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Selective harvesting in a two species fishery model

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Abstract

A two species combined harvesting fishery model with selective harvesting by incorporating a discrete time delay (τ) in harvesting age and size of both the species have been considered. It has been observed that the otherwise asymptotically stable system undergoes Hopf bifurcation for some value of $\tau > \tau_0$ giving rise to a small amplitude oscillation around the non-zero equilibrium. Numerical analysis and computer simulation have been performed to investigate the global properties of the system.

Keywords: Fishery model; Selective harvesting

1. Introduction

The problem of interspecific competition between two species which obey the law of logistic growth has been considered by Gause (1935). But he did not study the effect of harvesting. Combined harvesting of two ecologically independent fish populations, obeying the same dynamics of Gause, has been considered by Clark (1976). Clark has also considered harvesting of a single species in a two fish ecologically competing population model. Modifying Clark's model, (Chaudhuri, 1986; Chaudhuri, 1988) has studied combined harvesting and considered the perspectives of bioeconomics and dynamic optimization of a two species fishery.

Several investigations have been made to study the effect of time lag on the stability of Lotka–Volterra population models (e.g., Wangersky and Cumingham, 1954; Goel et al., 1971; May, 1973; Beddington and May, 1975; Mac Donald, 1976; Mac Donald, 1978, to mention a few). Most of the studies mentioned above are based on linear stability analysis. To the author's knowledge no attempt has been made to investigate the effect of time delay in combined harvesting problems.

The problem of combined harvesting with time delay in a multi-species fishery is an important and practical subject for study. Maintaining a certain time delay in harvesting by restricting to harvest fishes above a certain age or size only (selective harvesting) can help in maintaining the fishery and prevent its extinction. It is customary for the fishermen to throw small fish back into the water and keep the larger fish caught in the net for consumption. This selectivity can also be made by adjusting the mesh size of the net so that when nets are placed in water, they capture all fish except those that are small enough to swim through the mesh. The concept of adding a time lag in the harvesting term in each of the equations of a two species fishery model and then investigating its dynamics to observe the role of time lag on the system came to us from the realistic observation.

In Section 2, we have presented the combined harvesting two-species fishery model with discrete time delay in harvesting terms followed by equilibria analysis in Section 3. In Section 4, we have performed the local stability analysis of the system. We have carried out numerical analysis and computer simulation of the model in Section 5.

2. The mathematical model

The combined harvesting of two competing fish species with discrete time delay τ (in months) can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r_1 x(1 - x/K_1) - \alpha_1 xy - q_1 Ex(t-\tau)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = r_2 y(1 - y/K_2) - \alpha_2 xy - q_2 Ey(t-\tau) \tag{1}$$

where r_1, r_2, α_1, α_2, K_1, K_2 are positive rate constants. Here r_1, r_2 denote the natural growth rates and K_1, K_2 the environmental carrying capacity of the two species. The natural growth obeys the law of logistic growth. In addition, the interaction terms $\alpha_1 xy$ and $\alpha_2 xy$ indicate that the two species compete for the use of the same resource. E denotes the combined harvesting effort and q_1, q_2 are the catchability coefficients of the species.

Clark (1976) studied a model with the harvesting effect on x-species only. Chaudhuri (1986) also dealt with the problem of the combined harvesting effect of the two competing fish species. In the present paper, we modified the harvesting terms of Chaudhuri's model by introducing a time delay in the rate equation. Thus the idea of harvesting each fish species at a specific age is incorporated in this model.

We shall now investigate the dynamics of the delay system (1) by local stability analysis.

3. Equilibria

The possible steady states of (1) are

$$E_0: (0, 0)$$
$$E_1: \left(\frac{K_1}{r_1}(r_1 - q_1 E), 0\right)$$
$$E_2: \left(0, \frac{K_2}{r_2}(r_2 - q_2 E)\right)$$
$$E^*: (\bar{x}, \bar{y}) \tag{2}$$

where

$$\bar{x} = K_1[\alpha_1 K_2(r_2 - q_2 E) - r_2(r_1 - q_1 E)]/(\alpha_1 \alpha_2 K_1 K_2 - r_1 r_2)$$
$$\bar{y} = K_2[\alpha_2 K_1(r_1 - q_1 E) - r_1(r_2 - q_2 E)]/(\alpha_1 \alpha_2 K_1 K_2 - r_1 r_2) \tag{3}$$

The existence and local stability properties of the equilibria in absence of any time delay have been elaborately discussed by Chaudhuri (1986). We shall investigate the above properties with reference to our system (1) with time lag. In the following section we shall perform only the local stability analysis of the interior equilibrium E^*.

4. Local stability analysis

We now investigate asymptotic stability of the equilibrium of the system (1). Let $\xi(t)=x(t)-\bar{x}$, $\eta(t)=y(t)-\bar{y}$ are the perturbed variables. After removing non-linear terms, we obtain the linear variational system, by using equilibria conditions, as

$$\frac{d\xi}{dt}=(-r_1\bar{x}/K_1+q_1E)\xi-\alpha_1\bar{x}\eta-q_1E\xi(t-\tau)$$

$$\frac{d\eta}{dt}=(-r_2\bar{y}/K_2+q_2E)\eta-\alpha_2\bar{y}\xi-q_2E\eta(t-\tau) \tag{4}$$

The associated characteristic equation $\Delta(\lambda,\tau)=0$ with eigenvalue λ can be written as

$$\lambda^2+a_1\lambda+a_2+(b_1\lambda+b_2)e^{-\lambda\tau}+c_2e^{-2\lambda\tau}=0 \tag{5}$$

where

$$a_1=r_1\bar{x}/K_1+r_2\bar{y}/K_2-(q_1+q_2)E$$
$$a_2=(r_1\bar{x}/K_1-q_1E)(r_2\bar{y}/K_2-q_2E)-\alpha_1\alpha_2\bar{x}\bar{y}$$
$$b_1=(q_1+q_2)E$$
$$b_2=[q_1(r_2\bar{y}/K_2-q_2E)+q_2(r_1\bar{x}/K_1-q_1E)]E$$
$$c_2=q_1q_2E^2 \tag{6}$$

Firstly, we consider the case when $\tau=0$, then the Eq. (5) reduces to

$$\Delta(\lambda,0)=\lambda^2+(a_1+b_1)\lambda+(a_2+b_2+c_2)=0 \tag{7}$$

It can be shown that the roots of (7) have negative real parts and the system is locally stable if $r_1r_2>\alpha_1\alpha_2K_1K_2$. (We shall not deal with the case $r_1r_2<\alpha_1\alpha_2K_1K_2$ as this will give to rise to saddle point instability (Chaudhuri, 1986) and we are investigating the effect of time lag on a stable system.) Now we rewrite Eq. (5) as

$$\Delta(\lambda,\tau)=P(\lambda)+Q_1(\lambda)e^{-\lambda\tau}+Q_2(\lambda)e^{-2\lambda\tau}=0 \tag{8}$$

where

$$P=\lambda^2+a_1\lambda+a_2$$
$$Q_1=b_1\lambda+b_2$$
$$Q_2=c_2 \tag{9}$$

and we define

$$Q(\lambda)=Q_1(\lambda)+Q_2(\lambda) \tag{10}$$

Lemma 1. *There exists a unique pair of ω_0, τ_0 with ω_0, $\tau_0\geq 0$, $\omega_0\tau_0<2\pi$ such that $\Delta(i\omega_0,\tau_0)=0$ if the sufficient condition $E<\gamma$ holds.*

Proof. Firstly, we have $\Delta(0, \tau) \neq 0$.

Now consider $|P(\mathrm{i}\omega)|^2 - |Q(\mathrm{i}\omega)|^2$ for $\omega \in R$, we have

$$|P(\mathrm{i}\omega)|^2 - |Q(\mathrm{i}\omega)|^2 = \omega^4 + k_1\omega^2 + k_2 \tag{11}$$

where

$$k_1 = a_1^2 - 2a_2 - b_1^2 \tag{12}$$

and

$$k_2 = a_2^2 - (b_2 + c_2)^2 \tag{13}$$

where a_1, a_2, b_1, b_2, c_2 are defined in (6).

Now, $k_1 > 0$ if $2q_1q_2E^2 + 2(r_1\bar{x}q_1/K_1 + r_2\bar{y}q_2/K_2)E - (r_1\bar{x}/K_1 - r_2\bar{y}/K_2)^2 < 0$, i.e. if

$$E < \gamma \tag{14}$$

where

$$\gamma = \left((-r_1\bar{x}q_1/K_1 + r_2\bar{y}q_2/K_2) + \sqrt{(r_1\bar{x}q_1/K_1 + r_2\bar{y}q_2/K_2)^2 + 2q_1q_2(r_1\bar{x}/K_1 - r_2\bar{y}/K_2)^2}\right) \times (2q_1q_2)^{-1}$$

Again, (14) is a sufficient condition for $k_2 < 0$.

Let

$$|P(\mathrm{i}\omega)|^2 - |Q(\mathrm{i}\omega)|^2 = \nu^2 + k_1\nu + k_2 = 0 \tag{15}$$

has a unique positive root

$$\nu_0 = \tfrac{1}{2}\left(-k_1 + \sqrt{k_1^2 - 4k_2}\right) > 0 \tag{16}$$

Consequently, $|P(\mathrm{i}\omega)|^2 - |Q(\mathrm{i}\omega)|^2 = 0$, $\omega \in R$ if $\omega = \pm\omega_0$, $\omega_0 = \sqrt{\nu_0} > 0$.

Therefore, $|P(\mathrm{i}\omega)|^2 - |Q(\mathrm{i}\omega)|^2 = 0$ implies that there is a unique $\tau_0 \geq 0$ such that $\omega_0\tau_0 < 2\pi$ and

$$\Delta(\mathrm{i}\omega_0, \tau_0) = P(\mathrm{i}\omega_0) + Q_1(\mathrm{i}\omega_0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_0} + Q_2(\mathrm{i}\omega_0)\mathrm{e}^{-2\mathrm{i}\omega_0\tau_0} = 0 \tag{17}$$

Moreover, it is to be noted that the critical value of τ_0 can be calculated from ω_0 after computing ν_0 from Eq. (16).

From condition (14) and lemma (1) we see that

$$\Delta(\mathrm{i}\omega, \tau) = 0, \quad \omega \in R, \tau \geq 0$$

if $\omega = \pm\omega_0$, $\tau = \tau_n = \tau_0 + 2\pi n/\omega_0$, $n = 0, 1, 2, \ldots$ where $\omega_0 > 0$ and $\tau_0 \geq 0$ as is defined in lemma (1). □

Lemma 2.

$$\mathrm{Re}\left[\mathrm{i}\omega_0 \frac{\partial\bar{\Delta}(\mathrm{i}\omega_0, \tau_n)}{\partial\lambda}\left(Q_1(\mathrm{i}\omega_0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_n} + 2Q_2(\mathrm{i}\omega_0)\mathrm{e}^{-2\mathrm{i}\omega_0\tau_n}\right)\right] > 0, \quad n = 1, 2, \ldots \tag{18}$$

Proof. Now,

$$
\begin{aligned}
&\mathrm{i}\,\omega_0 \frac{\partial \overline{\Delta}(\mathrm{i}\,\omega_0,\tau_n)}{\partial \lambda}\left(Q_1(\mathrm{i}\,\omega_0)\mathrm{e}^{-\mathrm{i}\,\omega_0\tau_n}+2Q_2(\mathrm{i}\,\omega_0)\mathrm{e}^{-2\mathrm{i}\,\omega_0\tau_n}\right)\\
&=\mathrm{i}\,\omega_0\left\{-2\mathrm{i}\,\omega_0+a_1+b_1\mathrm{e}^{\mathrm{i}\,\omega_0\tau_n}-\tau_n\left(\overline{Q}_1(\mathrm{i}\,\omega_0)\mathrm{e}^{\mathrm{i}\,\omega_0\tau_n}+2\overline{Q}_2(\mathrm{i}\,\omega_0)\mathrm{e}^{-2\mathrm{i}\,\omega_0\tau_n}\right)\right\}Q_1(\mathrm{i}\,\omega_0)\mathrm{e}^{-\mathrm{i}\,\omega_0\tau_n}\\
&\quad+2\mathrm{i}\,\omega_0\left\{-2\mathrm{i}\,\omega_0+a_1+b_1\mathrm{e}^{\mathrm{i}\,\omega_0\tau_n}-\tau_n\left(\overline{Q}_1(\mathrm{i}\,\omega_0)\mathrm{e}^{\mathrm{i}\,\omega_0\tau_n}+2\overline{Q}_2(\mathrm{i}\,\omega_0)\mathrm{e}^{2\mathrm{i}\,\omega_0\tau_n}\right)\right\}Q_2(\mathrm{i}\,\omega_0)\mathrm{e}^{-2\mathrm{i}\,\omega_0\tau_n}\\
&=\left(2\omega_0^2+\mathrm{i}\,\omega_0 a_1\right)\left(-P+Q_2\mathrm{e}^{-2\mathrm{i}\,\omega_0\tau_n}\right)+\mathrm{i}\,\omega_0 b_1\left(-P+Q_2\mathrm{e}^{-2\mathrm{i}\,\omega_0\tau_n}\right)\mathrm{e}^{\mathrm{i}\,\omega_0\tau_n}-\mathrm{i}\,\omega_0\tau_n\left(|Q_1|^2+4|Q_2|^2\right)\\
&\quad-2\mathrm{i}\,\omega_0\tau_n\left(Q_1\mathrm{e}^{\mathrm{i}\,\omega_0\tau_n}+\overline{Q}_1\mathrm{e}^{-\mathrm{i}\,\omega_0\tau_n}\right)Q_2\\
&=-2\omega_0^2\left(-\omega_0^2+a_2\right)+\omega_0^2a_1^2-\mathrm{i}\left(\omega_0^2+a_2\right)a_1\omega_0-\mathrm{i}\,\omega_0\tau_n\left(|Q_1|^2+4|Q_2|^2\right)\\
&\quad+\left(2\omega_0^2+\mathrm{i}\,\omega_0 a_1\right)c_2(\cos 2\omega_0\tau_n-\mathrm{i}\sin 2\omega_0\tau_n)\\
&\quad+\left[\omega_0^2a_1b_1+2\omega_0^2\tau_n b_1c_2-\mathrm{i}\,\omega_0\left\{\left(-\omega_0^2+a_2\right)b_1+2\tau_n b_2c_2\right\}\right](\cos\omega_0\tau_n+\mathrm{i}\sin\omega_0\tau_n)\\
&\quad-\left[2\omega_0^2\tau_n b_1c_2+\mathrm{i}\,\omega_0(b_1c_2+2b_2c_2\tau_n)\right](\cos\omega_0\tau_n-\mathrm{i}\sin\omega_0\tau_n)
\end{aligned}
$$

So,

$$
\begin{aligned}
&\mathrm{Re}\left[\mathrm{i}\,\omega_0 \frac{\partial \overline{\Delta}(\mathrm{i}\,\omega_0,\tau_n)}{\partial \lambda}\left(Q_1(\mathrm{i}\,\omega_0)\mathrm{e}^{-\mathrm{i}\,\omega_0\tau_n}+2Q_2(\mathrm{i}\,\omega_0)\mathrm{e}^{-2\mathrm{i}\,\omega_0\tau_n}\right)\right]\\
&=-2\omega_0^2\left(-\omega_0^2+a_2\right)+\omega_0^2a_1^2+2\omega_0^2c_2\cos 2\omega_0\tau_n+\omega_0a_1c_2\sin 2\omega_0\tau_n\\
&\quad+\left(2\omega_0^2b_1c_2+\omega_0^2a_1b_1\right)\cos\omega_0\tau_n+\omega_0\left\{b_1\left(\omega_0^2+a_2\right)+2b_2c_2\tau_n\right\}\sin\omega_0\tau_n-2\omega_0^2\tau_n b_1c_2\cos\omega_0\tau_n\\
&\quad-\omega_0(b_1c_2+2b_2c_2\tau_n)\sin\omega_0\tau_n\\
&=-\omega_0^2\left(-2\omega_0^2+2a_2-a_1^2\right)+\omega_0c_2(2\omega_0\cos 2\omega_0\tau_n+a_1\sin 2\omega_0\tau_n)+\omega_0^2a_1b_1\cos\omega_0\tau_n\\
&\quad+\omega_0\left(b_1\left(-\omega_0^2+a_2\right)-b_1c_2\right)\sin\omega_0\tau_n\\
&=-\omega_0^2\left(-2\omega_0^2+2a_2-a_1^2\right)+\omega_0c_2p_1\cos(2\omega_0\tau_n-\theta_1)\\
&\quad+\omega_0b_1p_2\cos(\omega_0\tau_n-\theta_2)>\omega_0^2\left(2\omega_0^2-2a_2+a_1^2\right)\\
&=\omega_0^2\left(2\omega_0^2-2a_2+a_1^2-b_1^2+b_1^2\right)=\nu_0\left(\sqrt{k_1^2-4k_2}+b_1^2\right)>0
\end{aligned}
$$

where

$$p_1^2=4\omega_0^2+a_1^2$$

$$p_2^2=a_1^2\omega_0^2+\left(-\omega_0^2+a_2-c_2\right)^2$$

$\tan\theta_1=a_1/2\omega_0$ and $\tan\theta_2=(-\omega_0^2+a_2-c_2)/a_1$.

Hence the lemma. □**Lemma 3.** *If $E<\gamma$ and ω_0, τ_n, $n=0, 1, 2, \ldots$ be defined as in lemma 1, then for each τ_n, there exists a neighbourhood $I_n\subset R$ of τ_n and a continuously differentiable function $\lambda_n : I_n\to C$ such that*

(i) $\lambda_n(\tau_n)=i\,\omega_0$

(ii) $\Delta(\lambda_n(\tau),\tau)=0,\quad \tau\in I_n$

(iii) $Re\left(\left.\dfrac{\partial\lambda_n(\tau)}{\partial\tau}\right|_{\tau=\tau_n}\right)>0$

Table 1
Different values of E and τ with corresponding maximum and minimum values of x and y

E	τ	x-maximum	x-minimum	y-maximum	y-minimum
0	—	48.21	25	37.89	20
1	0	25	23.38	20	18.95
1	23	25	21.99	20	18.46
1	24.49994	25	21.87	20.13	18.15
1	25	27.99	19.42	23.54	12.53

The values of the other parameters of the system (1) are as follows: $r_1 = 0.1$, $K_1 = 71$, $\alpha_1 = 0.0009$, $q_1 = 0.05$, $r_2 = 0.08$, $K_2 = 80$, $\alpha_2 = 0.0009$, $q_2 = 0.04$. The initial values of x and y are chosen as $x = 25$ and $y = 20$.

Proof. It is clear from lemma 1, that $\partial\Delta(\mathrm{i}\,\omega_0,\ \tau_n)/\partial\lambda \neq 0,\quad n = 0, 1, 2, \ldots$. From the implicit function theorem there exist a neighbourhood I_n and a continuously differentiable function λ_n which satisfies the condition (i) and (ii) of lemma 3.

Now, differentiating the condition (ii), we have

$$\frac{\partial\Delta(\mathrm{i}\,\omega_0,\ \tau_n)}{\partial\lambda}\frac{\partial\lambda_n}{\partial\tau} - \mathrm{i}\,\omega_0 Q_1(\mathrm{i}\,\omega_0)\mathrm{e}^{-\mathrm{i}\,\omega_0\tau_n} - 2\mathrm{i}\,\omega_0 Q_2(\mathrm{i}\,\omega_0)\mathrm{e}^{-2\mathrm{i}\,\omega_0\tau_n} = 0$$

Therefore, it follows from lemma 1 and lemma 2 that

$$\mathrm{Re}\left(\frac{\partial\lambda_n(\tau_n)}{\partial\tau}\right) = \mathrm{Re}\left[\frac{1}{\left(\partial\Delta(\mathrm{i}\,\omega_0,\ \tau_n)/\partial\lambda\right)^2}\left\{\mathrm{i}\,\omega_0\frac{\partial\bar{\Delta}(\mathrm{i}\,\omega_0,\ \tau_n)}{\partial\lambda}\left(Q_1(\mathrm{i}\,\omega_0)\mathrm{e}^{-\mathrm{i}\,\omega_0\tau_n} + 2Q_2(\mathrm{i}\,\omega_0)\mathrm{e}^{-2\mathrm{i}\,\omega_0\tau_n}\right)\right\}\right]$$

$$= \frac{1}{\left(\partial\Delta(\mathrm{i}\,\omega_0,\ \tau_n)/\partial\lambda\right)^2}\mathrm{Re}\left[\mathrm{i}\,\omega_0\frac{\partial\bar{\Delta}(\mathrm{i}\,\omega_0,\ \tau_n)}{\partial\lambda}\left\{Q_1(\mathrm{i}\,\omega_0)\mathrm{e}^{-\mathrm{i}\,\omega_0\tau_n} + 2Q_2(\mathrm{i}\,\omega_0)\mathrm{e}^{-2\mathrm{i}\,\omega_0\tau_n}\right\}\right] > 0$$

□

Next we state the following theorem due to Cooke and van den Driessche (1986) as modified by Boese (Kuang, 1993).

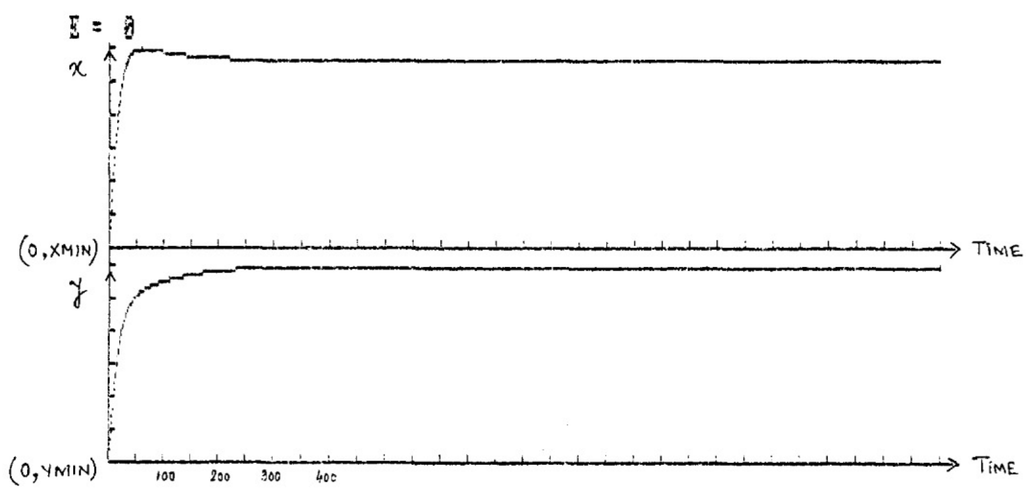

Fig. 1. In the absence of any fishing effort ($E = 0$), both species coexist.

Theorem 1. *If $P(\lambda)$ and $Q(\lambda)$ are analytic functions in $Re\lambda > 0$ and satisfy the following conditions (Here '$-$' denotes complex conjugate)*
(i) $P(\lambda)$ and $Q(\lambda)$ have no common imaginary root;
(ii) $\overline{P}(-iy) = \overline{P}(iy)$, $\overline{Q}(-iy) = Q(iy)$ for real y;
(iii) $P(O) + Q(O) \neq 0$;
(iv) $\lim\sup\{|Q(\lambda)/P(\lambda)|: |\lambda| \to \infty, Re\,\lambda \geq 0\} < 1$;
(v) $F(y) \equiv |P(iy)|^2 - |Q(iy)|^2$ for real y has at most a finite number of real zeros.
Then the following statements are true:
(a) If $F(y) = 0$ has no positive roots, then no stability switch may occur.
(b) If $F(y) = 0$ has at least one positive root and each of them is simple, then as τ increases, a finite number of stability switches may occur, and eventually the considered equation becomes unstable.

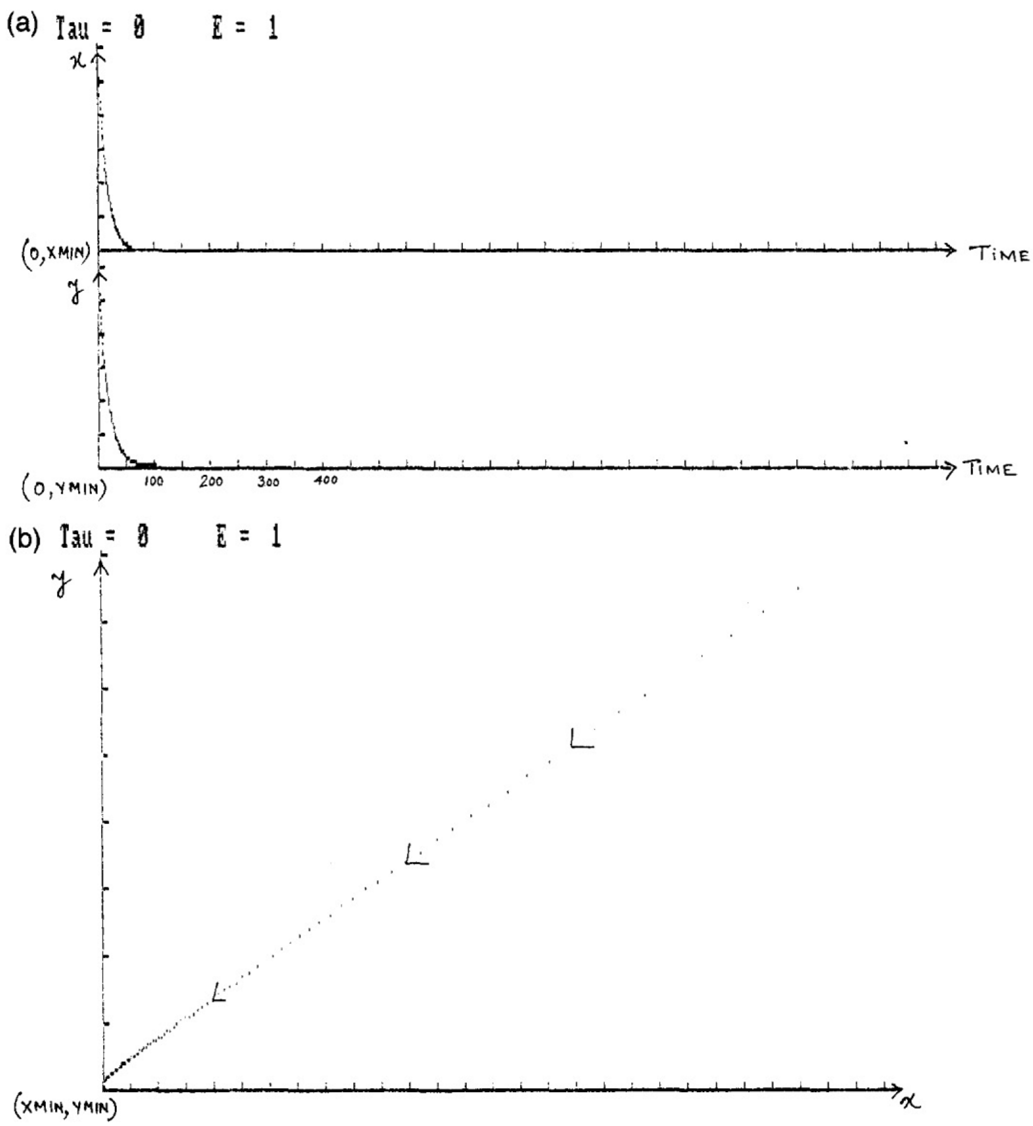

Fig. 2. When $E = 1$, $\tau = 0$, both species are extinct. (a) Time dependent solution of x, y. (b) Phase portrait in x–y plane.

By applying lemma 1, lemma 3 and the above theorem, we finally conclude:

Theorem 2. *Let $0 < E < \gamma$, then the equilibrium $(\bar{x}, \bar{y})$ is locally asymptotically stable if $0 \leq \tau < \tau_0$ and unstable if $\tau > \tau_0$, where τ_0 is defined in lemma 1.*

Remark 1. As τ passes through the value τ_0, the equilibrium $(\bar{x}, \bar{y})$ loses its stability and Hopf bifurcation occurs with emergence of a small amplitude periodic oscillation.

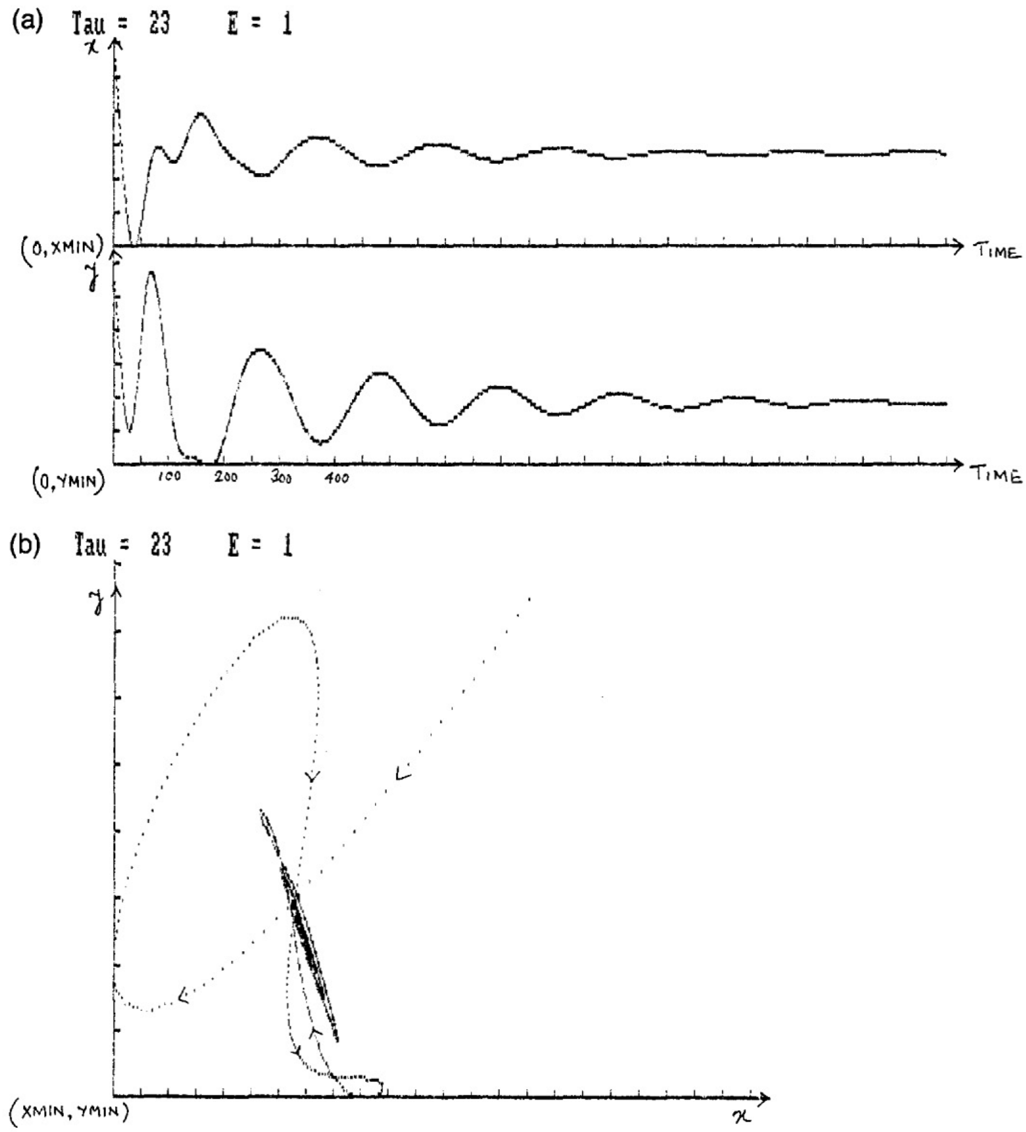

Fig. 3. With the same value of $E(E = 1)$ but $\tau = 23$ both species coexist with decaying oscillation. (a) Time dependent solution of x, y. (b) Phase portrait in the x–y plane.

5. Numerical analysis

The model (1) has been analysed numerically using fourth-order Runge–Kutta method modified to incorporate time delay and then simulated on a micro computer (IBM compatible). The values of the parameters were chosen as in Table 1.

In the absence of harvesting ($E = 0$), both the species are found to be coexisting as obtained from Fig. 1. As effort of harvesting increases the equilibrium population of both the species tends to be extinct and finally becomes completely extinct at $E = 1$ when there is no time delay ($\tau = 0$). This is exhibited in Fig. 2(a) and Fig. 2(b). Now applying some constraints in harvesting all fishes of all ages and sizes such as restricting to harvesting fish of each species at or above certain age or size this state of extinction can be removed and the

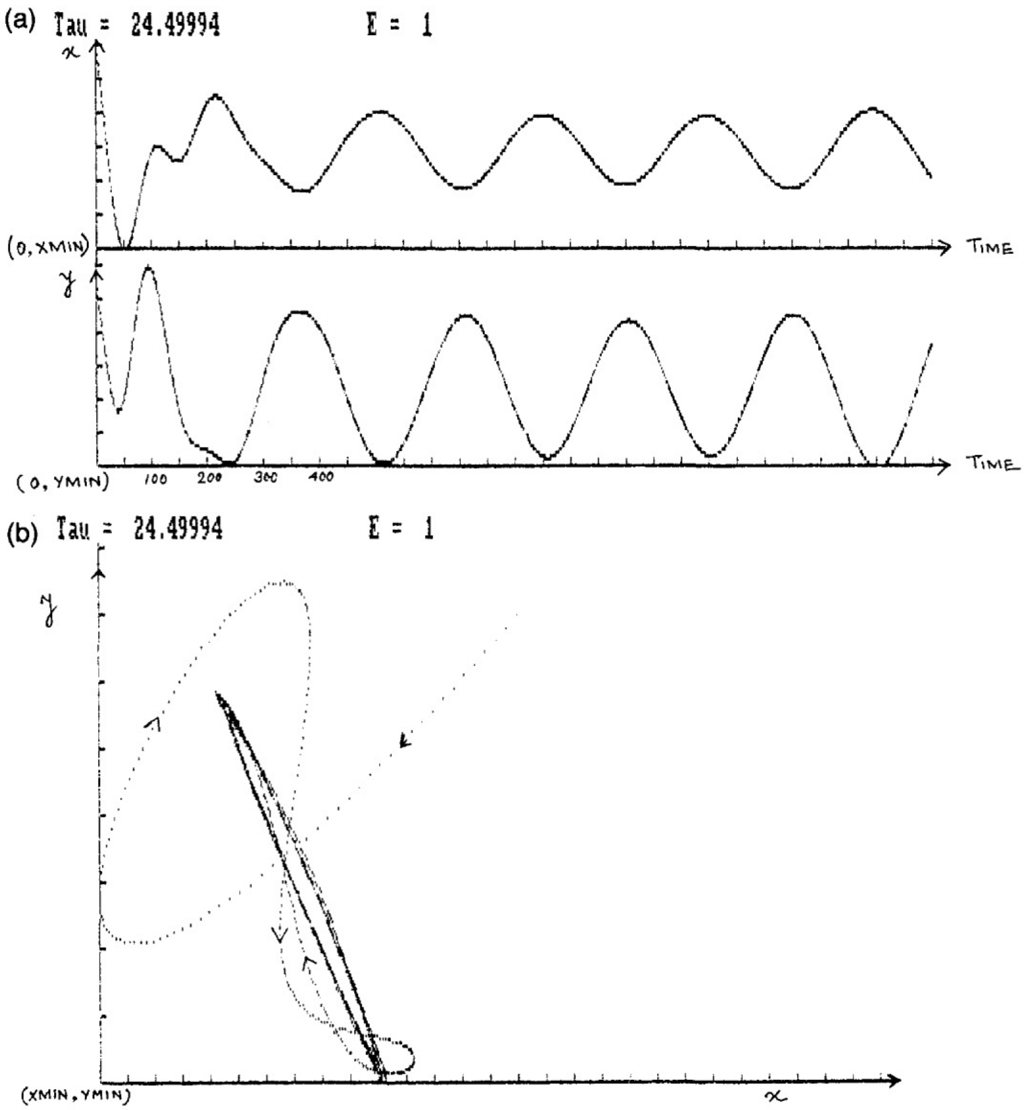

Fig. 4. With the same value of $E (E = 1)$, the system bifurcates at $\tau = 24.49994$ to a stable limit cycle oscillation. (a) Time dependent solution of x, y. (b) Trajectory in the x–y plane.

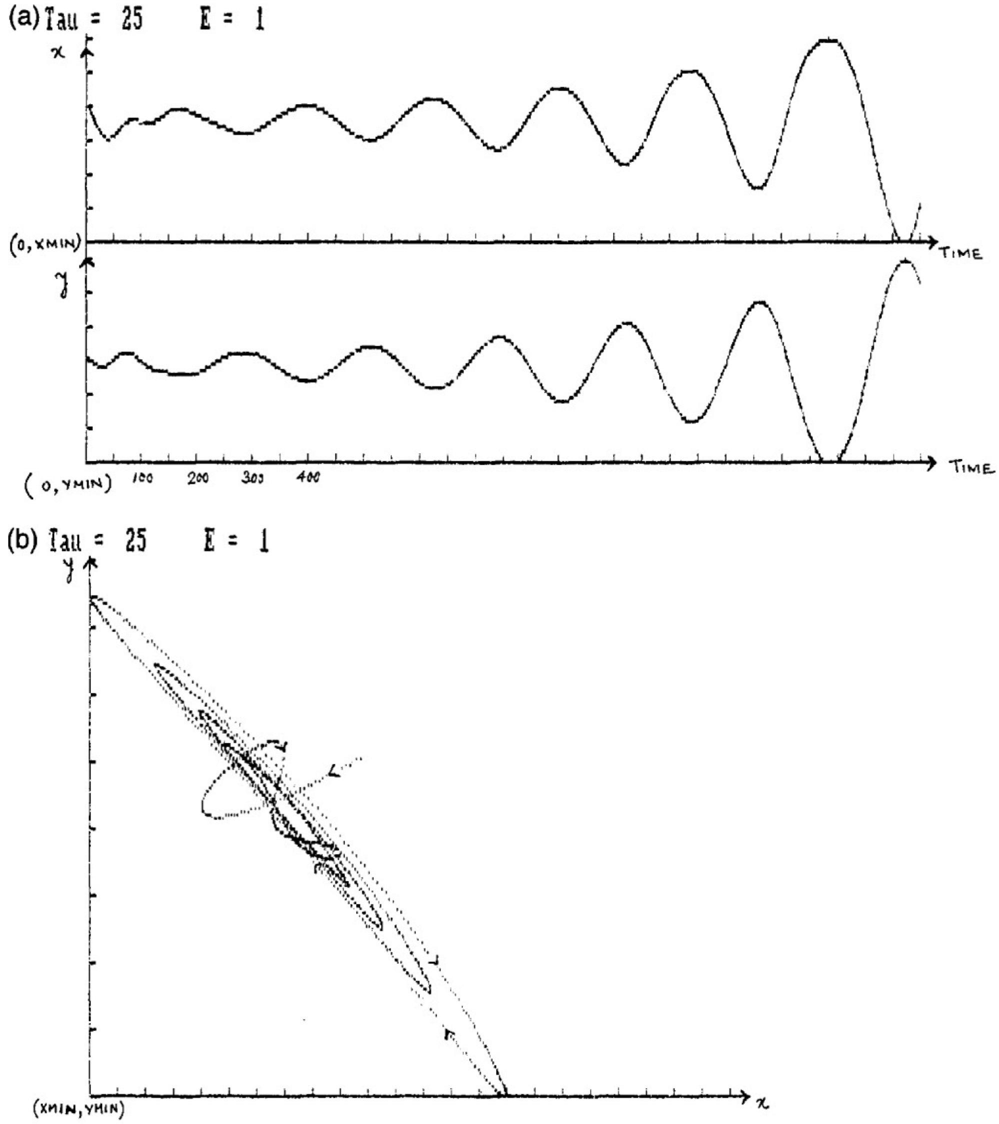

Fig. 5. With the same $E(E=1)$ but τ further raise to $\tau = 25$, an unstable periodic solution ensures. (a) Time dependent solution of x, y. (b) Phase portrait in the x–y plane.

fishery can be restored. This is exhibited in Fig. 3(a) and Fig. 3(b) where for $E = 1$ and $\tau = 23$ months, a decaying oscillation is observed in the fish population. At $\tau = 24.49994$ ($E = 1$), the computer simulation shows the emergence of a small amplitude periodic solution (limit cycle) in Fig. 4(a) and Fig. 4(b) and $\tau = 25$ a growing (unstable) periodic orbit is exhibited in Fig. 5(a) and Fig. 5(b).

The reality and utility of the time-delay model of a combined harvesting system are justified for prevention of extinction of fishery. It is also observed from Figs. 3–5 that maximum population of one species correspond to minimum population of the other showing thereby the effect of competition for the same resource. The oscillation is due to selective harvesting whereas the yield of harvesting is alternatively rich in either of the species.

6. Discussion

In this paper we have studied a two-species combined harvesting fishery model with time delay in harvesting of each of the species. Combined harvesting irrespective of age of the species has been considered by (Clark, 1976 and Chaudhuri, 1986; Chaudhuri, 1988).

On logical consideration random fishing of all ages of fishes is not advisable for the persistence of the fishery. Catching of fishes above a certain age is recommended and is also, in fact, in practice so that the fishery can thrive well. Investigation of fishery models with time delay in combined harvesting has not been dealt with before as to the knowledge of the authors.

The present study shows that incorporation of discrete time lags in the harvesting terms drive the otherwise stable system into Hopf bifurcation and emergence of small amplitude periodic solution around the non-zero equilibrium point when the time lag τ reaches a certain value (τ_0). It has been observed numerically on a computer simulation that while an increase effort of harvesting drives the fishery to extinction, addition of time lag in the form of selective harvesting can save the situation and help to maintain persistence, that is permanent coexistence of both the species. The computer simulation also establishes the persistence of the selective combined harvesting system (1) in a global sense for suitable values of τ. The model thus shows temporal waxing and waning of both the species due to a combined effect of competition and harvesting but never any phenomenon of extinction of any of the species for suitable τ. We have also worked out the length of the delay associated with the effort required for Hopf bifurcation and persistence of the system.

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