KEIO ECONOMIC STUDIES 34(2), 17–31 (1997)

RELATIVE DEPRIVATION AND SATISFACTION ORDERINGS

Satya R. CHAKRAVARTY

Indian Statistical Institute, Calcutta, India

First version received September 1995; final version accepted May 1997

Abstract: The concern of this paper is the relationship between social welfare and relative deprivation/satisfaction. We first identify the class of social welfare functions whose orderings of alternative distributions of a fixed total income agree with the rankings generated by non-intersecting relative deprivation/satisfaction curves. We then show that for the variable mean income case the welfare ordering can be implemented by seeking a dominance relation in terms of the generalized satisfaction curve. Finally, the existence of a relationship between a summary measure of deprivation and a social welfare function is demonstrated.

JEL Classification Numbers: D31, D63.

Key-words: Relative deprivation, satisfaction, social welfare, ethical index.

1. INTRODUCTION

A person's feeling of relative deprivation in a society arises out of the comparison of his situation with those of better off persons. Runciman (1996) used the example of promotion to illustrate an individual's feeling of relative deprivation. In Runciman's view the extent of deprivation felt by an individual for not being promoted is an increasing function of the number of persons who have been promoted. Yitzhaki (1979) considered relative deprivation in terms of income and showed that in the Runciman framework one plausible index of average relative deprivation in a society is the product of the mean income and the Gini index for the society.

Hey and Lambert (1980) provided an alternative derivation of Yitzhaki's result. Their derivation is based on the following remark of Runciman (1966): 'The magnitude of a relative deprivation is the extent of the difference between the desired situation and that of the persons desiring it' (op. cit., p. 10). Interesting variations and generalizations of the Yitzhaki index have also been proposed.

Acknowledgements. For comments and suggestions, I am grateful to an anonymous referee and H. N. Roy.

For encouraging discussions, I wish to thank the participants of the Third International Conference of the Society for Social Choice and Welfare, Maastrict, The Netherlands, June 22–25, 1996; and the seminar participants at the University of Karlsruhe and the University of Kiel, Germany; and also at the Kagawa University, Japan.

(See, for example, Chakravarty and Chakraborty, 1984; Berrebi and Silber, 1985; Chakravarty, 1990, 1995; Paul, 1991 and Chakravarty and Chattopadhyay, 1994).

The Hey-Lambert demonstration is quite similar to Sen's (1973) interpretation of the Gini index. According to Sen, in any pairwise comparison, the person with lower income will suffer from depression upon discovering that his income is lower. The average of all such depressions in all pairwise comparisons turns out to be the Gini coefficient if the extent of depression is proportional to the differences in incomes. Kakwani (1980) proved that if a person's depression is proportional to the square of income differences, the resulting index of average deprivation becomes the coefficient of variation. Kakwani (1984) defined the plot of the sum of income share shortfalls of different individuals from richer individuals against the cumulative proportions of persons as the Relative Deprivation Curve (RDC) and showed that the area under this curve is the Gini coefficient for the society.

It may be interesting to note that the interpretation of relative deprivation in terms of income differences is formally equivalent to the Temkin (1986) approach to inequality measurement. According to Temkin a person has a complaint if he has income less than others and inequality can be formulated in terms of such complaints. For any person in the society the greater is the gap between his income and incomes of those richer than him, the greater is his complaint. Likewise, the greater is the number of persons richer than him, the greater is his complaint. Societal inequality is defined as an increasing function of the total number and size of complaints of different individuals in the society. 'This idea of aggregate complaint differs from the conventional approach in at least one important respect.... We may not be able to deal with just a set of pure income transfers, as in the conventional approach' (Amiel and Cowell, 1994, p. 5). In fact, it has been demonstrated that the relative deprivation ordering implies but is not implied by the Loenz ordering. More precisely, given any two income distributions x and y of a fixed total over a fixed population size, if the RDC of x dominates that of y (that is, the RDC of x lies nowhere below that of y), then y Lorenz dominates x. But the converse is not true. (See Chakravarty, Chattopadhyay and Majumder, 1995.)

Now, it is well-known that of two income distributions x and y of a fixed total over a given population size, if y Lorenz dominates x, then y is regarded as at least as good as x by any equity oriented (in the sense of being S-concave) Social Welfare Function (SWF). The converse is also true (Dasgupta, Sen and Starrett, 1973). Since the Lorenz ordering does not imply the deprivation ordering, S-concavity of an SWF cannot be the property which will ensure that a less deprived distribution is not socially worse than a more deprived one. Given that relative deprivation is an alternative way of looking at inequality, it seems worthwhile to isolate the class of SWFs that will rank income distributions in exactly opposite way as the deprivation ordering does. This is one purpose of this paper. We refer to the identified SWFs as rational. Clearly, rationality is stronger than S-concavity, this property implies S-concavity but the converse is not ture.

We then consider the problem of ranking income distributions with variable total incomes. In this case in addition to satisfying rationality, the SWF has to exhibit a clear preference for higher efficiency also (that is, preference for higher total income). For this purpose following Shorrocks (1983) we consider three different notions of efficiency preference. It may be important to note that in the variable total income case the welfare orderings rely on the generalized satisfaction curve which is obtained by scaling up the relative satisfaction curve by the mean income. (The relative satisfaction curve is generated by taking complement to unity of the RDC.) The results demonstrated along this line, as well as the one developed in the context of fixed total income, can be extended for comparing distributions over variable population sizes also. Finally, we give an example to show how a Runciman (1966)–Kakwani (1984)–Temkin (1986) type deprivation index can be related to a rational SWF in a negative monotonic way.

The paper is organized as follows. Section 2 discusses the relative deprivation ordering. Section 3 presents the main results and finally section 4 concludes.

2. THE RELATIVE DEPRIVATION AND SATISFACTION ORDERINGS

For a population of size n, the set of income distributions is denoted by D^n with a typical element $x=(x_1, x_2, \cdots x_n)$, where D^n is the non-negative orthant of the Euclidean n-space R^n with the origin deleted. The set of all possible income distributions is $D=\bigcup_{n\in N} D^n$, where N is the set of positive integers. Note that by deleting origin from the domain D^n, $n\in N$, we ensure that there is at least one person with positive income. Throughout the paper we will adopt the following notation. For all $n\in N$, $x\in D^n$, we write $\lambda(x)$ for the mean of x, z for the income share vector $x/n\lambda(x)$ corresponding to x and $\bar{x}=(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ for the illfare ranked permutation of x, that is, $\bar{x}_1\leq\bar{x}_2\leq\cdots\leq\bar{x}_n$. For all $n\in N$, an n-coordinated vector of ones is denoted by 1^n. For any function $H: D\rightarrow R^1$, we denote the restriction of H on D^n by H^n. For $n=1$, the concept of relative deprivation is vacuous. We therefore assume that $n\geq 2$.

Following Runciman's (1966) remark mentioned in the introduction, the deprivation d_{ij} felt by an individual with income x_i relative to jth person's income x_j, where $x_j\geq x_i$, can be taken as their income share differential (see also Kakwani, 1984 and Temkin, 1986). That is,

$$\begin{aligned} d_{ij} &= (x_j - x_i)/n\lambda(x) \quad &&\text{if} \quad x_j \geq x_i \\ &= 0 &&\text{if} \quad x_j < x_i \end{aligned} \tag{1}$$

If $x_j<x_i$, person i's feeling of deprivation does not arise at all. It is therefore reasonable to define d_{ij} as being zero in this case. Note that d_{ij} is normalised over [0, 1], it is continuous in x, increasing in x_j and decreasing in x_i.

Now, an individual with income $\bar{x}_i$ in the ordered profile $\bar{x}$ is deprived of the incomes $\bar{x}_{i+1}, \cdots, \bar{x}_n$. Therefore, the total deprivation felt by this person is

$$d_i(x) = \sum_{j=i+1}^{n} (\bar{x}_j - \bar{x}_i)/n\lambda(x) \tag{2}$$

The individual deprivation function d_i possesses many interesting properties:

(i) d_i is decreasing in $\bar{x}_i$.
(ii) d_i is independent of incomes smaller than $\bar{x}_i$.
(iii) An increase in any income higher than $\bar{x}_i$ increases d_i.
(iv) d_i decreases under a rank preserving income transfer from someone with income higher than $\bar{x}_i$ to someone with income smaller than x_i.
(v) An equiproportionate variation in all incomes does not change d_i.
(vi) d_i decreases under equal absolute augmentations in all incomes.
(vii) d_i is continuous in its arguments.
(viii) d_i remains unaffected when rank preserving income transfers take place among persons with incomes higher than $\bar{x}_i$.

Kakwani (1984) defined the Relative Deprivation Curve (RDC) corresponding to the distribution x as the plot of $d_i(x)$ against the cumulative proportion of population i/n, where $i = 0, 1, \cdots, n$, and $d_0(x) = 1$. The extension $d_0(x) = 1$ ensures that the RDC is a closed graph. Evidently, the RDC is downward sloping ($d_i(x)$ decreases as i/n increases).[1]

If incomes are equally distributed, there is no feeling of deprivation by any person ($d_i(x) = 0$ for all i). In contrast, the maximum deprivation arises if the entire income is monopolized by the richest person ($d_i(x) = 1$ for $1 \le i \le n-1$ and $d_n(x) = 0$).

We can rewrite $d_i(x)$ in (2) as follows:

$$d_i(x) = 1 - L_i(x) - (n-i)\bar{x}_i/n\lambda(x) \tag{3}$$

where $L_i(x)$ $(=\sum_{j=1}^{i} \bar{x}_j/n\lambda(x))$ is the cumulative share of the total income $n\lambda(x)$ enjoyed by the bottom i/n $(0 \le i \le n)$ fraction of the population. The graph of $L_i(x)$ against i/n, where $i = 0, 1, \cdots, n$ and $L_0(x) = 0$, gives us the well-known Lorenz Curve (LC). We regard the complement

$$s_i(x) = L_i(x) + (n-i)\bar{x}_i/n\lambda(x) \tag{4}$$

of $d_i(x)$ (to 1) as the relative satisfaction function of the person with income $\bar{x}_i$. The function $s_i(x)$ can be interpreted in the following way. Since person i (in the ordered profile $\bar{x}$) is not deprived of incomes $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{i-1}$, he may be regarded as being satisfied if he compares these incomes with his own income $\bar{x}_i$. Therefore, the first term on the right hand side of (4) is based on the truncated distribution $(\bar{x}_1, \cdots, \bar{x}_i)$ (more precisely, on the truncated illfare ranked income share distribution $(\bar{z}_1, \cdots, \bar{z}_i)$). Next, since this person is deprived of the incomes

[1] It has been demonstrated rigorously that no definite conclusion can be drawn regarding the curvature of the relative deprivation curve. See Chakravarty, Chattopadhyay and Majumder (1995).

$\bar{x}_{i+1}, \cdots, \bar{x}_n$, to eliminate his feeling of deprivation about these $(n-i)$ incomes we replace each of them by $\bar{x}_i$. Thus, this approach ignores actual information on incomes of persons richer than i but counts them in with his income level $\bar{x}_i$. Given that in addition to person i there are now $(n-i)$ persons with income $\bar{x}_i$ and since all these persons are treated identically, we simply add the term $(n-i)\bar{x}_i$ (as a fraction of the total income $n\lambda(x)$) to $L_i(x)$ for arriving at (3). Therefore, for each i, $s_i(x)$ is based on the censored income distribution $(\bar{x}_1, \cdots, \bar{x}_{i-1}, \cdots, \bar{x}_i)$ (more precisely, on $(\bar{z}_1, \cdots, \bar{z}_{i-1}, \bar{z}_i, \cdots, \bar{z}_i)$).[2] We can develop properties for $s_i(x)$ which are analogous to properties (i)–(viii) of $d_i(x)$. By plotting $s_i(x)$ against i/n, where $i=0, 1, \cdots, n$, we can generate the Relative Satisfaction Curve (RSC) of x. Since $L_0(x)=s_0(x)=0$, $L_n(x)=s_n(x)=1$ and $s_i(x)>L_i(x)$ for $i=1, \cdots, n-1$, the RSC always lies above the LC.

Given any two income distributions $x, y \in D^n$, we say that x dominates y by the relative deprivation criterion ($x >_{-d} y$, for short) if the RDC of x lies nowhere below that of y. Formally $x >_{-d} y$ means that

$$d_i(x) \geq d_i(y) \tag{5}$$

for all $i=1, 2, \cdots, n$. From (3), (4) and (5) it follows that $x >_{-d} y$ holds if and only if y dominates x by the relative satisfaction criterion, that is, $s_i(y) \geq s_i(x)$ for $1 \leq i \leq n$ ($y >_{-s} x$, for short).[3]

The following result, which we will use later, gives some implications of the deprivation dominance relation for income distributions with a fixed total over a given population size.

THEOREM 1. *(Chakravarty, Chattopadhyay and Majumder, 1995). Let $x, y \in D^n$, where $\lambda(x)=\lambda(y)$, be arbitrary. Then $x >_{-d} y$ (or, $y >_{-s} x$) implies that*

(a) *y is not Lorenz inferior to x ($y >_{-L} x$, for short), that is, $L_i(y) \geq L_i(x)$ for all $i=1, 2, \cdots, n$.*

(b) *$W^n(y) \geq W^n(x)$ for all SWFs $W^n: D^n \to R^1$ that satisfy S-concavity.*[4]

(c) *$\bar{y}$ can be obtained from $\bar{x}$ by a finite sequence of transformations of the form*

$$\bar{x}_i^{\alpha+1} = \bar{x}_i^{\alpha} + c^{\alpha} \leq \bar{x}_j^{\alpha}$$

[2] For further discussions on censored income distributions and their role in the construction of poverty indices, see Hamada and Takayama (1977), Takayama (1979) and Chakravarty (1983, 1990, 1997).

[3] In Chakravarty, Chattopadhyay and Majumder (1995) the deprivation ordering considered was strict in the sense that in (5) strict inequality was required for at least one $i<n$. The weak ordering considered in (5) is reflexive and transitive, but the strict ordering is transitive only. In this paper, for simplicity, we concentrate on the weak ordering and the results developed here can be extended easily to the strict set up. Mukherjee (1996) identified the income tax functions that make the post-tax income distribution not more deprived than the pre-tax one according to the ordering in (5).

[4] $W^n: D^n \to R^1$ is S-concave if $W^n(Bx) \geq W^n(x)$ for all $x \in D^n$ and all bistochastic matrices B of order n. An $n \times n$ non-negative matrix is a bistochastic matrix of order of n if each of its rows and columns sums to one. An S-concave SWF is symmetric, that is, it remains invariant under any permutation of incomes. A standard example of an S-concave SWF is the Gini SWF $W_G^n(x) = \sum_{i=1}^n [2(n-i)+1]\bar{x}_i/n^2$.

$$x_j^{\alpha+1} = \bar{x}_j^\alpha - c^\alpha \geq \bar{x}_i^\alpha$$

where $i < j$, $c^\alpha > 0$ *and* $\bar{x}_k^{\alpha+1} = \bar{x}_k^\alpha$ *for* $k \neq i, j$.

Equivalence between conditions (a), (b) and (c) were demonstrated by Dasgupta, Sen and Starrett (1973), (See also Kolm, 1969; Atkinson, 1970 and Rotheschild and Stiglitz, 1973.) We have already discussed conditions (a) and (b). Condition (c) means that $\bar{x}$ can be transformed into $\bar{y}$ by a sequence of rank preserving transfers from the rich (person j) to the poor (person i).

We may take $x = (10, 20, 30, 40)$ and $y = (10, 24, 26, 40)$ to see that $y >_{-L} x$ does not imply $x >_{-d} y$. The intuitive reasoning behind this is as follows. We get y from x by transferring 4 units of income from the 2nd richest person to the 3rd richest person. In view of equivalence between (a) and (c), we have $y >_{-L} x$. But the transfer while reduces the deprivation of the recipient increases that of the donor, and, consequently, the net effect becomes ambiguous.

The following proposition, which has been demonstrated by Chakravarty, Chattopadhyay and Majumder (1995), will also be useful for our future analysis.

PROPOSITION 2. *Let* $x, y \in D^n$, *where* $\lambda(x) = \lambda(y)$, *be arbitrary. Then* $x >_{-d} y$ (*or*, $y >_{-s} x$) *implies that*

(a) y *is regarded as at least as good as* x *by the Rawlsian maximin criterion* (*Rawls*, 1971), *that is*, $\min_i \{y_i\} \geq \min_i \{x_i\}$.

(b) x *is not regarded as worse than* y *by the maximax criterion, that is*, $\max_i \{x_i\} \geq \max_i \{y_i\}$.

3. RATIONAL SOCIAL WELFARE FUNCTIONS

The purpose of this section is to develop a social welfare ordering consistent with the relative deprivation (satisfaction) criterion. For this, we say that given $x, y \in D^n$, y is obtained from x by a mean preserving transformation $b \in R^n$ if $y = x - b$ where $\sum_{i=1}^n b_i = 0$. In this definition, person i is called a donor, recipient or unaffected according as b_i is positive, negative or zero. Since b is not identically zero (x is different from y), there is at least one donor and one recipient. Also, since $y \in D^n$, b satisfies the feasibility condition $b \leq x$. A mean preserving transformation can be explained in many ways. For instance, if y is obtained from x through a fiscal program that satisfies the balanced budget principle $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, then b is the corresponding tax-subsidy vector (Fei, 1981).

We call a mean preserving transformation b to be fair if for each i, $1 \leq i \leq n$, $b_i \leq \sum_{j \in S_i} b_j / |S_i|$, where S_i is the set of persons richer than i in the original distribution x and $|S_i|$ is the number of persons in S_i. To explain this, let $\sum_{j \in S_i} b_j < 0$. Then fairness demands that the amount received by person i under the mean preserving feasible transformation b is not smaller than the average recipt of persons richer than him. A similar explanation can be given for the case $\sum_{j \in S_i} b_j \geq 0$. It may be interesting to note that fairness is weaker than Fei's (1981) minimal progressivity condition which requires that if $x_i \geq x_j$, then $b_i \geq b_j$.

Now, since fairness represents one notion of equity, it may be possible to use this as a criterion for choosing among alternative distributions of income. We will say that an SWF $W^n: D^n \to R^1$ is regular if $W^n(y) \geq W^n(x)$ where y is obtained from $x \in D^n$ by a fair transformation. Thus, reqularity requires social welfare to be non-decreasing under a fair transformation. We also presume that all persons are treated similarly, that is, the SWF is symmetric. An SWF will be called rational if it satisfies regularity and symmetry. One implication of rationality is that we can define the welfare function directly on the ordered profile $\bar{x}$.

We now show that for comparing distributions of the same amount of income among the same number of people three seemingly unrelated conditions are equivalent.

THEOREM 3. *Let $x, y \in D^n$, where $\lambda(x) = \lambda(y)$, be arbitrary. Then the following statements are equivalent*:

(a) $x >_{-d} y$ *(or,* $y >_{-s} x$*).*

(b) $\bar{x}$ *can be transformed into* $\bar{y}$ *by the following sequence of transformations*

$$\bar{x}^i_{n-i} = \bar{x}^{i-1}_{n-i} + a^i \leq \bar{x}^{i-1}_{n-i+1}$$

$$\bar{y}_{n-i+1} = \bar{x}^i_{n-i+1} = \bar{x}^{i-1}_{n-i+1} - a^i \geq \bar{x}^{i-1}_{n-i}$$

with

$$\bar{x}^i_j = \bar{x}^{i-1}_j, \qquad j \neq n-i,\ n-i+1$$

$$a^i = \bar{x}^{i-1}_{n-i+1} - \bar{y}_{n-i+1} \geq 0, \qquad \bar{x}^0_i = x_i,$$

$a^i = 0$ *at any stage implies that the transformation is complete.*

(c) $W^n(y) \geq W^n(x)$ *for all rational SWFs* $W^n: D^n \to R^1$.

Theorem 3 parallels the results established in the inequality literature on equivalence between the Lorenz ordering and other egalitarian principles (see, for instance, statements (a), (b), (c) in theorem 1). Condition (b) in theorem 3 is analogous to condition (c) of theorem 1. The difference between the sequence described in (b) above and the type which arises in the Lorenz case is that in the case of deprivation dominance, we start transferring income from the richest person and the transfer operation stops only when the remaining lower parts of the distributions are identical. This will follow from the fact that proposition 2(b) need not hold for the Lorenz situation in general.

The proof of theorem 3 relies on the following lemma.

LEMMA 4. *Suppose that the income distribution $\bar{y}$ is obtained from $\bar{x} \in D^n$ through the fair transformation α. Then if four some i, $1 \leq i \leq n-1$, $\sum_{j=i+1}^n \alpha_j = 0$, we have $\alpha_j = 0$ for all $j = 1, 2, \cdots, i$.*

Proof. By fairness, we have for all $i = 1, \cdots, n-1$, $\alpha_i \leq \sum_{j=i+1}^n \alpha_j/(n-i)$. Now, if $\sum_{j=i+1}^n \alpha_j = 0$ for some i, then $\alpha_i \leq 0$. In this case either $\alpha_i = 0$ or $\alpha_i < 0$.

Suppose $\alpha_i<0$. Then $\sum_{j=i}^{n}\alpha_j<0$ and $\alpha_{i-1}\leq\sum_{j=i}^{n}\alpha_j/(n+1-i)$. Thus, $\alpha_i<0$ implies $\alpha_{i-1}<0$. Repeating the above argument, we have $\alpha_j<0$ for all $j=1,\cdots,i-2$. This shows that $\sum_{j=1}^{n}\alpha_j=\sum_{j=1}^{i}\alpha_j+\sum_{j=i+1}^{n}\alpha_j<0$, which is a contradiction. Hence $\alpha_i=0$ and $\sum_{j=i}^{n}\alpha_j=0$ which in turn implies that $\alpha_{i-1}\leq 0$. Arguing similarly have the desired result.

Proof of theorem 3. We prove that (a)$\Leftrightarrow$(b) and (a)$\Leftrightarrow$(c), which in view of transitivity of an equivalence relation show that (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c).

(a)$\Rightarrow$*(b)*: Since $x>_{-d}y$ holds, we have $\sum_{j=i+1}^{n}(\bar{y}_j-\bar{y}_i)\leq\sum_{j=i+1}^{n}(\bar{x}_j-\bar{x}_i)$ for all $i=1,\cdots,n$. This in turn implies that $\alpha_i\leq\sum_{j=i+1}^{n}\alpha_j/(n-i)$ for $1\leq i\leq n-1$, where $\alpha=\bar{x}-\bar{y}$. Thus, $\bar{y}$ is obtained from $\bar{x}$ by a fair transformation. We can therefore employ lemma 4.

By proposition 2(b), we have $\bar{x}_n\geq\bar{y}_n$. If $\bar{x}_n=\bar{y}_n$, then by lemma 4, $\bar{x}=\bar{y}$ and there is nothing to prove. So, let us assume that $\bar{x}_n>\bar{y}_n$. Now, $\bar{y}_n=\bar{x}_n-(\bar{x}_n-\bar{y}_n)=\bar{x}_n^0-a^1$. Also,

$$\begin{aligned}\bar{x}_{n-1}^1&=\bar{x}_{n-1}^0+a^1\\&=\bar{x}_{n-1}+(\bar{x}_n-\bar{y}_n)\geq\bar{y}_{n-1}\end{aligned}\tag{6}$$

The last inequality on the right hand side of (6) is a consequence of $\sum_{j=i}^{n}\bar{x}_j\geq\sum_{j=i}^{n}\bar{y}_j$ for all $i=1,\cdots,n$, which follows from condition (a) in theorem 1. Now, if (6) holds with equality, we have $\sum_{j=n-1}^{n}\alpha_j=0$. Hence from lemma 4, $\bar{x}_{n-i}=\bar{y}_{n-i}$ for $i=2,\cdots,n-1$. If (6) holds with strict inequality

$$\begin{aligned}\bar{y}_{n-1}&=\bar{x}_{n-1}^1-(\bar{x}_{n-1}^1-\bar{y}_{n-1})\\&=\bar{x}_{n-1}^1-a^2\end{aligned}$$

and

$$\begin{aligned}\bar{x}_{n-2}^2&=\bar{x}_{n-2}^1+a^2\\&=\bar{x}_{n-2}+(\bar{x}_{n-1}^1-\bar{y}_{n-1})\\&=\bar{x}_{n-2}+(\bar{x}_{n-1}+(\bar{x}_n-\bar{y}_n)-\bar{y}_{n-1})\\&\geq\bar{y}_{n-2}\end{aligned}\tag{7}$$

where the inequality in (7) is again based on $\sum_{j=i}^{n}\bar{x}_j\geq\sum_{j=i}^{n}\bar{y}_j$, $1\leq i\leq n$. We can complete the demonstration by repeating the above argument.

(b)$\Rightarrow$*(a)* Given that the sequence in (b) takes us from $\bar{x}$ to $\bar{y}$, by substituting the terms of the sequence one can check that $x>_{-d}y$ holds.

(a)$\Rightarrow$*(c)* $x>_{-d}y$ implies that $\bar{y}$ is obtained from $\bar{x}$ through a fair transformation (see the demonstration (a)$\Rightarrow$(b)). Therefore for all regular SWFs $W^n: D^n\to R^1$, $W^n(\bar{y})\geq W^n(\bar{x})$. Symmetry of W^n implies that $W^n(\bar{y})=W^n(y)$ and $W^n(\bar{x})=W^n(x)$. Therefore $W^n(x)\leq W^n(y)$.

(c)$\Rightarrow$*(a)* In proving this part we follow Rothschild and Stiglitz (1973). Consider the SWF $\sum_{i=1}^{k}\bar{x}_i/n+(n-k)\bar{x}_k/n$, where $1\leq k\leq n$. This welfare function is regular, non-decreasing and symmetric (symmetry follows from the fact $\bar{x}$ is ordered).

Thus, $\sum_{i=1}^{k} \bar{y}_i/n + (n-k)\bar{y}_k/n \geq \sum_{i=1}^{k} \bar{x}_i/n + (n-k)\bar{x}_k/n$ for $1 \leq k \leq n$, which in turn implies that $d_k(x) \geq d_k(y)$ for all $k = 1, 2, \cdots, n$. ■

From theorem 3 it follows that a regular, symmetric SWF is S-concave but the converse is not true. It should also be noted that the welfare ordering given by (c) is a quasi-ordering: it is transitive and reflexive but not complete. When two RDCs cross we can get two rational SWFs that will rank the associated income distributions in different directions.

We may now relate $x >_{-d} y$ to a theorem proved by Fei (1981). According to Fei's theorem, if $\bar{y}$ is obtained from $\bar{x}$ through a fiscal program that satisfies the balanced budget principle and minimal progressivity then $y >_{-L} x$. It is clear from theorem 3 that $x >_{-d} y$ holds if and only if $\bar{y} - \bar{x}$ is fair. The intuitive reasoning behind this is quite clear. Any redistribution that decreases the average income of the persons richer than i by $c > 0$, will decrease $d_i(x)$ in (2) by c. Consequently person i can at most give up an amount less than or equal to c and vice-versa. (Note that the sequence in condition (b) of theorem 3 is fair.) Since fairness is weaker than minimal progressivity and $x >_{-d} y$ implies $y >_{-L} x$, we can say that (rank preserving) fairness is weaker for Lorenz domination than Fei's (rank preserving) minimal progressivity.

Intercountry comparisons of deprivation (welfare) usually involve different population sizes and means, as do intertemporal comparisons for the same country. For ranking distributions with the same mean over differing population sizes, we assume, following Sen (1976), that welfare is population replication invariant. That is, for all $n \in N$, $x \in D^n$,

$$W^n(x) = W^{mn}(x^1, \cdots, x^m) \tag{8}$$

where each $x^i = x$ and $m \geq 2$ is arbitrary. That is, an m-fold replication of the population leaves social welfare unchanged. (The welfare function considered in footnote 4 meets this property.) Since RDC (or, RSC) also satisfies population invariance, we have

THEOREM 5. *Let $x \in D^m$ and $y \in D^n$, where $\lambda(x) = \lambda(y)$, be arbitrary. Then the following statements are equivalent:*

(a) *$x >_{-d} y$ (or, $y >_{-s} x$).*

(b) *$W^n(y) \geq W^m(x)$ for all SWFs $W: D \to R^1$ which satisfy rationality and population replication invariance.*

For comparing distributions with different means, we consider the Generalized Satisfaction Curve (GSC) which is produced by scaling up the RSC by the mean income. Thus, the GSC of $x \in D^n$ is the plot of $\sum_{i=1}^{k} \bar{x}_i/n + (n-k)\bar{x}_k/n$ against k/n, $k = 0, 1, \cdots, n$. Since the graph of $\sum_{i=1}^{k} \bar{x}_i/n$ against k/n is Shorrocks' (1983) Generalized Lorenz Curve (GLC) for x, the GSC always lies above the GLC. We will say that $y \in D^n$ dominates $x \in D^n$ by the generalized satisfaction criterion ($y >_{-g} x$, for short), if the GSC of y lies nowhere below that of x. Similarly, we write $y >_{-G} x$ to mean that y generalized Lorenz dominates x. The following result, whose proof is easy, give us an implication of the relation $y >_{-g} x$ in terms of $y >_{-G} x$.

THEOREM 6. *Let $x, y \in D^n$ be arbitrary. Then $y >_{-g} x$ implies that*
(a) $y >_{-G} x$.
(b) $W^n(y) \geq W^n(x)$ *for all SWFs* $W^n: D^n \to R^1$ *which are non-decreasing and S-concave.*

The equivalence between (a) and (b) in theorem 6 was proved by Shorrocks (1983). We note that now in addition to S-concavity the SWF is required to satisfy non-decreasingness also. Non-decreasingness of an SWF is the requirement that social welfare should not decrease when any one of the incomes is increased, keeping all other incomes constant. This assumption, represents one concept of efficiency preference.

Using $y = (10, 24, 26, 41)$ and $x = (10, 20, 30, 40)$ one can check that $y >_{-G} x$ does not imply $y >_{-g} x$. The following theorem, however, shows that the comparison of distributions by rational, non-decreasing SWFs implies and is implied by the relation $>_{-g}$.

THEOREM 7. *Let $x, y \in D^n$ be arbitrary. Then the following statements are equivalent*:
(a) $y >_{-g} x$.
(b) $W^n(y) \geq W^n(x)$ *for all non-decreasing, rational SWFs* $W^n: D^n \to R^1$.

Proof.
(a) ⇒ (b) We observe that $y >_{-g} x$ implies $\lambda(y) \geq \lambda(x)$. Define $u \in D^n$ by $u_i = x_i$, $i = 1, 2, \cdots, n-1$; $u_n = x_n + n(\lambda(y) - \lambda(x))$. Then by non-decreasingness of W^n, $W^n(u) \geq W^n(x)$. Further, $\lambda(u) = \lambda(y)$ and $y >_{-g} u$. Hence $y >_{-s} u$ and by theorem 3, $W^n(y) \geq W^n(u)$ which shows that $W^n(y) \geq W^n(x)$.
(b) ⇒ (a) This part of the proof is identical to that of (c) ⇒ (a) of theorem 3 and hence omitted. ∎

Non-decreasingness of an SWF captures the desire for higher incomes. But this condition may come into conflict with the desire for higher satisfaction. For instance, an increase in the income of the richest person indicates efficiency gain, but at the same time the relative satisfactions of all the other persons decrease. We therefore consider alternative notions of efficiency. Following Shorrocks (1983) we first consider 'scale improvement', which demands that welfare improves if all the incomes are increased equiproportionally. Formally, $W^n: D^n \to R^1$ satisfies scale improvement if for all $x \in D^n$, for all $k \geq 1$,

$$W^n(kx) \geq W^n(x)\,. \tag{9}$$

Clearly, (9) corresponds to a preference for higher incomes without altering all the individual relative deprivations that depend on income shares. Evidently, (9) is weaker than non-decreasingness.

We then have

THEOREM 8. *Let x, $y \in D^n$ be arbitrary. Then the following statements are equivalent*:

(a) $W^n(y) \geq W^n(x)$ *for all SWFs* $W^n: D^n \to R^1$ *that satisfy rationality and the scale improvement condition.*

(b) $\lambda(y) \geq \lambda(x)$ *and* $y >_{-s} x$.

Proof.

(a) ⇒ *(b)* Suppose $W^n(y) = (\lambda(y))^\theta f(y/\lambda(y))$, where $\theta \geq 0$ and f is regular and symmetric. Note that W^n satisfies condition (9).

By choosing $f(y/\lambda(y)) = 1$, $W^n(y) \geq W^n(x)$ implies $\lambda(y) \geq \lambda(x)$. Alternatively if $\theta = 0$, then the inequality $W^n(x) \leq W^n(y)$ gives $f(x/\lambda(x)) \leq f(y/\lambda(y))$. But f is an arbitrary regular, symmetric function and $x/\lambda(x)$ and $y/\lambda(y)$ have the same mean $(=1)$. Hence by theorem 3, $y/\lambda(y) \geq_{-s} x/\lambda(x)$. Since RSC is homogeneous of degree zero in incomes, we have $y >_{-s} x$.

(b) ⇒ *(a)* Let $u = (\lambda(y)/\lambda(x))x$. Then by (9) $W^n(u) \geq W^n(x)$. Note that the RSC of u coincides with that of x. Therefore $y >_{-s} x$ implies that $y >_{-s} u$. Since $\lambda(u) = \lambda(y)$, by theorem 3, $W^n(y) \geq W^n(u)$. This along with $W^n(u) \geq W^n(x)$ implies that $W^n(y) \geq W^n(x)$. ■

Thus, theorem 8 provides a welfare rationalization of the technique of comparing two income distributions when one has higher mean and higher RSC.

When nominal income differences become a source of envy, individual deprivations depend on absolute income differences only. (We can assume that the individual deprivation functions are of the type $d_i'(x) = \sum_{j=i+1}^{n} (\bar{x}_j - \bar{x}_i)/n$). The following efficiency preference proposed by Shorrocks (1983) indicates preference for higher incomes keeping deprivations of this type constant:

$$W^n(x + \alpha 1^n) \geq W^n(x) \tag{10}$$

where $W^n: D^n \to R^1$ and $\alpha \geq 0$ is arbitrary. Condition (10) is called incremental improvement condition. Note that (10) while maintains the same absolute differentials, reduces income ratios. (Condition (9) increases income differentials.)

We can now prove the following

THEOREM 9. *Let x, $y \in D^n$ be arbitrary. Then the following conditions are equivalent*:

(a) $W^n(y) \geq W^n(x)$ *for all SWFs* $W^n: D^n \to R^1$ *that satisfy rationality and incremental improvement principle.*

(b) $\mathrm{GS}(y, i/n) - \mathrm{GS}(x, i/n) \geq (\lambda(y) - \lambda(x)) \geq 0$ *for all* $i = 1, \cdots, n$, *where* $GS(x, i/n) = \sum_{j=1}^{i} \bar{x}_j/n + (n-i)\bar{x}_i/n$ *is the ordinate of the GSC (of x) corresponding to the population fraction i/n.*

Proof.

$(a) \Rightarrow (b)$ Suppose $W^n(y) = (\lambda(y))^\theta . f(y + (\mu - \lambda(y))1^n)$, where $\theta \geq 0$, $\mu > 0$ and f is regular and symmetric. Observe that W^n meets (10). If $f(y) = 1$, $W^n(y) \geq W^n(x)$ implies $\lambda(y) \geq \lambda(x)$. Next, if $\theta = 0$, we have $f(y + (\mu - \lambda(y))1^n) \geq f(x + (\mu - \lambda(x))1^n)$ for any regular, symmetric f. But $x + (\mu - \lambda(x))1^n$ and $y + (\mu - \lambda(y))1^n$ have the same mean (equal to μ). Hence by theorem 3, $(y + (\mu - \lambda(y))1^n) >_{-s} (x + (\mu - \lambda(x))1^n)$ from which we have $(y + (\mu - \lambda(y))1^n) >_{-g} (x + (\mu - \lambda(x))1^n)$. The result now follows from the definition of $>_{-g}$.

$(b) \Rightarrow (a)$ Define $u = x + (\lambda(x) - \lambda(y))1^n$. By condition (10), $W^n(u) \geq W^n(x)$. Observe that $\lambda(u) = \lambda(y)$. Now, $\text{GS}(u, i/n) = \text{GS}(x, i/n) + (\lambda(y) - \lambda(x)) \leq \text{GS}(y, i/n)$ for all $i = 1, \cdots, n$. Therefore, $y >_{-s} u$, and by theorem 3, $W^n(y) \geq W^n(u) \geq W^n(x)$. ■

Condition (b) in theorem 9 shows that for welfare dominance given by (a), not only the GSC of y will lie everywhere above that of x but the vertical distance $\lambda(y) - \text{GS}(y, i/n)$ must also be smaller everywhere. Clearly, using SWFs that meet (8) we can extend theorems 7–9 for comparing distributions over variable population sizes.

We wind up this section by demonstrating how a Runciman (1966)–Kakwani (1984)–Temkin (1986) type index can be related to a particular regular SWF in a negative monotonic way. As a general index of deprivation let us consider

$$I_r^n(x) = 1 - \frac{\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\left(\sum_{j=1}^{i}\bar{x}_j + (n-i)\bar{x}_i\right)\right)^r\right)^{1/r}}{\lambda(x)}, \quad r \leq 1, \quad r \neq 0,$$

$$= 1 - \frac{\prod_{i=1}^{n}\left(\frac{1}{n}\left(\sum_{j=1}^{i}\bar{x}_j + (n-i)\bar{x}_i\right)\right)^{1/n}}{\lambda(x)}, \quad r = 0. \tag{11}$$

Note that $(1 - I_r^n)$ is the symmetric mean of order r (≤ 1) of individual satisfaction levels of the form (4). These individual satisfaction levels as well as their symmetric mean take on the maximal value 1 when all the persons enjoy the same income. Therefore, I_r^n in (11) can be interpreted as the shortfall of the actual (average) satisfaction of the society from its maximum attainable value. I_r^n is continuous, symmetric, invariant under equiproportionate variations in incomes, population replication invariant and bounded between zero and one, where the lower bound is achieved whenever incomes are equal.

To provide an ethical interpretation of I_r^n we consider its dual

$$W_r^n(x) = \lambda(x)(1 - I_r^n(x)) \tag{12}$$

as an SWF. W_r^n is continuous, symmetric, non-decreasing in individual incomes, regular, linearly homogeneous and population replication invariant. When efficiency considerations are absent (that is, mean income is fixed), an increase in

W_r^n is equivalent to a reduction in I_r^n and vice-versa. Therefore, the SWF W_r^n ranks income profiles in exactly negative way as the deprivation index I_r^n. The parameter r in (12) (11)) represents different perceptions of welfare (deprivation). As r decreases, I_r^n becomes more sensitive to the deprivations of the poorer persons. For $r=1$, I_r^n is the Gini index ($=1-\sum_{i=1}^{n}(2(n-i)+1)\bar{x}_i/n^2\lambda(x)$) and the associated SWF is the Gini SWF. As $r\to-\infty$, $I_r^n(x)\to\sum_{j=2}^{n}(\bar{x}_j-\bar{x}_1)/n\lambda(x)$, the deprivation of the poorest person. This can be regarded as the maximin index of deprivation, since as $r\to-\infty$, $W_r^n(x)\to\min_i\{x_i\}$, the Rawlsian maximin criterion. In fact, the lower is the value of r, the higher are the implicit ethics to the maximin rule.

Clearly, we can employ different averaging principles of individual deprivation/satisfaction levels to generate alternative indices of deprivation and interpret them ethically by relating negatively to appropriate SWFs.

4. CONCLUSIONS

Runciman (1966) and Temkin (1986) argued that a person's feeling of relative deprivation generates from existence of differentials between incomes of the richer persons and his own income. The graphical representation of the sum of income share shortfalls of a person from richer individuals against the cumulative proportions of persons is called the relative deprivation curve (Kakwani, 1984). In this paper we first isolate the class of social welfare functions whose ordering of income distributions of a fixed total agree with that generated by two non-intersecting relative deprivation curves. Since the deprivation ordering is sufficient but not necessary for Lorenz ordering (Chakravarty, Chattopadhyay and Majumder, 1995), all these welfare functions are S-concave, but the converse does not follow. In the variable total income case the welfare ordering is shown to coincide with that based on the generalized satisfaction curve which is produced by scaling up the relative satisfaction curve by the mean income. (The relative satisfaction curve is defined by taking complement (to unity) of the relative deprivation curve.) Possibility of connecting an average deprivation index to a social welfare function of this type in a negative monotic way is also explored. Since this area of research is quite young, a great deal remains to be done in the context of complete taxonomy of deprivation/satisfaction indices according to their properties (both ethical and descriptive). Axiomatic foundations of alternative indices may be of some interest here.

In theorem 6 of this paper we state that the generalized satisfaction ordering implies but is not implied by the generalized Lorenz ordering. To relate this to a result of Hey and Lambert (1980) let us suppose that income Y follows a continuous type distribution with distribution function $F:[0,\infty]\to[0,1]$. ($F(t)$ is the proportion of persons with income $\leq t$, F is non-decreasing, $F(0)=0$ and $F(\infty)=1$). Yitzhaki (1979, 1982), Hey and Lambert (1980) and Stark and Yitzhaki (1988) defined $s_t(F)=\int_0^t(1-F(u))du$ as the relative satisfaction function of the person with income t. It is shown that if G is another income distribution function, then

$s_t(F) \geq s_t(G)$ holds for all $t \in [0, \infty]$ if and only F is not worse than G by the utilitarian rule, that is, $\int_0^\infty u(t)dF(t) \geq \int_0^\infty u(t)dG(t)$ for all increasing, concave u. This follows from the fact that the above notion of satisfaction dominance is equivalent to the condition that F weakly dominates G by the second order stochastic dominance criterion, that is, $\int_0^t F(u)du \leq \int_0^t G(u)du$ for all $t \in [0, \infty]$, which is same as the requirement that F generalized Lorenz dominates G (see Hey and Lambert, 1980).[5] Given that the Runciman–Kakwani–Temkin satisfaction function in (4) is different from the Yitzhaki–Hey–Lambert–Stark formulation, we have one way implication only and this is why non-decreasingness and S-concavity of a welfare function are not sufficient for our generalized satisfaction ordering.

REFERENCES

1. Amiel, Y. and F. A. Cowell (1994): Inequality Changes and Income Growth, in W. Eichhorn (ed.) Models and Measurement of Welfare and Inequality (Springer-Verlag, London), 3–27.
2. Atkinson, A. B. (1970): On the Measurement of Inequality, Journal of Economic Theory, 2, 244–263.
3. Berrebi, Z. M. and J. Silber (1985): Income Inequality Indices and Deprivation: A Generalization, Quarterly Journal of Economics, 100, 807–810.
4. Chakraverty, S. R. (1983): Ethically Flexible Measures of Poverty, Canadian Journal of Economics, 16, 74–85.
5. Chakravarty, S. R. (1990): Ethical Social Index Numbers Springer-Verlag, London.
6. Chakravarty, S. R. (1995): Measuring Inequality: The Axiomatic Approach, in J. Silber (ed.) Income Inequality Measurement: From Theory to Practice (Kluwer Academic, Boston), Forthcoming.
7. Chakravarty, S. R. (1997): On Shorrocks' Reinvestigation of the Sen Poverty Index, Econometrica, Forthcoming.
8. Chakravarty, S. R. and A. B. Chakraborty (1984): On Indices of Relative Deprivation, Economics Letters, 14, 283–287.
9. Chakravarty, S. R. and N. Chattopadhyay (1994): An Ethical Index of Relative Deprivation, Research on Economic Inequality, 5, 231–240.
10. Chakravarty, S. R., N. Chattopadhyay and A. Majumder (1995): Income Inequality and Relative Deprivation, Keio Economic Studies, 32, 1–15.
11. Dasgupta, P., A. K. Sen and D. Starrett (1973): Notes on the Measurement of Inequality, Journal of Economic Theory, 6, 180–187.
12. Fei, J. C. H. (1981): Equity Oriented Fiscal Programs, Econometrica, 49, 869–881.
13. Foster, J. E., and A. F. Shorrocks (1988): Poverty Orderings, Econometrica, 56, 173–178.
14. Hamada, K. and N. Takayama (1977): Censorded Income Distributions and The Measurement of Poverty, Bulletin of the International Statistical Institute, 47, 617–630.
15. Hey, J. D. and P. J. Lambert (1980): Relative Deprivation and the Gini Coefficient: Comment, Quarterly Journal of Economics, 95, 567–573.
16. Kakwani, N. C. (1980): Income Inequality and Poverty: Methods of Estimation and Policy Applications, Oxford University Press, London.
17. Kakwani, N. C. (1984): The Relative Deprivation Curve and Its Applications, Journal of Business and Economic Statistics, 2, 384–405.

[5] For further discussions on stochastic dominance, see Foster and Shorrocks (1988) and Chakravarty (1990).

18. Kolm, S. C. (1969): The Optimal Production of Social Justice, in J. Margolis and H. Guitton (eds.) Public Economics (Macmillan, London), 145–200.
19. Mukherjee, D. (1996): Deprivation Reducing Income Tax Function, Research on Economic Inequality, Forthcoming.
20. Paul, S. (1991): An Index of Relative Deprivation, Economics Letters, 36, 337–341.
21. Rawls, J. (1971): A Theory of Justice, Harvard University Press, Camabridge.
22. Rothschild, M. and J. E. Stiglitz (1973): Some Further Results on the Measurement of Inequality, Journal of Economic Theory, 6, 188–204.
23. Runciman, W. G. (1966): Relative Deprivation and Social Justice, Routledge, London.
24. Sen, A. K. (1973): On Economic Inequality, Oxford University Press, London.
25. Sen, A. K. (1976): Real National Income, Review of Economic Studies, 43, 19–39.
26. Shorrocks, A. F. (1983): Ranking Income Distributions, Economica, 50, 3–17.
27. Stark, O. and S. Yitzhaki (1988): Labour Migration as a Response to Relative Deprivation, Journal of Population Economics, 1, 57–70.
28. Takayama, N. (1979): Poverty, Income Inequality and Their Measures: Professor Sen's Axiomatic Approach Reconsidered, Econometrica, 47, 747–759.
29. Temkin, E. (1986): Inequality, Philosophy and Public Affiars, 15, 99–121.
30. Yitzhaki, S. (1979): Relative Deprivation and the Gini Coefficient, Quarterly Journal of Economics, 93, 321–324.
31. Yitzhaki, S. (1982): Relative Deprivation and Economic Welfare, European Economic Review, 17, 99–113.