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THE CRITICAL PROBABILITY FOR THE FROG MODEL IS NOT A MONOTONIC FUNCTION OF THE GRAPH

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Abstract

We show that the critical probability for the frog model on a graph is not a monotonic function of the graph. This answers a question of Alves, Machado and Popov. The nonmonotonicity is unexpected as the frog model is a percolation model.

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1. Introduction

We study a property of the critical probability for percolation of a discrete-time particle system on a graph, known as the frog model with death. In this model, particles move as a discrete-time independent simple symmetric random walk (SSRW) on the vertices of a graph $\mathcal{G}$, dying after a geometrically distributed random lifetime. Initially there is an independent random number of particles at each site of $\mathcal{G}$. A site of $\mathcal{G}$ is singled out and called its root. All particles are inactive at time zero, except for those that are placed at the root. At each instant of time, each active particle may die with probability $1 - p$. If an active particle survives, it jumps, along an edge, to one of its nearest neighbor sites, chosen with uniform probability, performing an SSRW on the vertices of $\mathcal{G}$. Up to the time it dies, it activates all inactive particles that it meets along its way. From the moment that they are activated onwards, every such particle starts to walk, performing exactly the same dynamics, independent of everything else.

The motivation for studying this model is practical, as this model has been proposed for the study of both information spreading and virus transmission over a network of computers; one of the authors learned this from K. Ravishankar. The original idea is that every moving particle has some information and it shares that information with a sleeping particle at the time the former jumps onto the site at which the latter is. Particles that have the information move freely, helping in the process of spreading information. This model has experienced a growing interest and progress recently.

Let us define the model in a formal way. We denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ an infinite connected nonoriented graph of locally bounded degree. Here, $\mathcal{V} := \mathcal{V}(\mathcal{G})$ is the set of vertices (sites) of $\mathcal{G}$ and $\mathcal{E} := \mathcal{E}(\mathcal{G})$ is the set of edges of $\mathcal{G}$. Sites are said to be neighbors if they belong to

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a common edge. The *degree* of a site x is the number of edges which have x as an endpoint. A graph is *locally bounded* if all its sites have finite degree. In addition, a graph has *bounded degree* if its maximum degree is finite. Fix a site $\mathbf{0} \in \mathcal{V}$ and call it the root of $\mathcal{G}$.

With the usual abuse of notation, by $\mathbb{Z}^d$ we mean the graph with the vertex set $\mathbb{Z}^d$ and edge set

$$\{(x, y) : x, y \in \mathbb{Z}^d \text{ and } \|x - y\|_1 = 1\},$$

where $\|\cdot\|_1$ is the L_1 norm; that is, for $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, we take $\|x\|_1 = \sum_{i=1}^d |x_i|$. For $d \geq 1$, we denote by $\mathbb{T}_d$ the degree $d + 1$ homogeneous tree.

Let η be a random variable taking values in $\mathbb{N} = \{0, 1, 2, \ldots\}$ such that $\mathrm{P}[\eta \geq 1] > 0$. Let $\{\eta(x);\ x \in \mathcal{V}\}$, $\{(S_n^x(i))_{n\in\mathbb{N}};\ i \in \{1, 2, 3, \ldots\}, x \in \mathcal{V}\}$ and $\{(\Xi_p^x(i));\ i \in \{1, 2, 3, \ldots\}, x \in \mathcal{V}\}$ be independent sets of i.i.d. random variables defined as follows. For each $x \in \mathcal{V}$, $\eta(x)$ has the same law as η and gives the initial number of particles at site x. If $\eta(x) \geq 1$, then, for each $i \in \{1, \ldots, \eta(x)\}$, $(S_n^x(i))_{n\in\mathbb{N}}$ is a discrete-time SSRW on $\mathcal{G}$ starting from x (it describes the trajectory of the ith particle from x), and $\Xi_p^x(i)$, which denotes the lifetime of the ith particle at the site x, is a random variable whose law is given by $\mathrm{P}[\Xi_p^x(i) = k] = (1 - p)p^{k-1}$ for $k = 1, 2, \ldots$, where $p \in [0, 1]$ is a fixed parameter.

Thus, the ith particle at site x follows the SSRW $(S_n^x(i))_{n\in\mathbb{N}}$ and dies (disappears) $\Xi_p^x(i)$ units of time after being activated.

Observe that, from the moment the particle disappears, it is unable to activate other particles (as first we decide whether the particle survives, and only after the particle survives is it allowed to jump). Notice that there is no interaction between active particles, which means that each active particle moves independently of everything else. We denote by $\mathrm{FM}(\mathcal{G}, p, \eta)$ the frog model on the graph $\mathcal{G}$ with survival parameter p and initial configuration given by independent copies of η at each site of $\mathcal{G}$. We denote by $\mathbf{1}$ the case where $\eta = 1$ almost surely.

Let us consider the following definition.

Definition 1. A particular realization of the frog model *survives* if there is at least one active particle at every instant of time. Otherwise, we say that it *dies out*.

Now we observe that $\mathrm{P}[\mathrm{FM}(\mathcal{G}, p, \eta) \text{ survives}]$ is nondecreasing in p and define

$$p_c(\mathcal{G}, \eta) := \inf\{p : \mathrm{P}[\mathrm{FM}(\mathcal{G}, p, \eta) \text{ survives}] > 0\},$$

with the convention that $\inf \varnothing = 1$. As usual, we say that $\mathrm{FM}(\mathcal{G}, p, \eta)$ exhibits *phase transition* if

$$0 < p_c(\mathcal{G}, \eta) < 1.$$

Before going further, let us emphasize that in fact we are dealing with a percolation model. Indeed, let

$$\mathcal{R}_x^i = \{S_n^x(i) : 0 \leq n < \Xi_p^x(i)\} \subset \mathcal{G}$$

be the 'virtual' set of sites visited by the ith particle originally placed at x. The set $\mathcal{R}_x^i$ becomes 'real' in the case when x is actually visited (and thus all the sleeping particles from there are activated). We define the (virtual) range of site x by

$$\mathcal{R}_x := \begin{cases} \displaystyle\bigcup_{i=1}^{\eta(x)} \mathcal{R}_x^i & \text{if } \eta(x) > 0, \\ \{x\} & \text{if } \eta(x) = 0. \end{cases}$$

Notice that the frog model survives if and only if there exists an infinite sequence of distinct sites $\mathbf{0} = x_0, x_1, x_2, \ldots$ such that, for all j,

$$x_{j+1} \in \mathcal{R}_{x_j}. \tag{1}$$

The last observation shows that the extinction of the frog model is equivalent to the finiteness of the cluster of $\mathbf{0}$ in the following oriented percolation model: from each site x the oriented edges are drawn to all the sites of the set $\mathcal{R}_x$.

2. Nonmonotonicity of $p_c(\mathcal{G}, \eta)$ in $\mathcal{G}$

We show the nonmonotonicity of $p_c(\mathcal{G}, \eta)$ in $\mathcal{G}$ by presenting graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ such that $\mathcal{G}_1 \subset \mathcal{G}_2$ and $\mathcal{G}_3 \subset \mathcal{G}_4$. We show that, for these graphs, $p_c(\mathcal{G}_1, \mathbf{1}) > p_c(\mathcal{G}_2, \mathbf{1})$ yet $p_c(\mathcal{G}_3, \mathbf{1}) < p_c(\mathcal{G}_4, \mathbf{1})$. This answers a question posed by Alves *et al.* [1]. The nonmonotonicity may be considered an unexpected fact as the frog model is a percolation model. To be more specific, $\mathcal{G}_1 = \mathbb{T}_2$, $\mathcal{G}_2 = \mathbb{T}_{12}$ and $\mathcal{G}_3 = \mathbb{Z}^2$. The graph $\mathcal{G}_4$ is a little tricky to describe and is defined after the proof of the fact that $p_c(\mathbb{Z}^2, \mathbf{1}) < 1$.

2.1. $\mathcal{G}_1 \subset \mathcal{G}_2$ and $p_c(\mathcal{G}_2, \mathbf{1}) < p_c(\mathcal{G}_1, \mathbf{1})$

First of all let us show that $p_c(\mathbb{T}_2, \mathbf{1}) > p_c(\mathbb{T}_{12}, \mathbf{1})$. For this, by comparison with a Galton–Watson branching process, we present a lower and an upper bound to $p_c(\mathcal{G}, \eta)$.

Lemma 1. *Suppose that $\mathcal{G}$ is a graph of maximum degree k and η is such that* $\mathrm{E}\,\eta < \infty$. *Then*

$$p_c(\mathcal{G}, \eta) \geq \frac{k}{1 + (k-1)(\mathrm{E}\,\eta + 1)}.$$

Proof. Consider a Galton–Watson branching process where particles produce no offspring with probability $1 - p$, one offspring with probability p/k and the random number $\eta + 1$ of offspring with probability $p(k-1)/k$. Observing that every site with at least one active particle at time $n > 0$ has at least one neighbor site whose original particle or particles have been activated prior to time n, we find that the frog model is dominated by the Galton–Watson process just defined. An elementary calculation shows that, if $p < k(1 + (k-1)(\mathrm{E}\,\eta + 1))^{-1}$, then the mean offspring in the Galton–Watson process defined above is less than 1; therefore, it dies out almost surely. Consequently, the same happens to the frog model.

Lemma 2. *We have that*

$$p_c(\mathbb{T}_d, \mathbf{1}) \leq \frac{d+1}{2(d-1)}.$$

Proof. Let ξ_n be the set of active particles of $\mathrm{FM}(\mathbb{T}_d, p, \mathbf{1})$ which are at level n (i.e. at distance n from the root) at time n. Next we present a discrete-time supercritical Galton–Watson branching process, which is dominated by ξ_n. We do this by constructing an auxiliary process $\tilde{\xi}_n \subset \xi_n$. First of all, consider the particle at $\mathbf{0}$ belonging to $\tilde{\xi}_0$. In general, the process $\tilde{\xi}_n$ is constructed by the following rules. If at time $n-1$ the set of particles $\tilde{\xi}_{n-1}$ (which lives on the level $n-1$) is constructed, then at time n the set of particles $\tilde{\xi}_n$ (which are all at level n) is constructed in the following way. Introduce some ordering of the particles of $\tilde{\xi}_{n-1}$; they will be allowed to jump according to that order. Now, if the current particle survives, then

- if the particle jumps to some site of level n and does not encounter any particles that already belong to $\tilde{\xi}_n$ there, then this particle and the particle activated by it enter $\tilde{\xi}_n$;
- otherwise it is deleted.

The particles of $\tilde{\xi}_{n+1}$ activated by some particle from $\tilde{\xi}_n$ are considered to be the offspring of that particle; note that, due to the asynchronous construction of the process $\tilde{\xi}_n$, each particle has exactly one ancestor and, comparing with a Galton–Watson process, may have either zero or two offspring. So, it follows that the process $\tilde{\xi}_n$ dominates a Galton–Watson process with mean offspring being greater than or equal to

$$\frac{2(d-1)p}{d+1}.$$

From this we conclude that, if $p > (d+1)/2(d-2)$, the frog model survives on $\mathbb{T}_d$ with positive probability. This means that $p_c(\mathbb{T}_d) < (d+1)/2(d-2)$.

Now we are done with the first part of our task.

Corollary 1. *We have that*

$$p_c(\mathbb{T}_{12}, \mathbf{1}) < p_c(\mathbb{T}_2, \mathbf{1}).$$

Proof. The result follows from Lemmas 1 and 2. All we have to do is to notice that the upper bound presented by Lemma 2 for $\mathbb{T}_{12}$ is smaller than the lower bound presented by Lemma 1 for $\mathbb{T}_2$.

2.2. $\mathcal{G}_3 \subset \mathcal{G}_4$ but $p_c(\mathcal{G}_3, \mathbf{1}) < p_c(\mathcal{G}_4, \mathbf{1})$

In this subsection we present a graph $\mathcal{G}_4$ which contains $\mathbb{Z}^2$ but is such that $p_c(\mathbb{Z}^2, \mathbf{1}) < p_c(\mathcal{G}_4, \mathbf{1})$. The following theorem is proved for the more general condition of random initial configuration in [1]. To keep the paper self-contained, we present here a simpler proof which works well for the one-particle-per-site initial configuration.

Theorem 1. *We have that* $p_c(\mathbb{Z}^2, \mathbf{1}) < 1$.

Proof. For $n > 1$, let

$$f_n = \mathrm{P}[\{S_n^0 = 0\} \cap \{S_m^0 \neq 0 \text{ for all } m \in \{1, \ldots, n-1\}\}]$$

and let $A(k, p)$ be the event {the first particle to be woken returns to the origin at least k times before dying}. Notice that

$$\mathrm{P}[A(k, p)] = \left[\sum_{n=1}^{\infty} f_n p^n\right]^k$$

Since the SSRW is recurrent on $\mathbb{Z}^2$, we have that $\sum_{n=1}^{\infty} f_n = 1$. It follows that, for any fixed k,

$$\lim_{p \to 1} \mathrm{P}[A(k, p)] = 1.$$

Fix $k = 4r$, where $r \geq 1$. Then, by choosing p large enough, the probability that the first particle to be woken returns to the origin at least k times can be made arbitrarily large. After each instant the particle returns, it has a fixed positive probability of hitting the site $(0, 1)$ which is the same as that of hitting any other of its nearest neighbors. At each time $4i$, $i \geq 1$, after the particle returns, observe whether it hits the site $(1, 0)$ or not. Do the same for site $(0, 1)$ at time $4i + 1$, site $(-1, 0)$ at time $4i + 2$ and site $(0, -1)$ at time $4i + 3$. If r is large enough, the probability of hitting all nearest neighbors, given that the particle returns at least r times, can be made arbitrarily large. Denote this probability by $Q(r)$. Choose r so large that $Q(r) > \pi$, where π is the critical parameter for independent site percolation on $\mathbb{Z}_+^2$ (see [3]). Choose p so large that

$$P[A(2r, p)]Q(r) > \pi.$$

With this choice for p, the supercritical independent site percolation process is dominated by the set of active particles in the frog model. Therefore, with positive probability, the frog model survives on $\mathbb{Z}^2$.

Let us denote by $\mathcal{G}_4 = \mathcal{G}_4(n)$ the graph whose set of vertices is

$$\mathcal{V}(\mathcal{G}_4) = \bigcup_{x \in \mathbb{Z}^2} N_n(x),$$

where

$$N_n(x) = \{(x, 0), (x, 1), \dots, (x, n)\}.$$

The set of edges of $\mathcal{G}_4$ is such that, for any $x \in \mathbb{Z}^2$, the vertex (x, i) is a neighbor of (x, j) when $0 \le i < j \le n$. In addition, $(x, 0)$ is a neighbor of $(y, 0)$ if $\|x - y\|_1 = 1$.

We prove that $p_c(\mathbb{Z}^2, \mathbf{1}) < p_c(\mathcal{G}_4, \mathbf{1})$ by showing that $p_c(\mathcal{G}_4, \mathbf{1})$ can be made arbitrarily close to 1 by increasing the parameter n. This is done by dominating the model on $\mathcal{G}_4$ by a Galton–Watson branching process. With every active particle which leaves its original complete connected graph, we associate Y offspring, where Y is the random variable defined as follows: among the $n + 1$ particles which were in the complete connected graph to which the active particle jumps initially plus the jumping particle which activated it, Y is the number of particles which leave this graph. If two or more active particles jump to the same complete connected graph, we associate with one of them Y offspring and to each one of the others we associate independent copies of Y. As shown later, for any fixed value of p, as n increases, $E(Y)$ goes to zero. So it is eventually smaller than 1 which makes the Galton–Watson branching process, which dominates our model, die out with probability 1. Consequently, any p such that $p_c(\mathcal{G}_3, \mathbf{1}) < p < 1$ can be a strict lower bound to $p_c(\mathcal{G}_4, \mathbf{1})$ for a large value of n. As we also show later, the model on $\mathcal{G}_4$ has phase transition as well.

For a fixed n and $x \in \mathbb{Z}^2$, let $Y(x)$ be the random variable which counts the total number of particles originally from the complete connected graph connected to $(x, 0)$ whose virtual range includes some other site $y \in \mathbb{Z}^2$. Observe that, if the active particle from the origin in its first movement jumps to a site $(x, 0)$ for some $x \neq 0$, then $Y(0) \geq 1$. That event happens with probability $4p/(n + 4)$. In the event that the active particle initially from $(0, 0, 0)$ in its first movement jumps to a site $(0, 0, i)$ with $i \in \{1, \dots, n\}$, there will be two active particles at $(0, 0, i)$ at time 1.

Let us number the particles that are originally at the sites of the connected graph connected to the site $(x, 0)$ as $i = 1, \dots, n + 1$ and define

$$Y_i(x) = \begin{cases} 1 & \text{if } \mathcal{R}^1_{(x,i)} \text{ includes a site } (y, 0) \text{ such that } y \neq x, \\ 0 & \text{if not.} \end{cases}$$

Thus, $Y(x) = \sum_{i=1}^{n+1} Y_i(x)$. Now,

$$E\left(\sum_{i=1}^{n+1} Y_i(x)\right) = \sum_{i=1}^{n+1} E(Y_i(x)) = \sum_{i=1}^{n+1} P(Y_i(x) = 1)$$

and

$$P(Y_i(x) = 1) = \sum_{k=1}^{\infty} P(A_k^i),$$

where A_k^i is the event {it takes k steps for the ith particle to visit a neighboring complete connected graph}. Moreover, for $i = 2, \dots, n+1$,

$$P(A_1^i) = 0,$$

$$P(A_2^i) = \frac{4p^2}{n(n+4)},$$

$$P(A_3^i) = \frac{p(n-1)}{n} P(A_2^i)$$

and, for $k \geq 4$,

$$P(A_k^i) = \frac{p(n-1)}{n} P(A_{k-1}^i) + \frac{p}{n}\frac{p(n-1)}{n} P(A_{k-2}^i).$$

Thus,

$$\sum_{k=2}^{\infty} P(A_k^i) = P(A_2^i) + \frac{p(n-1)}{n} \sum_{k=2}^{\infty} P(A_k^i) + \frac{p^2(n-1)}{n^2} \sum_{k=2}^{\infty} P(A_k^i),$$

and so

$$\sum_{k=2}^{\infty} P(A_k^i) = \frac{4p^2}{n(n+4)[1 - p(n-1)/n - p^2(n-1)/n^2]}.$$

Now observe that

$$P(Y_1(x) = 1) = \frac{4p}{n+4} + \frac{np}{n+4} P(Y_2(x) = 1)$$

and finally we have that

$$E(Y(x)) \leq \frac{4p}{n+4} + \frac{4p^2(n+1)}{n(n+4)[1 - p(n-1)/n - p^2(n-1)/n^2]}. \tag{2}$$

Clearly, the right-hand side of (2) goes to 0 as n grows to infinity. To estimate the average number of offspring of the Galton–Watson branching process that we used, assume that there are two active particles at $\mathbb{Z}^2$ level in the complete connected graph connected to a fixed site $(x, 0)$. That is easy once we know that (2) holds.

Finally, we briefly argue that $p_c(\mathcal{G}_4, 1) < 1$. For this, define q to be the probability that a particle which is at $\mathbb{Z}^2$ level leaves its complete connected graph. That probability can be made arbitrarily close to 1 by making p close to 1. Now associate the event that a fixed particle initially in the complete connected graph of a given site of $\mathbb{Z}^2$ to the event that in $\mathrm{FM}(\mathbb{Z}^2, q, 1)$ the particle initially at that site survives and jumps from that fixed site in $\mathrm{FM}(\mathbb{Z}^2, q, 1)$. As seen in Theorem 1, that model exhibits phase transition.

3. Strict monotonicity for critical probability

Alves *et al.* [1] proved the following theorem which gives asymptotic values for critical parameters for the case of $\mathbb{Z}^d$ and regular trees.

Theorem 2. *We have*

$$\lim_{d\to\infty} p_c(\mathbb{T}_d, \eta) = \lim_{d\to\infty} p_c(\mathbb{Z}^d, \eta) = \tfrac{1}{2}.$$

Theorem 2 suggests that there is monotonicity of the critical probability in the dimension for regular classes of graphs (such as $\mathbb{Z}^d$ or $\mathbb{T}_d$ for $d \geq 1$). The example we have given makes clear that this is not true in general. A natural question to ask is whether it is true that $p_c(\mathbb{Z}^d, \eta) \geq p_c(\mathbb{Z}^{d+1}, \eta)$ for all d or at least for d large enough. In addition to that, can we replace '$\geq$' with '$>$'? What about the case of $\mathbb{T}_d$?

In fact, a more general question would be on necessary and sufficient conditions for a class of graphs such that, whenever $\mathcal{G}_1 \subset \mathcal{G}_2$ in that class, is it true that $p_c(\mathcal{G}_1, \mathbf{1}) \geq p_c(\mathcal{G}_2, \mathbf{1})$, as happens for the usual percolation models. Observe that, even though this model can be seen as a percolation model (see (1)), there is no natural way of coupling the constructions of the clusters of fixed sites for the process on $\mathbb{Z}^d$ and $\mathbb{Z}^{d+1}$ as can be done in independent percolation (e.g. [2]).

Finally, observe that the graph $\mathcal{G}_4$ defined in this paper is not an enhancement of $\mathbb{Z}^2$, so the critical probabilities for independent percolation on these two graphs are the same. For more details, see [4]. In order to distinguish them, consider another graph $\mathcal{G}_5$ made up from two copies of $\mathbb{Z}^2$ where each site (x, y) of the first copy is connected to each site of a complete connected graph as in $\mathcal{G}_4$. Moreover, each site of that complete connected graph is also connected to the site (x, y) of the second copy. From the computation above, we can see that, while for the frog model $p_c(\mathbb{Z}^2, \mathbf{1}) < p_c(\mathcal{G}_5, \mathbf{1})$, on the contrary (again from [4]) for independent percolation $p_c(\mathcal{G}_5) < p_c(\mathbb{Z}^2)$

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