Sankhyā : The Indian Journal of Statistics
1984, Volume 46, Series A, Pt. 1, pp. 24–34.

ESTIMATION OF FIRST CROSSING TIME DISTRIBUTION FOR BROWNIAN MOTION PROCESSES RELATIVE TO UPPER CLASS BOUNDARIES

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SUMMARY. $(X_t, t \geqslant 0)$ denotes a standard Brownian motion process. A nonnegative non-increasing function ϕ defined for small arguments and such that $\sqrt{t}\phi(t)$ is nondecreasing is said to be upper class at 0 if almost all sample paths exceed $\phi(t)$ only finitely often as $t \to 0$. The Kolmogorov integral test provides a necessary and sufficient condition. Here the related first-crossing time is considered and estimates for its distribution function near 0 are obtained, the estimate depending on the integral in the Kolmogorov test.

1. Introduction

Throughout the paper $\{X_t : t \geqslant 0\}$ will denote a standard Brownian motion process. Let ϕ be a nonnegative non-increasing function defined for small arguments. We say $\phi \in F$ if the function

$$g(t) = t^{1/2}\phi(t)$$

is non-decreasing near 0. $\phi \in F$ is said to be upper class at 0 ($\phi \in \mathcal{U}_0$) if for almost all sample paths S there is $\epsilon(S) > 0$ such that for all $t < \epsilon$

$$X_t < g(t).$$

Otherwise we say ϕ is lower class at 0. ($\phi \in \mathcal{L}_0$).

A necessary and sufficient condition for $\phi \in F$ being upper class at 0 is furnished by the Kolmogorov-Petrovski-Erdös-Feller (KPEF) integral test (See Sirao and Nisida, 1952)

$$\int_{0+} \exp(-\tfrac{1}{2}\phi^2(t))\phi(t)t^{-1}dt < \infty. \qquad \ldots \quad (1)$$

Khinchin's weak law of the iterated logarithm (LIL) follows directly from this.

We define the first crossing time T_g by

$$T_g = \inf\{t : X_t \geqslant g(t)\}. \qquad \ldots \quad (2)$$

AMS (1980) *subject classification* : Primary 60G 15.

Key words and phrases : Simple Brownian motion process, upper class function, first crossing time.

We are interested in the distribution of T_g. For $\tau > 0$ let us write

$$p_\tau = P(T_g \leqslant \tau) = P(X_t \geqslant g(t) \text{ for some } t \leqslant \tau) \qquad \ldots \quad (3)$$

$$I_g(\tau) = (2\pi)^{-1/2} \int_{0+}^{\tau} \exp(-\tfrac{1}{2}\phi^2(t))\phi(t)t^{-1}dt. \qquad \ldots \quad (4)$$

If ϕ is lower class at 0 then T_g has a degenerate distribution with all the mass at 0. On the other hand, if $\phi \in \mathcal{U}_0$ we shall show that the integrand in (1) acts like a pseudo-density for T_g in the sense of the following theorem :

Theorem 1 : *Let $\phi \in \mathcal{U}_0$. Then as $\tau \to 0$*

$$\cdot 0907(1+o(1))I_g(\tau) \leqslant p_\tau \leqslant e(1+o(1))I_g(\tau).$$

The method used consists of elementary quantification of the Borel-Cantelli arguments used in standard LIL-type results, involving the first and the second Bonferroni inequalities. The idea is that slight modification of the standard methods not only gives us the KPEF test but also provides an interpretation of the integral in the test in terms of the distribution of the first crossing time. It is to be noted here that the constants ·0907 and e in the theorem are artifacts of the methods used and we conjecture that (perhaps with a slight modification of the integrand in (1)) we can make the two bounds coincide at some intermediate value. In fact Strassen (1967) has done this using entirely different methods and under stronger assumptions. Our methods are quite general and have been used in Sen (1981) to obtain analogues for Brownian sheets on R and $R^d (d \geqslant 1)$.

2. Probability estimates

The following results will be used in the proof of the theorem. Lemma 1 is well-known. Lemmas 2, 3 and 4 are refinements of results due to Chung, Erdös and Sirao (1959); the fluctuation inequality, Lemma 5, is well-known (see Billingsley, 1979).

Lemma 1 : (Tail probability lemma) : *Let $U \sim N(0, 1)$ and $\lambda > 0$. Then*

$$(2\pi)^{-1/2}\lambda^{-1}(1+o(1))exp(-\tfrac{1}{2}\lambda^2) \leqslant P(U \geqslant \lambda) \leqslant (2\pi)^{-1/2}\lambda^{-1}exp(-\tfrac{1}{2}\lambda^2).$$

Proof : Use integration by parts.

Lemma 2 : (small ρ lemma) : *Let*

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

Then for each $\delta > 0$,

$$\lim_{a\to\infty} \sup_{a<b,\, \rho ab \leqslant \delta,\, \rho \geqslant 0} \frac{P(V \geqslant b \mid U \geqslant a)}{P(V \geqslant b)} \leqslant e^{2\delta}.$$

Proof: For $a < b$

$$P(U \geqslant a, V \geqslant b) = P(U \geqslant a, b \leqslant V < 2b) + P(U \geqslant a, V \geqslant 2b)$$
$$\leqslant P(U \geqslant a, b \leqslant V < 2b) + P(V \geqslant 2b).$$

For b large, Lemma 1 gives

$$P(V \geqslant 2b) \leqslant (2\pi)^{-\frac{1}{2}}(2b)^{-1}e^{-2b^2}$$
$$= ((2\pi)^{-1/2}b^{-1}(1+o(1))e^{-\frac{1}{2}b^2})^2(2\pi)^{1/2}(2b)^{-1}b^2e^{-b^2}$$
$$\leqslant o(1)(P(V \geqslant b))^2$$
$$\leqslant o(1)P(U \geqslant a)P(V \geqslant b).$$

For $a > 2\rho b$,

$$P(U \geqslant a, b \leqslant V < 2b)$$
$$= \int_{b \leqslant v < 2b} P(U \geqslant a \mid V = v)P(V \in dv)$$
$$= \int_{b \leqslant v < 2b} P(\rho v + (1-\rho^2)^{1/2}U \geqslant a)P(V \in dv)$$
$$\leqslant P(U \geqslant (a-2\rho b)(1-\rho^2)^{-1/2})P(b \leqslant V < 2b)$$
$$\leqslant P(U \geqslant a-2\rho b)P(V \geqslant b).$$

By Lemma 1

$$P(U \geqslant a-2\rho b) \leqslant (2\pi)^{-1/2}(a-2\rho b)^{-1}\exp(-\tfrac{1}{2}(a-2\rho b)^2)$$
$$\leqslant (2\pi)^{-1/2}a^{-1}e^{-\frac{1}{2}a^2}\left(1-\frac{2\rho ab}{a^2}\right)^{-1}e^{2\rho ab}$$
$$= (1+o(1))P(U \geqslant a)\, e^{2\delta}$$

for $\rho ab \leqslant \delta$ and a large. □

Lemma 3: (Moderate ρ lemma): *Let*

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

Then for each $0 < r < 1$, there exists a constant $\alpha = \alpha(r)$ such that

$$P(V \geqslant b \mid U \geqslant a) \leqslant exp\Big(-\frac{1}{12}b^2(1-\rho^2)\Big)$$

for $b \geqslant a \geqslant \alpha$ and $0 \leqslant \rho \leqslant r$.

Proof: For $0 \leqslant a \leqslant b$ and $\delta \equiv c(1-\rho^2) > 0$,

$$P(U \geqslant a, V \geqslant b) = P(a \leqslant U < b(1+\delta), V \geqslant b) + P(U \geqslant b(1+\delta), V \geqslant b)$$

$$\leqslant \int_{a \leqslant u < b(1+\delta)} P(V \geqslant b \mid U = u) P(U \in du) + P(U \geqslant b(1+\delta))$$

$$= \mathrm{I}+\mathrm{II}. \quad \ldots \quad (5)$$

Now for b large,

$$\mathrm{II} \leqslant b^{-1}(1+\delta)^{-1}(2\pi)^{-1/2}\exp(-\tfrac{1}{2}b^2(1+\delta)^2)$$

$$\leqslant (2\pi)^{-1/2}b^{-1}\exp(-\tfrac{1}{2}b^2)(1+\delta)^{-1}\exp(-\delta b^2)$$

$$\leqslant (1+o(1))P(U \geqslant a)(1+\delta)^{-1}\exp(-c(1-\rho^2)b^2) \quad \ldots \quad (6)$$

and, provided

$$p \equiv b(1-\rho(1+\delta))(1-\rho^2)^{-1/2} \equiv b\gamma$$

is positive,

$$\mathrm{I} = \int_{a \leqslant u < b(1+\delta)} P(V > (b-\rho u)(1-\rho^2)^{-1/2})P(U \in du)$$

$$\leqslant P(U \geqslant a)P(V \geqslant p) \leqslant P(U \geqslant a)(2\pi)^{-1/2}p^{-1}e^{-\frac{1}{2}p^2}$$

$$\leqslant P(U \geqslant a)(2\pi)^{-1/2}p^{-1}\exp\Big(-\tfrac{1}{2}b^2\Big(\frac{(1-\rho)^2}{1-\rho^2}-\frac{2\delta\rho(1-\rho)}{1-\rho^2}\Big)\Big)$$

$$= P(U \geqslant a)(2\pi)^{-1/2}b^{-1}\gamma^{-1}\exp\Big(-\tfrac{1}{2}b^2(1-\rho^2)\Big((1+\rho)^{-2}-\frac{2\rho c}{1+\rho}\Big)\Big).$$

$$\ldots \quad (7)$$

Now choose c so that

$$\tfrac{1}{2}((1+\rho)^{-2}-2\rho c(1+\rho)^{-1}) = c,$$

i.e.,

$$c = \tfrac{1}{2}(1+\rho)^{-1}(1+2\rho)^{-1}.$$

Note that

$$c \geqslant (2\times 2\times 3)^{-1} = 1/12 \quad (8)$$

and

$$\gamma = \frac{1-\rho-\rho\delta}{(1-\rho^2)^{1/2}} = \sqrt{\frac{1-\rho}{1+\rho}} - c\rho(1-\rho^2)^{1/2}$$

$$= (1-\rho^2)^{1/2}\frac{2+3\rho}{2(1+\rho)(1+2\rho)}$$

is positive. In fact, γ is bounded away from 0 for ρ bounded away from 1. So for $0 \leqslant \rho < r < 1$, the lemma follows from (5), (6), (7) and (8). □

Lemma 4 : (Large ρ lemma) : *Let*

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

Then for $0 < a \leqslant b < \infty$ *and* $0 \leqslant \rho < 1$,

$$P(V \geqslant b \mid U \geqslant a) \leqslant 4(2\pi)^{-1/2}(a\sqrt{1-\rho^2})^{-1}exp(-a^2(1-\rho^2)/8).$$

Proof: For $b \geqslant a$,

$$\begin{aligned} P(U \geqslant a,\ V \geqslant b) &\leqslant P(U \geqslant a,\ V \geqslant a) \\ &= 2P(a \leqslant U \leqslant V) \\ &= 2\int_{u \geqslant a} P(U \in du)P(u\rho+(1-\rho^2)^{1/2}V \geqslant u) \\ &\leqslant 2\int_{u \geqslant a} P(U \in du)P\left(V \geqslant \frac{u(1-\rho)}{(1-\rho^2)^{1/2}}\right) \\ &\leqslant 2\int_{u \geqslant a} P(U \in du)P(V \geqslant \tfrac{1}{2}a(1-\rho^2)^{1/2}) \\ &\leqslant 2\,P(U \geqslant a)(2\pi)^{-1/2}\frac{2}{a(1-\rho^2)^{1/2}}\exp(-a^2(1-\rho^2)/8). \end{aligned}$$ □

Lemma 5 : (Fluctuation equality) : *Let* X_t *be a standard Brownian motion process and* $t \geqslant 0$. *Then*

$$P\left(\sup_{s \leqslant t} X_s \geqslant a\right) = 2P(X_t \geqslant a).$$

3. Proof of the theorem

For the sake of better comprehension and organization the proof will be divided into several parts. First, we shall define a sequence $\{u_k\}$ to discretize time. Then the upper bound half of the theorem will be proved using Lemmas 1 and 5. Finally Lemmas 1, 2, 3 and 4 will be used to prove the lower bound half.

3.1. *The sequence* $\{u_k\}$. Given $\tau > 0$ and $0 < \alpha < \phi^2(\tau)$ we define u_k by induction as follows :

$$u_0 = \tau,\ u_{k+1} = u_k(1-\alpha/\phi^2(u_k)), \quad k \geqslant 0.$$

Lemma 6 : (properties of $\{u_k\}$) : *For* $\phi \in \mathcal{U}_0$ *and* $\{u_k\}$ *as above the following are true* :

(A) $u_k \downarrow 0$ *as* $k \to \infty$.

(B) $\phi(u_k) \uparrow \infty$ *as* $k \to \infty$.

(C) $u_{k+1} = u_k(1+o(1))$.

(D) $u_k - u_{k+1} \downarrow 0$ *as* $k \to \infty$.

(E) $\phi(u_{k+1}) = \phi(u_k)(1+o(1))$.

(F) $\phi^2(u_{k+1})(1-u_{k+1}/u_k) = \alpha(1+o(1))$.

(G) *For* $0 < f < 1$ *and* $fu_k \leqslant u_l < u_k$, $\phi^2(u_k)(1-u_l/u_k) \geqslant f^2\alpha(l-k)$.

(H) $\phi^2(u_{k+1}) - \phi^2(u_k) \leqslant 2\alpha(1+o(1))$.

Note. In this lemma, as in what follows, the $o(1)$ terms are to be interpreted in the sense "uniformly in k as $\tau \to 0$."

Proof : Facts (A) through (F) are straightforward consequences of the definitions and the fact that $\phi \in F$.

(G)
$$\begin{aligned}\phi^2(u_k)(1-u_l/u_k) &= \phi^2(u_k)(u_k-u_l)/u_k \\ &\geqslant \phi^2(u_k)(u_k-u_l)f/u_{l-1} \\ &\geqslant \phi^2(u_k)(l-k)(u_{l-1}-u_l)f/u_{l-1} \\ &= \phi^2(u_k)(l-k)f\alpha/\phi^2(u_{l-1}) \\ &\geqslant f\alpha(l-k)u_{l-1}/u_k \\ &\geqslant f^2\alpha(l-k).\end{aligned}$$

(H)
$$\begin{aligned}\phi^2(u_{k+1}) - \phi^2(u_k) &= (\phi(u_{k+1})+\phi(u_k))(\phi(u_{k+1})-\phi(u_k)) \\ &\leqslant 2\phi(u_{k+1})\phi(u_{k+1})(1-\phi(u_k)/\phi(u_{k+1})) \\ &\leqslant 2\phi^2(u_{k+1})(1-u_{k+1}^{1/2}/u_k^{1/2}) \\ &\leqslant 2\phi^2(u_{k+1})(1-u_{k+1}/u_k) \\ &= 2\phi^2(u_{k+1})\alpha/\phi^2(u_k) \\ &\leqslant 2\alpha(1+o(1)). \quad \square\end{aligned}$$

3.2. *The upper bound.* Using Lemmas 1, 5 and 6(F) and also the fact that g is nondecreasing,

$$p_\tau = P(X_t > g(t) \text{ for some } t \leqslant \tau)$$

$$\leqslant \sum_{k=0}^{\infty} P\left(\sup_{u_{k+1} \leqslant t < u_k} X_t > g(u_{k+1})\right)$$

$$\leqslant 2 \sum_{k=0}^{\infty} P(X_{u_k} > g(u_{k+1}))$$

$$\leqslant 2(2\pi)^{-1/2} \sum_{k=0}^{\infty} \exp\left(-\frac{1}{2}\frac{u_{k+1}}{u_k}\phi^2(u_{k+1})\right) \frac{\sqrt{u_k}}{\sqrt{u_{k+1}}\phi(u_{k+1})}$$

$$= 2(2\pi)^{-1/2} \sum_{k=0}^{\infty} \exp\left(-\frac{1}{2}\phi^2(u_{k+1})\right) \exp\left(\frac{1}{2}\phi^2(u_{k+1})(1-u_{k+1}/u_k)\right)$$

$$\times (u_k/u_{k+1})^{1/2}(\phi(u_{k+1}))^{-1}(u_k - u_{k+1})\frac{\phi^2(u_k)}{\alpha u_k}$$

$$\leqslant 2(1+o(1))(2\pi)^{-1/2}e^{\alpha/2}\alpha^{-1}\int_{0+}^{\tau}\exp\left(-\frac{1}{2}\phi^2(t)\right)(\phi(t))^{-1}\phi^2(t)t^{-1}dt$$

$$= (1+o(1))e^{\alpha/2}\left(\frac{1}{2}\alpha\right)^{-1} I_g(\tau).$$

Now, since

$$\inf_{\alpha > 0} e^{\alpha/2}\left(\frac{1}{2}\alpha\right)^{-1} = e.$$

the upper bound half of the theorem follows.

3.3. *The lower bound.* Let

$$A_k = \{X_{u_k} > g(u_k)\}.$$

Then by the second Bonferroni inequality,

$$p_\tau > P\left(\bigcup_{k=0}^{\infty} A_k\right)$$

$$\geqslant \sum_{k=0}^{\infty} P(A_k)\left(1 - \sum_{l > k} P(A_l \mid A_k)\right). \qquad \text{... (9)}$$

Now by Lemma 1 and Lemma 6 (C) and (E)

$$\sum_{k=0}^{\infty} P(A_k)$$

$$\geqslant (1+o(1))(2\pi)^{-1/2} \sum_{k=0}^{\infty} \exp\left(-\frac{1}{2}\phi^2(u_k)\right)(\phi(u_k))^{-1}$$

$$= (1+o(1))(2\pi)^{-1/2} \sum_{k=0}^{\infty} \exp\left(-\frac{1}{2}\phi^2(u_k)\right)(\phi(u_k))^{-1}(u_k-u_{k+1})\frac{\phi^2(u_k)}{\alpha u_k}$$

$$\geqslant (1+o(1))\alpha^{-1}I_\phi(\tau). \quad \ldots (10)$$

Set

$$\rho = \rho_{kl} = \mathrm{cor}(X_{u_k}, X_{u_l}) = \frac{\min(u_k, u_l)}{\sqrt{u_k}\sqrt{u_l}} = (u_l/u_k)^{1/2}$$

for $l > k$. To estimate terms like $P(A_l \mid A_k)$ in (9) we use Lemmas 2, 3 and 4 depending on the size of ρ.

3.3.1. *Low dependence.* By Lemma 2, for τ small and

$$(u_l/u_k)^{1/2}\phi(u_k)\phi(u_l) \leqslant 1 \quad \ldots (11)$$

we have

$$P(A_l \mid A_k) \leqslant 8P(A_l) \quad \ldots (12)$$

we shall call the set of all l's satisfying (11) the low dependence range R_L (for fixed k). By Lemmas 1 and 6(H)

$$\sum_{k=0}^{\infty} P(A_k) \leqslant (2\pi)^{-1/2} \sum_{k=0}^{\infty} (\phi(u_k))^{-1} \exp\left(-\frac{1}{2}\phi^2(u_k)\right)$$

$$\leqslant (2\pi)^{-1/2}e^{a}(1+o(1)) \sum_{k=0}^{\infty} (\phi(u_{k+1}))^{-1}$$

$$\times \exp\left(-\frac{1}{2}\phi^2(u_{k+1})\right) \times (u_k-u_{k+1})\frac{\phi^2(u_k)}{\alpha u_k}$$

$$\leqslant (1+o(1))\alpha^{-1}e^{a}I_\phi(\tau).$$

From this follows

$$\sum_{l>k} P(A_l) = o(1).$$

Hence

$$\sum_{l \in R_L} P(A_l \mid A_k) = o(1). \quad \ldots (13)$$

3.3.2. *Moderate dependence.* Let $0 < f < 1$. We denote by R_M, the moderate dependence range, the set of l's for which (11) does not hold and $u_l < fu_k$. For $l \in R_M$, $\rho^2 < f < 1$, so Lemma 3 applies and we have for τ small a constraut c such that

$$P(A_l \mid A_k) \leqslant \exp(-c\phi^2(u_l)) \leqslant \exp(-c\phi^2(u_k)).$$

Hence

$$\sum_{l \in R_M} P(A_l \mid A_k) \leqslant \sum_{l \in R_M} \exp(-c\phi^2(u_l))(u_{l-1}-u_l) \frac{\phi^2(u_{l-1})}{cu_{l-1}}$$

$$\leqslant \exp(-c\phi^2(u_k))+\alpha^{-1} \int_{\{t < fu_k,\ \sqrt{t}\phi(t) < \sqrt{u_k}/\phi(u_k)\}} \exp(-c\phi^2(t))\phi^2(t)t^{-1}dt$$

$$= o(1) \qquad \ldots \quad (14)$$

by Lemma 7 below.

Lemma 7 : *Let ϕ be nonnegative and nonincreasing near 0 and such that $\sqrt{t}\phi(t)$ is nondecreasing near 0. Suppose $\phi(t) \to \infty$ as $t \to 0$. Let $d > 0$. Then*

$$W = W_r(d) = \int_{\{t \leqslant r,\ \sqrt{t}\phi(t) > \sqrt{r}/\phi(r)\}} exp(-d\phi^2(t))\phi^2(t)t^{-1}dt = o(1)$$

uniformly in $0 < r \leqslant \tau$ as $\tau \to 0$.

Proof: Choose and fix θ, $0 < \theta < 1$ and let

$$T_k = \{t : t \leqslant r,\ a_k \leqslant t\phi^2(t) < b_k\}, \quad k = 0, 1, 2, \ldots, n,$$

where

$$a_k = \theta^{k+1} r\phi^2(r),\ b_k = \theta^{-1}a_k$$

and n is such that

$$a_n \leqslant r/\phi^2(r) < b_n.$$

Fix k for the moment and write $a = a_k$, $b = b_k$. Since ϕ is continuous,

$$\mu = \mu_k = \sup\{t : t \in T_k\}$$

belongs to T_k if T_k is nonempty. Also

$$b\mu^{-1}\phi^2(\mu) \geqslant \phi^2(r).$$

Now

$$W(T_k) = \int_{T_k} \exp(-d\phi^2(t))\phi^2(t)t^{-1}dt$$

$$\leqslant \int_{0^+}^{\mu} e^{-da/t}\, bt^{-2}dt$$

$$= b\int_{1/\mu}^{\infty} e^{-dau}\, du$$

$$= \frac{be^{-ad/\mu}}{ad}$$

$$\leqslant (\theta d)^{-1}\, e^{-\theta\phi^2(\tau)d}.$$

Hence

$$W \leqslant \sum_{k=0}^{n} W(T_k)$$

$$\leqslant (\theta d)^{-1}\,(n+1)\, e^{-\theta d\phi^2(\tau)}.$$

The lemma now follows, since

$$n \leqslant \frac{\log \phi^4(\tau)}{\log(1/\theta)}. \quad \square$$

3.3.3. *High dependence.* It remains to consider l in the high dependence range R_H, i.e., the set of l's for which (11) does not hold and

$$fu_k \leqslant u_l < u_k.$$

Using Lemmas 4 and 6(G),

$$\sum_{l\in R_H} P(A_l\,|\,A_k) \leqslant \sum_{l\in R_H} 4(2\pi)^{-1/2}(\phi^2(u_k)(1-u_l/u_k))^{-1/2}.$$

$$\times \exp(-\phi^2(u_k)(1-u_l/u_k)/8)$$

$$\leqslant 4(2\pi)^{-1/2}\sum_{l>k}(f^2\alpha(l-k))^{-1/2}\exp(-f^2\alpha(l-k)/8)$$

$$= 4(2\pi)^{-1/2}f^{-1}\alpha^{-1/2}\sum_{h>0} h^{-1/2}\exp(-f\alpha h/8). \quad \dots \quad (15)$$

3.4. *Coup de grace.* To get the final result we combine (9), (10), (13), (14) and (15), make $\tau \to 0$ and finally $f \to 1$ to obtain

$$p_\tau \geqslant (1+o(1))x^{-1}I_\theta(\tau)(1-4(2\pi)^{-1/2}x^{-1/2}P_\alpha)$$

for any $\alpha > 0$, where

$$P_\alpha = \sum_{h=1}^{\infty} h^{-1/2}\exp(-\alpha h/8). \quad \dots \quad (16)$$

Numerical computations show that

$$\alpha^{-1}(1-4(2\pi)^{-1/2}\alpha^{-1/2}P_{\alpha})$$

has its maximum of ·0907 near $\alpha = 7{\cdot}10$. This completes the proof.

Remark: An analogous analysis can be carried out for behavior of the sample paths near ∞ where we consider the last crossing time and a corresponding definition of upper class boundaries. The analogue to Theorem 1 is immediate, with the same constants, using the so called time-inversion: if X_t is a standard Brownian motion process, so is Y_t defined by

$$Y_t = \begin{cases} t X_{1/t} & t > 0 \\ 0 & t = 0. \end{cases}$$

For details see Sen (1981).

Acknowledgement. The results in this paper form part of the first author's doctoral work at the Department of Statistics, University of Chicago. The authors wish to thank the U.S. National Science Foundation for their financial support and the referee for his suggestions.

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Paper received: August, 1981.

Revised: June, 1983.