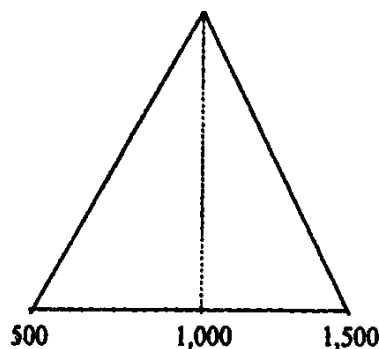Table I.
Numerical specification
of M

Figure 10.
Triangular fuzzy
number r_0

for improved diagnostics and therapeutic planning (Dutta Majumder and Bhattacharya, 1997, 1998a, b, c, 1999). The objective is to indicate an approach based on soft computing methods in uncertainty management for decision-making in context-dependent and context-independent systems of data fusion for medical images.

Let A_1 be the class of images of brain tissues of an Alzheimer's patient obtained by using T_1 weighted MR, A_2 be the class of images of the same brain tissues of the Alzheimer's patient obtained by T_2 weighted MR and A_3 be the class of images of the same brain tissues of the same Alzheimer's patient obtained by CT imaging. From Dempster-Shafer theory of evidence (Shafer, 1976), we obtain

$$\text{Bel}(A_1 \cup A_2 \cup A_3) = m(A_1) + m(A_2) + m(A_3) + m(A_1 \cup A_2) + m(A_2 \cup A_3)$$

$$+ m(A_3 \cup A1) + m(A_1 \cup A_2 \cup A_3),$$

where m is the measure of the basic probability assignment. Let the frame of discernment θ be the set of all elementary propositions $m(\theta)$, which is the basic probability assignment indicating to what extent a sensor is able to distinguish any elementary proposition.

Thus from the D-S theory a number in the interval [0,1] is used which indicates the degree of evidence to support a proposition. Thus, m: $2^\theta \rightarrow [0,1]$
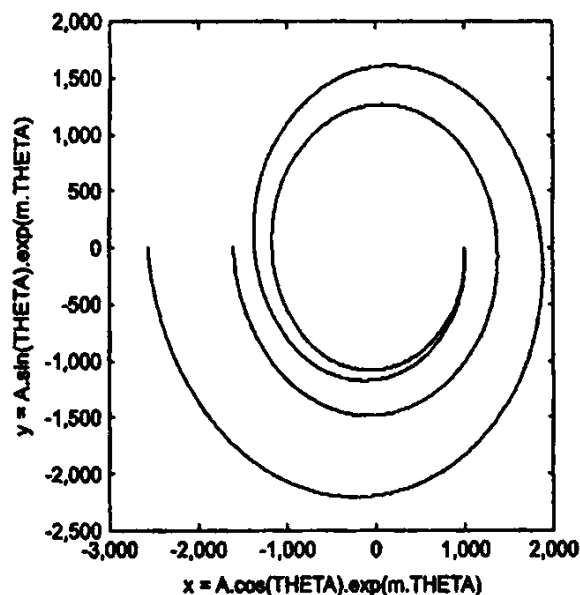

x = A.cos(THETA).exp(m.THETA)

Note: Where α = 1, that is, the phase space is showing the fuzzy flow (the entire region enclosed between the two curves including the curves also) representing the best system behaviour only. This flow tends to converge to a (fuzzy) point of the (fuzzy) phase space, which is known as the fuzzy attractor of the system. If this vortex ultimately matures into a severe cyclonic storm this fuzzy attractor will become the eye of the cyclone

and $\Sigma m(A) = 1$ for all A contained in θ and $m(\emptyset) = 0$, where $\emptyset$ is the null hypothesis. A are the focal elements of θ having nonzero m values.

The two images Im1 (Figure 12(a)) and Im2 (Figure 12(b)) are complementary to each other. By the D-S method of fusion the basic probability masses are assigned in an area where the union of classes defined is mixed.

Experimental procedure and result

In Im1 and Im2 the classes are discriminated in:

(1) gray matter;

(2) white matter;

(3) cerebrospinal fluid (CSF);

(4) ventricle;

(5) bones.

The gray matter and the white matter together constitute the brain region and according to our notation this is class C_1, while the ventricle and the CSF together are denoted by C_2. The overlapping regions of C_1 and C_2 are denoted by $C_1 \cap C_2 = C_3$ and the outer bony layer of each slice is denoted by C_4. So the four classes of brain matter are defined as C_1, C_2, C_3, C_4. To assign the probability mass for each class a central or focal element is chosen. This central element is related to each class of the Im1 and Im2. The null mass $m(0)$ is defined for each class which is not assigned. To determine the probability masses the frame of discernment is proposed as $\theta = (C_1, C_2)$. The power set of $\theta, 2^\theta = (\theta, C_1, C_2, C_1 \cup C_2)$. The mass functions are assigned for two images as m_1 and m_2, respectively, and mass functions assigned for the classes are $m_1(C_1), m_1(C_2), m_1(C_1 \cup C_2)$ for Im1 and as $m_2(C_1), m_2(C_2), m_2(C_1 \cup C_2)$ for Im2, such that, $m_i(C_1) + m_i(C_2) + m_i(C_1 \cup C_2) = 1$, where $1 = 1, 2$. $\mathrm{bel}(C_1) = m_i(C_1)$ and $\mathrm{bel}(C_2) = m_i(C_2)$, $\mathrm{bel}(C_1 \cup C_2) = m_i(C_1) + m_i(C_2) + m_i(C_1 \cup C_2) = 1$.

In pixel-based classification the distribution of different pixels over a specified gray level is considered. In Im1, the pixels are spread over the range 20-190. Primarily we have considered the distribution of pixels over the range below 130, between 130 and 190 and above 190.

(a) Image 1(Im1)   (b) Image 2(Im2)   (c) Fused image

Figure 12.
Fusion of medical images

When M is the number of pixels lying in the region over a specified gray level and if N is the dimension of the region ($N = N_1 \times N_2$) taken as small image of the brain:

(1) for pixels having gray values less than 130, $m_1(C_2) = 1$, $m_1(C_1) = 0$ and $m_1(C_1 \cup C_2) = 0$;

(2) for pixels with gray level values greater than 190, $m_1(C_1) = 1$, $m_2(C_2) = 0$, $m_1(C_1 \cup C_2) = 0$;

(3) for pixels with gray level values between 130 and 190, $m_1(C_1) = 0.49$; $m_1(C_2) = 0.3$ and $m_1(C_1 \cup C_2) = 0.193$.

Similarly for Im2 the range of gray levels distributed over the pixels:

(1) for pixels having gray values less than 90, $m_2(C_1) = 1$, $m_2(C_2) = 0$, $m_2(C_1 \cup C_2) = 0$;

(2) for pixels having gray values 90-135, $m_2(C_2) = 0$, $m_2(C_1) = 0.552$, $m_2(C_1 \cup C_2) = 0.498$;

(3) for pixels having gray values between 135 and 140, $m_2(C_1) = M'/N'$, $m_2(C_2) = M'_1/N'_1$ and $m_2(C_1 \cup C_2) = 1 - M'/N' + M'_1/N'_1$, thus $m_2(C_1) = 0.01$, $m_2(C_2) = 0$, $m_2(C_1 \cup C_2) = 0.99$;

(4) for pixel gray level values from 140 to 197 $m_2(C_1) = 0.03$, $m_2(C_2) = 0.07$ and $m_2(C_1 \cup C_2) = 0.9$;

(5) for pixel gray level values between 197 and 255, $m_2(C_1) = 0$, $m_2(C_2) = 0.9$ and $m_2(C_1 \cup C_2) = 0.7$.

The pixels at the specified gray level value ranges are classified from both the images of T_1 and T_2 weighted MR images of brain and are fused after the registration in a common reference frame using Dempster-Shafer accumulation theory. The fused images of both the modalities are shown in Figure 12(c). For a pixel the decision is taken in favor of either C_1 or C_2 as to whether bel(C_1) > bel(C_2) or vice versa. Thus, the belief measure indicates in which class a pixel belongs to the fused image (Figure 12(c)).

Case study III. A general system theoretic analysis of cancer self-remission
The dynamics of cancer development and regression are given by the following equation (Dutta Majumder and Roy (2000)):

$$\mathrm{d}M/\mathrm{d}t = q + [sM\{1 - (M/K)\}] - rf(M), \qquad (7.3)$$

where M is the density of tumour cells, q (a constant) is called malignant cellular transformation, i.e. the rate of normal cells to malignant ones. The term within the square bracket is a Fischer logistic growth term indicating the increase of tumour cells with replication rate s (constant) and maximum carrying or packing capacity K (constant). The last term $rf(M)$ denotes the rate of destruction of tumour cells by the immune cells, where r (constant) is

the rate of tumour cell destruction and $f(M)$ is a function in M. In a dimensionless form equation (7.3) can be reformulated using rescaled variables and a rescaled time $t = (s - q)t$:

$$dm/dt = v + m(1 - um) - r\{m/(1 + m)\}. \qquad (7.4)$$

Let r_t denote the fluctuation of r about a mean value of r. Then r_t is given by

$$r_t = r + \sigma H_t, \qquad (7.5)$$

where H_t is the statistical perturbation with standard deviation σ. It is clear that for normal cytotoxic or immunological interactions these fluctuations vary much more rapidly than the macroscopic evolution of the tumour. The probability distribution function $P(m)$ has been calculated by Dutta Majumder and Roy (2000) as

$$P(m) = \exp(2/\sigma^2)[-v/m + (v + 2 - u - r)m - (1 - 2u)m^2/2$$

$$- um^3/3 + (2v - 1 - r - \sigma^2)\ln m + \sigma^2 \ln(1 + m)]. \qquad (7.6)$$

We now wish to find the effect of increasing the fluctuation of the tumour cell destruction rate r, that is increasing its σ, and observe the consequent change of probability $P(m)$. Dutta Majumder and Roy (2000) have kept the range of increasing σ as $0 \le \sigma \le 3$. The cancer cell reduction rate r can be fluctuated by perturbing various parameters which influence r, such as perturbing any of the following parameters:

(1) radiation flux;

(2) cytotoxic chemical flux;

(3) immune cell concentration;

(4) tumour temperature;

(5) glucose level of the blood impinging on the tumour;

(6) oxygen partial pressure, pO_2, i.e. oxygen level in tumour matrix;

(7) haemeodynamic perfusion of the tumour; and so on.

We know that variations in these parameters are reflected as random variations of indices like r which give them a stochastic character (Lefever and Horsthemke, 1979). We see that, as σ increases from 0.5 to 0.854 to 2.83, the probability density function $P(m)$ exhibits a non-equilibrium phase transition, apropos the Glansdorff-Prigogine (1971) theorem. The peak probability density of tumour cells shifts towards very low value of tumour cell density X; for instance, tumour cell density X shifts from 4.3 (macro-cancer focus) to 0.46 (micro-cancer focus), i.e. there occurs the *phase transition*:

Macro-cancer focus $\xrightarrow[\text{Increase of } \sigma]{}$ Micro-cancer focus.

This corresponds to regression and elimination of malignancy. Hence, we infer that if one or more parameters such as, oxygenation, radiation or temperature, etc. are varied then the tumour may have a predisposition to regress and destabilize if the standard deviation σ of the parameter's variation crosses the following threshold.

The tumour regression threshold : $\sigma \leq 2.83$.

We can thus enunciate the corresponding stability principle for tumour regression and malignancy (Dutta Majumder and Roy, 2000).

General stability principle for tumour regression. A tumour may have predisposition to destabilize and regress if there is a sufficient fluctuation of the malignant cell reduction rate, so that σ is 2.83 or above, which may be achieved by correspondingly high variation of temperature, oxygenation, radiation, etc.

A successful implementation of the above principle has been christened as *multiplicative fluctuation* as a new multimodality therapy in Dutta Majumder and Roy (2000). Here perturbations have been used in the form of:

(1) increasing arterial pO_2 to 90 mmHg, decreasing venous pO_2 to 20 mmHg;

(2) using hyperthermia up to 105°F;

(3) hyperglycaemia up to 600 mg percent blood glucose level, which also produces pH perturbations (up to 6.5 from the normal tissue level of 7.8).

Owing to variation of oxygenation there is perturbation in the oxygen index of the blood η as $18.5 < \eta < 42$. The treatment was administered daily for 2 h for a period of 18 days. In terms of the number of tumour cells destroyed the technique presents a hundred-fold increase in cell kill leading to regression of a huge clear cell tumour above the knee with a size of 2 kg (Dutta Majumder and Roy, 2000).

Exploring new ways in this direction is still on. The treatment process towards this goal is *bi-thermia* (Dutta Majumder and Roy, 2000). It is a combination of *hyperglycaemia* (glucose level is varied between 400 mg/100 ml blood and normal level of 100 mg/100 ml blood) and *hyperthermia* (temperature variation used is from 98 to 103°F, using high-frequency inductive heating). pH fluctuation and bi-thermia have been devised in Dutta Majumder and Roy (2000). Much higher fluctuation can be achieved safely by temperature variation from 96 to 102°F with $\sigma = 5$. This has resulted in disappearance of metastatic spread in a lymph node in the waist (Dutta Majumder and Roy, 2000).

Conclusion

In this paper, we have studied various uncertain dynamical systems and cybernetics as convenient tools for modeling various complex biological, physical and natural phenomena. We have first discussed various uncertainties and then their roles in modeling various dynamical and cybernetic systems.

We have presented a very detailed discussion on FDSs. Although the mathematical theory of FDSs is the most fundamental of all fuzzy systems, of which fuzzy controllers are perhaps the most prominent examples, the development of this theory so far did not get the attention it deserves. In this paper along with presenting a review of the up to date important developments of the mathematical theory of FDSs we have attempted to extend the same on line of the development of the mathematical theory of classical crisp dynamical systems. In the process, we have made a slight extension to the Kloeden's (1982) definition of an FDS. Then we have introduced the notion of fuzzy dissipative dynamical system. We also have formulated an easy to implement criterion to test the dissipativeness of an FDS. Then we have given a very broad based definition of fuzzy attractors in terms of fuzzy attainability set mappings. We have defined stability of an FDS in terms of the attractor. We also have introduced the notion of robustness for an FDS in terms of its attractor. We have extended Devaney's (1989) definition of chaos to the FDSs. Along with this the concept of homoclinic points has been extended to the FDSs. Then we have defined fuzzy Liapunov exponent, fuzzy metric entropy and fuzzy Liapunov time. Fuzzy differential dynamical systems have been discussed in a separate section, where we have given an algorithm for solving one-dimensional FDIs. Solving one-dimensional FDIs by this algorithm takes much less number of computations (it only calculates the boundary of the solution set rather than the whole set as done in the existing one). A stability criterion has been formulated in terms of critical points for a fuzzy system represented by a set of FDIs.

Finally, we have presented three interesting case studies involving atmospheric and medical cybernetics to show how some of the theories described in the earlier part of this paper actually works in real life. In the second case study we have later extended the model of cycolgenesis from two dimension to three dimension (Majumdar and Dutta Majumder, 2004b). In the third case study, we have presented the bio-cybernetic model of cancer regression in terms of PU. We have already made an FDS model of the evolution of tumour in human tissue (Majumdar and Dutta Majumder, 2004a).

References

Bassanezi, R.C., De Barros, L.C., Pedro, A. and Tonelli, P.A. (2000), "Attractors and asymptotic stability for fuzzy dynamical systems", Fuzzy Sets and Systems, Vol. 113, pp. 473-83.

Bhattacharya, M. and Dutta Majumder, D. (2000), "Registration of CT and MR images of Alzheimer's patient: a shape theoretic approach", Pattern Recognition Letters, Vol. 21, pp. 531-48.

Birkoff, G.D. (1977), Dynamical Systems, Amer. Math. Soc. College Publ.

Buckley, J.J. and Feuring, T. (2000a), "Fuzzy differential equations", Fuzzy Sets and Systems, Vol. 110, pp. 43-54.

Buckley, J.J. and Feuring, T. (2000b), "Introduction to fuzzy partial differential equations", Fuzzy Sets and Systems, Vol. 110, pp. 241-8.

1182

Buckley, J.J. and Hayashi, Y. (1998), "Applications of fuzzy chaos to fuzzy simulation", *Fuzzy Sets and Systems*, Vol 99, pp. 151-7.

Cabrelli, C.A., Forte, B., Molter, U.M. and Vrscay, E.R. (1992), "Iterated fuzzy sets systems: a new approach to the inverse problem for fractals and other sets", *Journal of Mathematical Analysis and Applications*, Vol. 171, pp. 79-100.

Chang, C.L. (1968), "Fuzzy topological spaces", *Journal of Mathematical Analysis and Applications*, Vol. 24, pp. 182-90.

Chang, S.S.L. and Zadeh, L.A. (1972), "On fuzzy mappings and control", *IEEE Transactions on Systems, Man, and Cybernetics*, Vol. 2, pp. 30-4.

Coveney, P.V. (1988), "The second law of thermodynamics: entropy, irreversibility and dynamics", *Nature*, Vol. 333, pp. 409-15.

De Glas, M. (1983), "Theory of fuzzy systems", *Fuzzy Sets and Systems*, Vol. 10, pp. 65-77.

De Glas, M. (1984), "Invariance and stability of fuzzy systems", *Journal of Mathematical Analysis and Applications*, Vol. 99, pp. 299-319.

Devaney, R.L. (1989), *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley Publishing Company, Redwood City, CA.

Dutta Majumder, D. (1975), "Cybernetics, a science of engineering and biology", *Cybernetica*, Namur. Belgium, Vol. XVII No. 3.

Dutta Majumder, D. (1979), "Cybernetics and general systems – a unitary science?", *Kybernetes*, Vol. 8, pp. 7-15.

Dutta Majumder, D. (1993), "Fuzzy mathematics and uncertainty management for decision making in science and society", *Journal of Computer Science and Information*, Vol. 23, pp. 1-31.

Dutta Majumder, D. and Bhattacharya, M. (1997), "Uncertainty management in multimodal image fusion system for diagnosis and therapy planning", *International Conference on Recent Advances in Statistics and Probability*, 29 December-1 January, Indian Statistical Institute and Bernoulli Society for Mathematical Statistics and Probability, Calcutta.

Dutta Majumder, D. and Bhattacharya, M. (1998a), "Soft-computing methods for uncertainty management in multimodal image registration and fusion problems", *International Conference of Information Sciences JCIS-98*, October, for oral presentation and in proceedings, Duke University, NC.

Dutta Majumder, D. and Bhattacharya, M. (1998b), "Soft computing methods for uncertainty management in multimodal medical image data fusion", *16th International CODATA Conference*, 8-12 November, Delhi.

Dutta Majumder, D. and Bhattacharya, M. (1998c), "Multimodal data fusion for Alzheimer's patients using Dempster-Shafer theory of evidence", *3rd Asian Fuzzy Systems Symposium*, Korea, Published in Proceedings, pp. 713-19.

Dutta Majumder, D. and Bhattacharya, M. (1999), "Cybernetic approach to medical technology for diagnosis and therapy planning", *Proceedings of 11th International Congress of Cybernetics and Systems*, World Organization of Systems and Cybernetics, August, Brunel University, Uxbridge, Middlesex, pp. 103-8.

Dutta Majumder, D. and Bhattacharya, M. (2000), "Cybernetic approach to medical technology: application to cancer screening and other diagnostics", *Kybernetes*, Vol. 29, pp. 871-95.

Dutta Majumder, D. and Roy, P.K. (2000), "Cancer self-remission and tumour instability – a cybernetic analysis: towards a fresh paradigm of cancer treatment", *Kybernetes*, Vol. 29, pp. 896-927.

Dvorak, V.F. (1984), "Tropical cyclone intensity analysis using satellite data", *NOAA Technical Report NESDIS 11*, Satellite Applications Laboratory, Washington, DC.

Emanuel, K.A. (1988), "Toward a general theory of hurricanes", *American Scientist*, Vol. 76, pp. 370-9.

Glansdorff, P. and Prigogine, I. (1971), *Thermodynamic Theory of Stability, Structure and Fluctuation*, Wiley, London.

Hall, D.L. (1998), *Mathematical Techniques in Multiuser Data Fusion*, Artech Hause, Boston, MA.

Hullermeier, E. (1997), "An approach to modelling and simulation of uncertain dynamical systems", *International Journal of Uncertainty, Fuzziness, Knowledge-Based Systems*, Vol. 5, pp. 117-37.

Hullermeier, E. (1999), "Numerical methods for fuzzy initial value problems", *International Journal of Uncertainty, Fuzziness, Knowledge-Based Systems*, Vol. 7, pp. 1-23.

Kaleva, O. (1987), "Fuzzy differential equations", *Fuzzy Sets and Systems*, Vol. 24, pp. 301-17.

Kalman, R. (1962), "Mathematical descriptions of linear dynamical systems", *RIAS Technical Report*, Vol. 18 No. 62.

Katok, A. and Hasselblatt, B. (1995), *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, MA.

Klir, G.J. (1991), "Generalized information theory", *Fuzzy Sets and Systems*, Vol. 40, pp. 127-42.

Klir, G.J. and Folger, T.A. (1988), *Fuzzy Sets, Uncertainty, and Information*, Prentice-Hall of India.

Kloeden, P.E. (1982), "Fuzzy dynamical systems", *Fuzzy Sets and Systems*, Vol. 7, pp. 275-96.

Kloeden, P.E. (1991), "Chaotic iterations of fuzzy sets", *Fuzzy Sets and Systems*, Vol. 42, pp. 37-42.

Lefever, R. and Horsthemke, W. (1979), "Bi-stability in fluctuating environment: immunology", *Bull. Math. Biology*, Vol. 41, pp. 469-90.

Li, T.Y. and Yorke, J.A. (1975), "Period three implies chaos", *American Mathematical Monthly*, Vol. 82, pp. 985-92.

Lientz, R.P. (1972), "On time dependent fuzzy sets", *Information Science*, Vol. 4, pp. 367-76.

Majumdar, K.K. (2000a), "A mathematical analysis of time", *Science Philosophy Interface*, Vol. 5, pp. 61-9.

Majumdar, K.K. (2000b), "Generalized fuzzy vector space", *Proceedings of the 4th Asian Fuzzy Systems Symposium*, 31 May-3 June, Tsukuba, Japan, pp. 239-43.

Majumdar, K.K. (2001), "Fuzzy – another paradigm to tackle complexity", Accepted for presentation in the *10th FUZZ-IEEE Conference*, 2-5 December, Melbourne.

Majumdar, K.K. (2002a), "A fuzzy dynamical system modelling of a disturbance leading to cyclogenesis", *Journal of Intelligent and Fuzzy Systems*, Vol. 13, pp. 7-15.

Majumdar, K.K. (2002b), "A mathmatical model of the nascent cyclone", Accepted for presentation at the *International Geoscience and Remote Sensing Symposium 2002 and the 24th Canadian Symposium on Remote Sensing*, Toronto, 24-28 June, Also published in *IEEE Transactions on Geoscience and Remote Sensing* Vol. 41, pp. 1118-22.

Majumdar, K.K. (2002c), "One dimensional fuzzy differential inclusions", *Journal of Intelligent and Fuzzy Systems*, Vol. 13, pp. 1-5.

Majumdar, K.K., (2003a), "Some studies on uncertainty management in dynamical systems by fuzzy techniques with applications", PhD thesis, Indian Statistical Institute.

Majumdar, K.K. and Dutta Majumder, D. (2004a), "Fuzzy differential inclusions in atmospheric and medical cybernetics", *IEEE Transactions on Systems, Man, and Cybernetics, Part-B: Cybernetics*, Vol. 34 No. 2, pp. 877-87.

Majumdar, K.K. and Dutta Majumder, D. (2004b), "Some studies on uncertainty management in dynamical systems using cybernetic approach and fuzzy techniques with applications", *International Journal of Systems Science*.

Markov, A.A. (1931), "Sur une propriete generale des ensembles minimaux de M. Birkoff", *C.R. Academic Science Paris*, Vol. 193, pp. 823-5.

Milnor, J. (1985), "On the concept of attractor", *Communications in Mathematical Physics*, Vol. 99, pp. 177-95.

Nazaroff, G.J. (1973), "Fuzzy topological polysystems", *Journal of Mathematical Analysis and Applications*, Vol. 41, pp. 485-7.

Ooyama, K. (1969), "Numerical simulation of life cycle of tropical cyclones", *Journal of Atmospheric Science*, Vol. 26, pp. 3-40.

Pal, N.R. and Bezdek, J.C. (1994), "Measuring fuzzy uncertainty", *IEEE Transactions on Fuzzy Systems*, Vol. 2 No. 2, pp. 107-18.

Palis, J. (2002), "Chaotic and complex systems", *Current Science*, Vol. 82 No. 4, pp. 403-6.

Puri, M.L. and Ralescu, D.A. (1983), "Differentials of fuzzy functions", *Journal of Mathematical Analysis and Applications*, Vol. 91, pp. 552-8.

Seikkala, S. (1987), "On the fuzzy initial value problem", *Fuzzy Sets and Systems*, Vol. 24, pp. 319-30.

Shafer, G. (1976), *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, NJ.

Smale, S. (1967), "Differentiable dynamical systems", *Bulletin of American Mathematical Society*, Vol. 13, pp. 747-817.

Tathachar, M.A.L. and Viswanath, P. (1997), "On the stability of fuzzy systems", *IEEE Transactions on Fuzzy Systems*, Vol. 5, pp. 145-51.

Teodorescu, H.N. (1992), "Chaos in fuzzy systems and signals", *Proceedings of 2nd International Conference on Fuzzy Logic and Neural Networks*, Iizuka, Japan, pp. 21-50.

Warren, R.H. (1976), "Optimality in fuzzy topological polysystems", *Journal of Mathematical Analysis and Applications*, Vol. 54, pp. 309-15.

Wiener, N. (1949), *Cybernetics: or Control and Communication in the Animal and the Machine*, Wiley, New York, NY.

Yajnik, K.S. (1999), "Coexistence of regularity and irregularity in a nonlinear dynamical system", *Current Science*, Vol. 77, pp. 783-93.

Yen, J. (1990), "Generalizing the Dempster-Shafer theory to fuzzy sets", *IEEE Transactions on SMC*, Vol. 20, pp. 559-70.

Zadeh, L.A. (1965), "Fuzzy sets and systems", *Proceedings of Symposium on Systems Theory*, Polytechnic Institute Press, Brooklyn, NY.

Zadeh, L.A. (1978), "Fuzzy sets as a basis for theory of possibility", *Fuzzy Sets and Systems*, Vol. 1, pp. 3-28.

Further reading

Kandel, A., Friedman, M. and Ming, M. (1996), "On fuzziness and duality of fuzzy differential equations", *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, Vol. 4, pp. 553-60.

Majumdar, K.K. (2003b), "An alternative definition of fuzzy vector space", Electronics and Communication Sciences Unit Technical Report, Indian Statistical Institute.

Zadeh, L.A. and Desoer, C.A. (1963), *Linear Systems Theory - the State-Space Approach*, McGraw-Hill Book Co., New York, NY.