Water wave diffraction by a surface strip

By RUPANWITA GAYEN[1], B. N. MANDAL[1,†] AND A. CHAKRABARTI[2]

[1]Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700 108, India

[2]Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

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The two-dimensional problem of wave diffraction by a strip of arbitrary width is investigated here in the context of linearized theory of water waves by reducing it to a pair of Carleman-type singular integral equations. These integral equations have been solved earlier by an iterative process which is valid only for a sufficiently wide strip. A new method is described here by which solutions of these integral equations are determined by solving a set of four Fredholm integral equations of the second kind, and the process is valid for a strip of arbitrary width. Numerical solutions of these Fredholm integral equations are utilized to obtain fairly accurate numerical estimates for the reflection and transmission coefficients. Previous numerical results for a wide strip are recovered from the present analysis. Additional results for the reflection coefficient are presented graphically for moderate values of the strip width which exhibit a less oscillatory nature of the curve than the case of a wide strip.

1. Introduction

There is a considerable amount of interest in investigating surface wave interaction with sea ice. In Antarctica, a region between the ocean and the shore-fast sea ice exists, known as Marginal Ice Zone, which consists of continuous sheets of ice as well as broken ice. The latter can be viewed as consisting of non-interacting floating materials having no elasticity, i.e. it can be modelled as an inertial surface. In the present work we consider diffraction of surface waves by discontinuities in the surface boundary conditions which arise due to the presence of two types of inertial surfaces. The intermediate surface is finite in width and is surrounded by another inertial surface of different surface density. Both the surfaces extend uniformly and infinitely in one horizontal direction. Therefore, from a two-dimensional point of view this problem falls into the category of wave diffraction by a strip.

The problems of wave diffraction involving strips or slits have been the subject of several investigations in acoustics, electromagnetism, elasticity and hydrodynamics (Stoker 1957; Jones 1964; Williams 1982). Although several analytical and numerical treatments are available, the Wiener–Hopf (WH) technique (Noble 1958) is perhaps the key tool for solving this class of problems. When the strip is semi-infinite, the boundary value problem is reduced to a two-part WH problem which can be solved to give an exact solution. For a finite strip the corresponding boundary value problem transforms to a three-part WH problem whose solution can be obtained only approximately on the assumption of a wide strip. Shanin (2001, 2003) considered diffraction by a single strip as well as by two identical strips. His technique involves reduction of the WH functional relationship to solving certain ordinary differential equations. A residue calculus technique was used by Linton (2001) and Chung & Linton (2005) to examine scattering of water waves by a rigid finite dock and by a finite gap in a floating elastic plate in water of uniform finite depth.

In the context of linearized theory of water waves the problem of wave diffraction by a floating semi-infinite strip was solved by employing the WH technique by Weitz & Keller (1950), Gabov, Sveshnikov & Shatov (1989), Goldshtein & Marchenko (1989) and others. Weitz & Keller (1950) considered wave scattering by a semi-infinite inertial surface floating on finite-depth water. The case of two infinitely extended immiscible superposed fluids for which half the interface is covered by an inertial surface and the other half is free, was considered by Gabov *et al.* (1989). Peters (1950) investigated Weitz & Keller's (1950) problem for normal incidence of the incoming wave and infinite depth of water. The last two problems were extended to the case of a finite strip of inertial surface by Kanoria, Mandal & Chakrabarti (1999). They reduced the problem to a three-part WH problem whose solution was derived asymptotically for large width of the strip. Evans (1985) presented a method based on solving a functional equation for investigating water wave scattering by an infinite inertial surface with a continuously varying surface density.

In order to avoid the complexities of the WH technique, Chakrabarti (2000) derived an alternative method to study the problem of wave scattering by a semi-infinite inertial surface floating on deep water. This method used a Fourier analysis which converted the problem into solving a Carleman singular integral equation. This was solved explicitly by a technique associated with solving a Riemann–Hilbert problem, and the exact expressions for the reflection and the transmission coefficients were finally derived. A similar approach was recently followed by Gayen(Chowdhury), Mandal & Chakrabarti (2005) to investigate wave diffraction by an ice strip modelled as a thin elastic plate floating on deep water, resulting in two coupled singular integral equations. These were solved approximately by an iterative process, assuming that the breadth of the strip is wide. Numerical estimates of the reflection and transmission coefficients were obtained and presented graphically against the wavenumber, revealing their rapid oscillatory nature. This was attributed due to multiple reflections and transmissions by the two distant ends of the strip.

Here the problem of diffraction by a strip of inertial surface of a particular surface density lying sandwiched between other inertial surfaces of a different surface density, is reduced to two Carleman-type singular integral equations following the above approach. Employing a method involving inversion of the Carleman singular operator followed by some mathematical analysis, it is found that the solutions of the above coupled singular integral equations can be determined from the solutions of a set of four Fredholm integral equations of the second kind. Since these Fredholm integral equations can be solved numerically in a standard manner, it is obvious that the original two coupled singular integral equations are now solved for any width of the strip and thus the original diffraction problem is solved for any strip width. The mathematical formulation of the problem involves four constants, including the reflection and transmission coefficients, and these satisfy a system of four linear equations which are solved numerically. Numerical estimates for the reflection coefficient $|R|$ are presented graphically. The curve for $|R|$ for a wide strip plotted following the present analysis almost coincide with the corresponding curves for $|R|$ plotted following an earlier analysis valid only for wide strips. Curves for $|R|$

for moderate values of the strip width are also depicted against the wavenumber, and display a less oscillatory nature than a wider strip. Curves for $|R|$ for different situations such as when a strip of inertial surface lies sandwiched between other inertial surfaces, and when a strip of free surface lies between two inertial surfaces of the same density, are also depicted graphically.

2. Formulation of the problem

We consider two semi-infinite inertial surfaces of the same surface density $\epsilon_1\rho$ floating on deep water and in between them lies another inertial surface of surface density $\epsilon_2\rho(\epsilon_1 \neq \epsilon_2)$ in the form of a strip of width l. Here ρ is the density of water; ϵ_1, ϵ_2 are two constants which denote the depth of submergence of the immersed parts of the two types of inertial surfaces and thus have dimension of length. All the three surfaces are infinitely extended in one horizontal direction, say the z-direction so that the problem is two-dimensional in (x, y) coordinates, the x-axis being along the width of the strip and the y-axis being vertically downwards into the liquid. Thus the intermediate strip-like surface and the surfaces on its two sides occupy the regions $0 < x < l, y = 0$ and $(x < 0) \bigcup (x > l), y = 0$ respectively. Under the usual assumption of inviscid, incompressible and homogeneous fluid and the motion in it to be time-harmonic and irrotational, there exists a velocity potential $\mathrm{Re}\{\phi(x, y)\mathrm{e}^{-i\omega t}\}$ where ϕ satisfies

$$\nabla^2\phi = 0, \quad y \geqslant 0, \quad -\infty < x < \infty, \tag{2.1}$$

with the bottom condition

$$\nabla\phi \to 0 \text{ as } y \to \infty, \tag{2.2}$$

and the surface boundary conditions (cf. Appendix I, Weitz & Keller 1950)

$$K_1\phi + \phi_y = 0 \text{ on } y = 0 \; (-\infty < x < 0) \bigcup (x > l), \tag{2.3}$$

$$K_2\phi + \phi_y = 0 \text{ on } y = 0, \; 0 < x < l, \tag{2.4}$$

where $K_j = K/(1 - \epsilon_j K)(j = 1, 2)$ with $K = \omega^2/g$ being the wavenumber. For $0 < \epsilon_j K < 1$, the form of the boundary conditions (2.3) and (2.4) is merely a modification of the usual free surface condition $K\phi + \phi_y = 0$ on $y = 0$, and as such it allows progressive waves to propagate along the inertial surfaces. However, for $\epsilon_j K \geqslant 1$, the form of (2.3) or (2.4) becomes different and does not allow progressive waves along the inertial surfaces. This means that the case $\epsilon_j K \geqslant 1$ is not important physically. Thus it will be assumed that the inequality $\epsilon_j < g/\omega^2$ is always satisfied. This is equivalent to the assumption that the inertial surfaces are sufficiently light to support small time-harmonic progressive waves of given angular frequency ω. It may be remarked here that though there are limitations on the use of the present model involving floating ice, the use of the boundary conditions (2.3) and (2.4) provides a useful check of limiting cases for more general boundary value problems involving Laplace equation.

The edge conditions at the two discontinuity points (0, 0) and (0, l) can be expressed as

$$\left.\begin{aligned} \phi &= O(r) \\ \nabla\phi &= O(1) \end{aligned}\right\} \quad \text{as } r \to 0 \tag{2.5}$$

where r denotes the distance from either of the edges (0, 0) and (0, l). These edge conditions make the solution of the boundary value problem unique.

When a train of surface waves represented by $\exp(-K_1 y + \mathrm{i}K_1 x)$ travelling from the direction of $x = -\infty$ is normally incident on the strip at $(0, 0)$ part of it is reflected back into the region $x < 0$, the remaining part being transmitted through the strip $0 < x < l$. In this region it undergoes multiple reflection and transmission and finally is transmitted into the region $x > l$ through the point $(0, l)$. Thus the far-field behaviour of $\phi(x, y)$ is given by

$$\phi \to \begin{cases} \mathrm{e}^{-K_1 y+\mathrm{i}K_1 x} + R\mathrm{e}^{-K_1 y-\mathrm{i}K_1 x} & \text{as } x \to -\infty, \\ T\mathrm{e}^{-K_1 y+\mathrm{i}K_1(x-l)} & \text{as } x \to \infty \end{cases} \tag{2.6}$$

while in the strip region it has the form

$$\phi = \alpha\mathrm{e}^{-K_2 y+\mathrm{i}K_2 x} + \beta\mathrm{e}^{-K_2 y-\mathrm{i}K_2(x-l)} + \psi(x, y), \; 0 < x < l. \tag{2.7}$$

In the conditions (2.6) and (2.7), R, T, α, β are unknown constants and $\psi(x, y)$ is the local non-wavy solution of Laplace's equation. R and β are the reflection coefficients due to the discontinuities at the points $(0, 0)$ and $(0, l)$ respectively while α and T represent the transmission coefficients due to the same discontinuities. It may be noted that the constants α and β appearing in (2.7) are uniquely defined.

3. Reduction to singular integral equations

In this section, the mixed boundary value problem described above is reduced to two Carleman singular integral equations over a semi-infinite range. For this purpose $\phi(x, y)$ is represented in the three regions $x < 0, 0 < x < l$ and $x > l$ by making use of Havelock's expansion of the water wave potential (cf. Ursell 1947) in the forms

$$\phi \equiv \phi_1(x, y) = \mathrm{e}^{-K_1 y+\mathrm{i}K_1 x} + R\mathrm{e}^{-K_1 y-\mathrm{i}K_1 x} + \frac{2}{\pi}\int_0^\infty \frac{A(\xi)}{\xi^2 + K_1^2} L_1(\xi, y)\mathrm{e}^{\xi x}\,\mathrm{d}\xi, \; x < 0, \tag{3.1}$$

$$\begin{aligned}\phi \equiv \phi_2(x, y) = {} & \alpha\mathrm{e}^{-K_2 y+\mathrm{i}K_2 x} + \beta\mathrm{e}^{-K_2 y-\mathrm{i}K_2(x-l)} \\ & + \frac{2}{\pi}\int_0^\infty \frac{B(\xi)\mathrm{e}^{\xi(x-l)} + C(\xi)\mathrm{e}^{-\xi x}}{\xi^2 + K_2^2} L_2(\xi, y)\,\mathrm{d}\xi, \; 0 < x < l,\end{aligned} \tag{3.2}$$

$$\phi \equiv \phi_3(x, y) = T\mathrm{e}^{-K_1 y+\mathrm{i}K_1(x-l)} + \frac{2}{\pi}\int_0^\infty \frac{D(\xi)}{\xi^2 + K_1^2} L_1(\xi, y)\mathrm{e}^{-\xi(x-l)}\,\mathrm{d}\xi, \; x > l, \tag{3.3}$$

where $A(\xi)$, $B(\xi)$, $C(\xi)$ and $D(\xi)$ are unknown functions and are such that the integrals in (3.1)–(3.3) are convergent, and

$$L_j(\xi, y) = \xi\cos\xi y - K_j \sin\xi y, \; j = 1, 2.$$

In order to determine the unknown constants R, T, α, β and the unknown functions $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$, the conditions of continuity of $\phi(x, y)$ and $\phi_x(x, y)$ across the lines $x = 0$ and $x = l$ $(y > 0)$ are used. These give rise to the following four equations involving the unknown constants and the functions, valid for $y > 0$:

$$\begin{aligned}&\mathrm{e}^{-K_1 y}(1 + R) + \frac{2}{\pi}\int_0^\infty \frac{A(\xi)}{\xi^2 + K_1^2} L_1(\xi, y)\,\mathrm{d}\xi \\ &\quad = \mathrm{e}^{-K_2 y}(\alpha + \beta\mathrm{e}^{\mathrm{i}K_2 l}) + \frac{2}{\pi}\int_0^\infty \frac{B(\xi) + C(\xi)}{\xi^2 + K_2^2} L_2(\xi, y)\,\mathrm{d}\xi, \; y > 0,\end{aligned} \tag{3.4}$$

$$\mathrm{i}K_1\mathrm{e}^{-K_1y}(1-R)+\frac{2}{\pi}\int_0^\infty\frac{\xi A(\xi)}{\xi^2+K_1^2}L_1(\xi,y)\,\mathrm{d}\xi$$
$$=\mathrm{i}K_2\mathrm{e}^{-K_2y}(\alpha-\beta\mathrm{e}^{\mathrm{i}K_2l})+\frac{2}{\pi}\int_0^\infty\frac{\xi(B(\xi)-C(\xi))}{\xi^2+K_2^2}L_2(\xi,y)\,\mathrm{d}\xi,\quad y>0,\qquad(3.5)$$

$$\mathrm{e}^{-K_2y}(\alpha\mathrm{e}^{\mathrm{i}K_2l}+\beta)+\frac{2}{\pi}\int_0^\infty\frac{B(\xi)\mathrm{e}^{\xi l}+C(\xi)\mathrm{e}^{-\xi l}}{\xi^2+K_2^2}L_2(\xi,y)\,\mathrm{d}\xi$$
$$=T\mathrm{e}^{-K_1y}+\frac{2}{\pi}\int_0^\infty\frac{D(\xi)}{\xi^2+K_1^2}L_1(\xi,y)\,\mathrm{d}\xi,\quad y>0,\qquad(3.6)$$

$$\mathrm{i}K_2\mathrm{e}^{-K_2y}(\alpha\mathrm{e}^{\mathrm{i}K_2l}-\beta)+\frac{2}{\pi}\int_0^\infty\frac{\xi(B(\xi)\mathrm{e}^{\xi l}-C(\xi)\mathrm{e}^{-\xi l})}{\xi^2+K_2^2}L_2(\xi,y)\,\mathrm{d}\xi$$
$$=\mathrm{i}K_1T\mathrm{e}^{-K_1y}-\frac{2}{\pi}\int_0^\infty\frac{\xi D(\xi)}{\xi^2+K_1^2}L_1(\xi,y)\,\mathrm{d}\xi,\quad y>0.\qquad(3.7)$$

All the above four equations are basically of the form

$$\chi(y)=\psi_0\mathrm{e}^{-Ky}+\int_0^\infty\hat{\chi}(\xi)L(\xi,y)\,\mathrm{d}\xi,\quad y>0\qquad(3.8)$$

with

$$L(\xi,y)=\xi\cos\xi y-K\sin\xi y,$$

where the function $\chi(y)$ defined for $y>0$, is piecewise-continuously differentiable and absolutely integrable on $(0,\infty)$. The constant ψ_0 and the function $\hat{\chi}(\xi)$ are given by (cf. Ursell 1947)

$$\psi_0=2K\int_0^\infty\psi(u)\mathrm{e}^{-Ku}\,\mathrm{d}u,\qquad(3.9a)$$

$$\hat{\chi}(\xi)=\frac{2}{\pi}\frac{1}{\xi^2+K^2}\int_0^\infty\psi(u)L(\xi,u)\,\mathrm{d}u.\qquad(3.9b)$$

This is generally referred to as the Havelock's inversion theorem in the water wave literature.

Application of this theorem to equations (3.4) to (3.7) (using the results (3.9*b*) and (3.9*a*) in each equation) produces the following eight relations (after some simplifications):

$$A(\xi)=-\frac{(K_1-K_2)\xi}{\xi^2+K_2^2}(\alpha+\beta\mathrm{e}^{\mathrm{i}K_2l})+\frac{(\xi^2+K_1K_2)\{B(\xi)\mathrm{e}^{-\xi l}+C(\xi)\}}{\xi^2+K^2}$$
$$+\frac{2}{\pi}(K_1-K_2)\xi\,⨍_0^\infty\frac{u\{B(u)\mathrm{e}^{-ul}+C(u)\}}{(u^2+K_2^2)(u^2-\xi^2)}\,\mathrm{d}u,\qquad(3.10)$$

$$\xi A(\xi)=-\frac{\mathrm{i}K_2(K_1-K_2)(\alpha-\beta\mathrm{e}^{\mathrm{i}K_2l})\xi}{\xi^2+K_2^2}+\frac{\xi(\xi^2+K_1K_2)}{\xi^2+K_2^2}\{B(\xi)\mathrm{e}^{-\xi l}-C(\xi)\}$$
$$+\frac{2}{\pi}(K_1-K_2)\xi\,⨍_0^\infty\frac{u^2\{B(u)\mathrm{e}^{-ul}-C(u)\}}{(u^2+K_2^2)(u^2-\xi^2)}\,\mathrm{d}u,\qquad(3.11)$$

$$D(\xi) = -\frac{(K_1-K_2)(\alpha e^{iK_2 l}+\beta)\xi}{\xi^2+K_2^2} + \frac{(\xi^2+K_1K_2)\{B(\xi)+C(\xi)e^{-\xi l}\}}{\xi^2+K_2^2} + \frac{2}{\pi}(K_1-K_2)\xi \, ⨍_0^\infty \frac{\{B(u)+C(u)e^{-ul}\}}{(u^2+K_2^2)(u^2-\xi^2)} u \, du, \tag{3.12}$$

$$\xi D(\xi) = \frac{iK_2(K_1-K_2)(\alpha e^{iK_2 l}-\beta)\xi}{\xi^2+K_2^2} - \frac{\xi(\xi^2+K_1K_2)\{B(\xi)-C(\xi)e^{-\xi l}\}}{\xi^2+K_2^2} - \frac{2}{\pi}(K_1-K_2)\xi \, ⨍_0^\infty \frac{u^2\{B(u)-C(u)e^{-ul}\}}{(u^2+K_2^2)(u^2-\xi^2)} du, \tag{3.13}$$

$$\frac{1+R}{2K_1} = \frac{\alpha+\beta e^{iK_2 l}}{K_1+K_2} + \frac{2}{\pi}(K_1-K_2)\int_0^\infty \frac{\xi\{B(\xi)e^{-\xi l}+C(\xi)\}}{(\xi^2+K_1^2)(\xi^2+K_2^2)} d\xi, \tag{3.14}$$

$$\frac{i(1-R)}{2} = \frac{iK_2(\alpha-\beta e^{iK_2 l})}{K_1+K_2} + \frac{2}{\pi}(K_1-K_2)\int_0^\infty \frac{\xi^2\{B(\xi)e^{-\xi l}-C(\xi)\}}{(\xi^2+K_1^2)(\xi^2+K_2^2)} d\xi, \tag{3.15}$$

$$\frac{T}{2K_1} = \frac{\alpha e^{iK_2 l}+\beta}{K_1+K_2} + \frac{2}{\pi}(K_1-K_2)\int_0^\infty \frac{\xi\{B(\xi)+C(\xi)e^{-\xi l}\}}{(\xi^2+K_1^2)(\xi^2+K_2^2)} d\xi, \tag{3.16}$$

$$\frac{iT}{2} = \frac{iK_2(\alpha e^{iK_2 l}-\beta)}{K_1+K_2} + \frac{2}{\pi}(K_1-K_2)\int_0^\infty \frac{\xi^2\{B(\xi)-C(\xi)e^{-\xi l}\}}{(\xi^2+K_1^2)(\xi^2+K_2^2)} d\xi. \tag{3.17}$$

The integrals in (3.10) to (3.13) are in the sense of Cauchy principal value (CPV). It will be seen later that the last four relations will serve the purpose of determination of the four unknown constants.

Elimination of $A(\xi)$ between (3.10) and (3.11), and $D(\xi)$ between (3.12) and (3.13) yields

$$\lambda(\xi)B_1(\xi) + \frac{1}{\pi}⨍_0^\infty \frac{B_1(u)}{u-\xi} du - \frac{1}{\pi}\int_0^\infty \frac{C_1(u)}{u+\xi} e^{-ul} du = F_B(\xi), \ \xi > 0, \tag{3.18}$$

$$\lambda(\xi)C_1(\xi) + \frac{1}{\pi}⨍_0^\infty \frac{C_1(u)}{u-\xi} du - \frac{1}{\pi}\int_0^\infty \frac{B_1(u)}{u+\xi} e^{-ul} du = F_C(\xi), \ \xi > 0 \tag{3.19}$$

where

$$(B_1(\xi), C_1(\xi)) = \frac{\xi}{\xi^2+K_2^2}(B(\xi), C(\xi)), \tag{3.20}$$

$$\lambda(\xi) = \frac{\xi^2+K_1K_2}{\xi(K_1-K_2)}, \tag{3.21}$$

$$F_B(\xi) = \frac{\alpha}{2}\frac{e^{iK_2 l}}{\xi-iK_2} + \frac{\beta}{2}\frac{1}{\xi+iK_2}, \tag{3.22}$$

$$F_C(\xi) = \frac{\beta}{2}\frac{e^{iK_2 l}}{\xi-iK_2} + \frac{\alpha}{2}\frac{1}{\xi+iK_2}. \tag{3.23}$$

Equations (3.18) and (3.19) are two coupled Carleman-type singular integral equations for determining the unknown functions $B(\xi)$ and $C(\xi)$. One way to decouple these equations and to solve them approximately is to assume the width of the strip l to be sufficiently large. Then the integrals involving negative exponentials can be neglected in the first approximation. Gayen(Chowdhury) *et al.* (2005) utilized this idea to solve similar integral equations arising in the problem of water wave scattering by an ice strip modelled as a thin elastic plate, by an iterative process, and also obtained numerical estimates for the eight unknown constants occurring in them. This was valid for a sufficiently wide strip. However, if the breadth of the strip is moderate, this iterative process is not applicable, and here a new method is presented which enables us to solve the coupled singular integral equations for any width of the strip.

It may be mentioned here that equations (3.18) and (3.19) can also be decoupled by addition and subtraction. In that case also the decoupled equations can either be solved approximately by the assumption of a wide strip or by the method to be described in the next section. A brief description of this is given in §6.

4. Solution after reducing to Fredholm integral equations

Our aim is to solve the two singular integral equations (3.18) and (3.19) for any l, the width of the strip. To do this we write (3.18) and (3.19) in operator form as

$$\mathscr{S}B_1(\xi)+\mathscr{S}'C_1(\xi)=F_B(\xi),\quad \xi>0, \tag{4.1}$$

$$\mathscr{S}C_1(\xi)+\mathscr{S}'B_1(\xi)=F_C(\xi),\quad \xi>0, \tag{4.2}$$

where the operators $\mathscr{S}$ and $\mathscr{S}'$ are defined by

$$\left.\begin{aligned}\mathscr{S}f(\xi)&=\lambda(\xi)f(\xi)+\frac{1}{\pi}\int_0^\infty \frac{f(u)}{u-\xi}\,\mathrm{d}u,\\ \mathscr{S}'f(\xi)&=-\frac{1}{\pi}\int_0^\infty \frac{f(u)\mathrm{e}^{-ul}}{u+\xi}\,\mathrm{d}u\end{aligned}\right\},\quad \xi>0. \tag{4.3}$$

We note that the operator $\mathscr{S}$ involves a CPV integral while $\mathscr{S}'$ is a regular integral operator.

It is observed that the Carleman singular integral equation

$$\mathscr{S}f(\xi)=h(\xi),\xi>0 \tag{4.4}$$

can be solved by reducing it to the following Riemann–Hilbert problem (RHP):

$$[\lambda(\xi)+\mathrm{i}]\Lambda^+(\xi)-[\lambda(\xi)-\mathrm{i}]\Lambda^-(\xi)=h(\xi),\quad \xi>0, \tag{4.5}$$

where $\Lambda^\pm(\xi)$ are the limiting values of the sectionally analytic function $\Lambda(\zeta)$ defined by the relation

$$\Lambda(\zeta)=\frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{f(u)}{u-\zeta}\,\mathrm{d}u \tag{4.6}$$

in the complex ζ-plane ($\zeta=\xi+\mathrm{i}\eta$) cut along the real axis from $\xi=0$ to ∞. Solving the RHP (4.5) in the usual manner (see Gakhov 1966) we find that the solution of the Carleman singular integral equation (4.4) is obtained as

$$f(\xi)=\mathscr{S}^{-1}h(\xi)=\frac{\Lambda_0^+(\xi)}{\lambda(\xi)-\mathrm{i}}\widehat{\mathscr{S}}\left[\frac{h(\xi)}{\Lambda_0^+(\xi)(\lambda(\xi)-\mathrm{i})}\right],\quad \xi>0, \tag{4.7}$$

where the operator $\widehat{\mathscr{S}}$ is defined by

$$\widehat{\mathscr{S}}f(\xi)=\lambda(\xi)f(\xi)-\frac{1}{\pi}\int_0^\infty \frac{f(u)}{u-\xi}\,\mathrm{d}u,\quad \xi>0, \tag{4.8}$$

and

$$\Lambda_0^+(\xi)=\lim_{\zeta\to\xi+\mathrm{i}0}\Lambda_0(\zeta),$$

where

$$\Lambda_0(\zeta) = \exp\left[\frac{1}{2\pi i}\left\{\int_0^\infty \left(\ln\frac{t - iK_1}{t + iK_1} - 2\pi i\right)\frac{dt}{t - \zeta} - \int_0^\infty \left(\ln\frac{t - iK_2}{t + iK_2} - 2\pi i\right)\frac{dt}{t - \zeta}\right\}\right], \quad \zeta \notin (0, \infty) \tag{4.9}$$

is a solution of the homogeneous problem corresponding to the RHP (4.5).

We now apply the operator $\mathscr{S}^{-1}$ to (4.1) to obtain

$$B_1(\xi) = \mathscr{S}^{-1}[F_B(\xi) - \mathscr{S}'C_1(\xi)], \quad \xi > 0, \tag{4.10}$$

which when substituted into (4.2) produces

$$\mathscr{S}C_1(\xi) + \mathscr{S}'[\mathscr{S}^{-1}(F_B - \mathscr{S}'C_1)](\xi) = F_C(\xi), \quad \xi > 0. \tag{4.11}$$

Applying the operator $\mathscr{S}^{-1}$ to both sides of (4.11), we find

$$[I - \mathscr{L}^2]C_1(\xi) = r(\xi), \quad \xi > 0, \tag{4.12}$$

where the operator $\mathscr{L} = \mathscr{S}^{-1}\mathscr{S}'$ is ultimately given by (see Appendix A)

$$\mathscr{L}m(\xi) = -\frac{1}{\pi}\frac{\Lambda_0^+(\xi)}{\lambda(\xi) - i}\int_0^\infty \frac{m(u)e^{-ul}\,du}{(u + \xi)\Lambda_0(-u)} \tag{4.13}$$

and

$$r(\xi) = \mathscr{S}^{-1}[F_C - \mathscr{S}'\mathscr{S}^{-1}F_B](\xi), \quad \xi > 0. \tag{4.14}$$

It may be noted that the operator $\mathscr{S}^{-1}\mathscr{S}'$ is not commutative.

Now $F_B(\xi)$ and $F_C(\xi)$ are substituted from (3.22) and (3.23) into (4.14) to obtain $r(\xi)$ in the form

$$r(\xi) = \alpha r_1(\xi) + \beta r_2(\xi)$$

where

$$r_1(\xi) = \frac{1}{2c}\frac{\Lambda_0^+(\xi)}{\lambda(\xi) - i}\left[\frac{1}{\xi + iK_2} + \frac{e^{iK_2 l}}{\pi}\int_0^\infty \frac{\Lambda_0^+(u)e^{-ul}\,du}{(\lambda(u) - i)(u + \xi)(u - iK_2)\Lambda_0(-u)}\right] \tag{4.15}$$

and

$$r_2(\xi) = \frac{1}{2c}\frac{\Lambda_0^+(\xi)}{\lambda(\xi) - i}\left[\frac{e^{iK_2 l}}{\xi - iK_2} + \frac{1}{\pi}\int_0^\infty \frac{\Lambda_0^+(u)e^{-ul}\,du}{(\lambda(u) - i)(u + \xi)(u + iK_2)\Lambda_0(-u)}\right] \tag{4.16}$$

with

$$c = \Lambda_0(\pm iK_2) = \left(\frac{2K_2}{K_1 + K_2}\right)^{1/2}.$$

We now define two functions $U(\xi)$ and $V(\xi)$ for $\xi > 0$ such that

$$[I + \mathscr{L}]C_1(\xi) = U(\xi), [I - \mathscr{L}]C_1(\xi) = V(\xi), \quad \xi > 0 \tag{4.17}$$

so that

$$C_1(\xi) = \tfrac{1}{2}[U(\xi) + V(\xi)], \ \mathscr{L}\,C_1(\xi) = \tfrac{1}{2}[U(\xi) - V(\xi)], \quad \xi > 0. \tag{4.18}$$

Then the integral equation (4.12) can be written either as

$$[I + \mathscr{L}]V(\xi) = r(\xi), \quad \xi > 0 \tag{4.19}$$

or as

$$[I - \mathscr{L}]U(\xi) = r(\xi), \quad \xi > 0. \tag{4.20}$$

Since

$$r(\xi) = \alpha r_1(\xi) + \beta r_2(\xi),$$

we may express $U(\xi)$, $V(\xi)$ as

$$U(\xi) = [I - \mathscr{L}]^{-1} r(\xi) = \alpha u_1(\xi) + \beta u_2(\xi), \tag{4.21}$$

$$V(\xi) = [I + \mathscr{L}]^{-1} r(\xi) = \alpha v_1(\xi) + \beta v_2(\xi), \tag{4.22}$$

where $u_j(\xi)$, $v_j(\xi)$ $(j = 1, 2)$, $\xi > 0$ are unknown functions.

The integral equation (4.19) along with the relation (4.21), and the integral equation (4.20) along with the relation (4.22) are satisfied if $u_j(\xi)$, $v_j(\xi)$ $(j = 1, 2)$ satisfy

$$\left.\begin{array}{ll} [I - \mathscr{L}]u_1(\xi) = r_1(\xi), \quad \xi > 0, & [I - \mathscr{L}]u_2(\xi) = r_2(\xi), \quad \xi > 0, \\ [I + \mathscr{L}]v_1(\xi) = r_1(\xi), \quad \xi > 0, & [I + \mathscr{L}]v_2(\xi) = r_2(\xi), \quad \xi > 0. \end{array}\right\} \tag{4.23}$$

These are in fact Fredholm integral equations with regular kernels, the integral operator $\mathscr{L}$ being defined in (4.13). These integral equations are solved numerically by Nystrom's method and then the functions $u_j(\xi)$, $v_j(\xi)$ $(j = 1, 2)$ are found numerically. It may be noted that considerable analytical calculations are required to reduce the functions $r_j(\xi)$ $(j = 1, 2)$ to forms suitable for numerical computation. This is described in the Appendix B.

The functions $B_1(\xi)$ and $C_1(\xi)$ which satisfy the two coupled singular integral equations (3.18) and (3.19) are now found in a straightforward manner as

$$\begin{aligned} B_1(\xi) &= (\mathscr{S}^{-1} F_B)(\xi) - \mathscr{L} C_1(\xi) \\ &= (\mathscr{S}^{-1} F_B)(\xi) - \tfrac{1}{2}\{U(\xi) - V(\xi)\} \\ &= \alpha B_1^{\alpha}(\xi) + \beta B_1^{\beta}(\xi), \end{aligned} \tag{4.24}$$

$$C_1(\xi) = \tfrac{1}{2}\{U(\xi) + V(\xi)\} = \alpha C_1^{\alpha}(\xi) + \beta C_1^{\beta}(\xi), \tag{4.25}$$

where

$$B_1^{\alpha}(\xi) = \frac{1}{2}\left[\frac{\Lambda_0^{+}(\xi) e^{iK_2 l}}{c(\lambda(\xi) - i)(\xi - iK_2)} - u_1(\xi) + v_1(\xi)\right], \tag{4.26}$$

$$B_1^{\beta}(\xi) = \frac{1}{2}\left[\frac{\Lambda_0^{+}(\xi)}{c(\lambda(\xi) - i)(\xi + iK_2)} - u_2(\xi) + v_2(\xi)\right], \tag{4.27}$$

$$C_1^{\alpha}(\xi) = \tfrac{1}{2}\{u_1(\xi) + v_1(\xi)\}, \tag{4.28}$$

$$C_1^{\beta}(\xi) = \tfrac{1}{2}\{u_2(\xi) + v_2(\xi)\}; \tag{4.29}$$

the value of the constant c being given after equation (4.16).

Thus $B_1(\xi)$ and $C_1(\xi)$ are obtained in terms of the unknown constants α and β. We now replace these functions in equations (3.14)–(3.17) by their expressions in (4.24) and (4.25). This results in a system of four linear equations in α, β, R and T. These equations are solved numerically and the numerical estimates for the reflection and transmission coefficients are computed for different sets of prescribed parameters. Details about how the four linear equations are obtained are given in Appendix C.

5. Numerical results

Owing to the principle of conservation of energy, $|R|^2 + |T|^2 = 1$. Because of this, we present only the numerical estimates for the reflection coefficient $|R|$. This energy

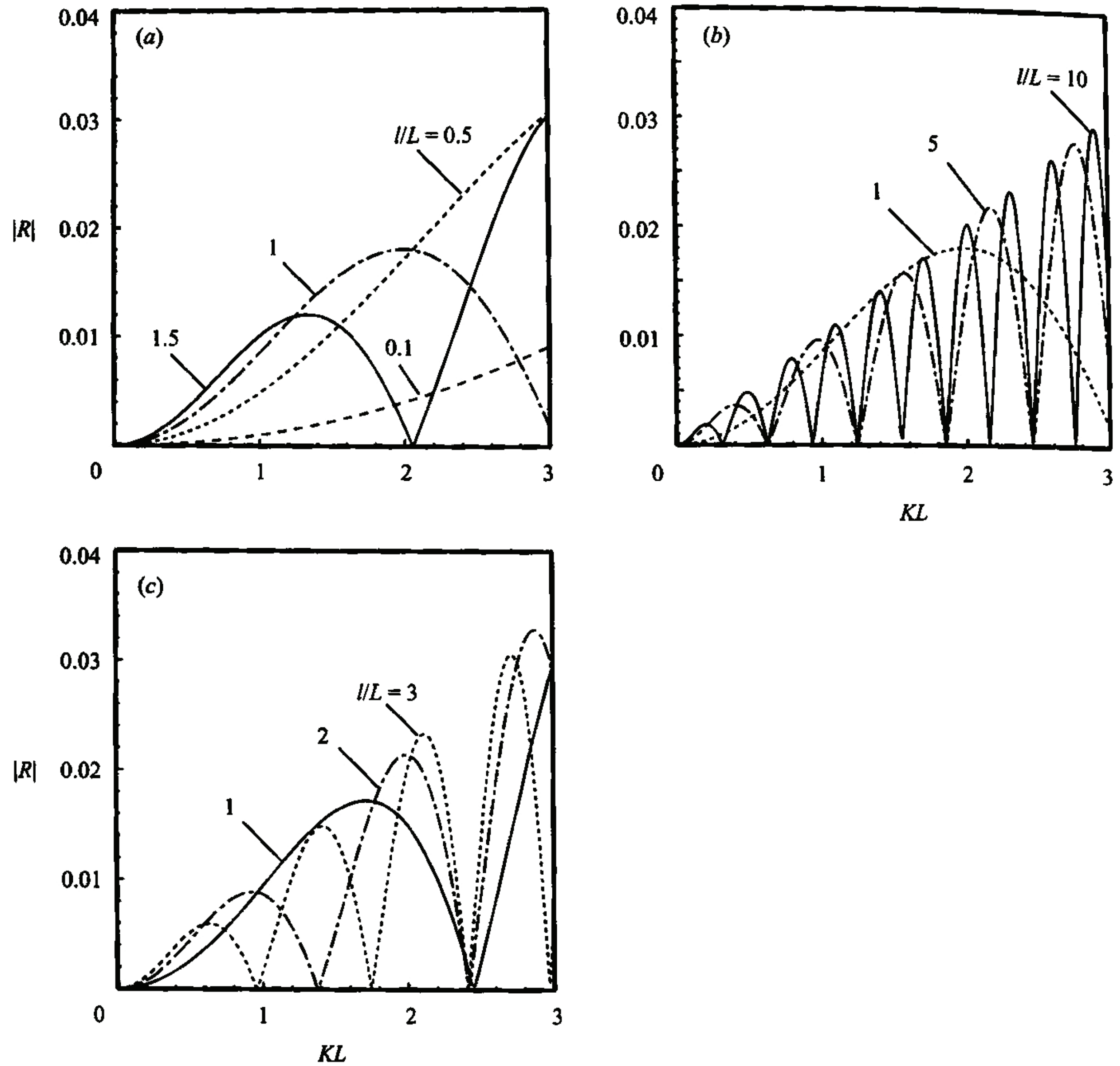

FIGURE 1. $|R|$ for different values of the strip width and $\epsilon_1/L = 0$: (*a*) $\epsilon_2/L = 0.01$ and $l/L = 0.1, 0.5, 1.0$ and 1.5; (*b*) $\epsilon_2/L = 0.01$ and $l/L = 1, 5$ and 10; (*c*) $\epsilon_2/L = 0.1$ and $l/L = 1, 2$ and 3.

identity can also be used as a check on the numerical results, which has been done here for all the data points. A characteristic length L, with respect to which the strip width can be regarded as wide or moderate (l/L large or moderate), is introduced to non-dimensionalize the quantities ϵ_j ($j = 1, 2$), l and K^{-1}. Figure 1(*a–c*) represents $|R|$ against the non-dimensional wavenumber KL for a strip of inertial surface floating sandwiched between two semi-infinite free surfaces, i.e. $\epsilon_1 = 0$ and $\epsilon_2/L = 0.01$ for figures 1(*a*) and 1(*b*) and $\epsilon_2/L = 0.1$ for figure 1(*c*). Figure 1(*a*) depicts $|R|$ for smaller values of the strip width l/L. The overall values of $|R|$ are less than 0.03 in figure 1(*a*), showing that for a strip of sufficiently small width, only a small amount of the incident wave energy is reflected back. As the strip width increases $|R|$ fluctuates and the occurrence of zeros of $|R|$ is observed. This feature is prominent in figure 1(*b*), showing that for larger strip widths ($l/L = 5, 10$) the number of zeros of $|R|$ increases. In figure 1(*c*) a heavier strip is considered ($\epsilon_2/L = 0.1$). Here also an increase in the number of zeros of $|R|$ with the increase in strip width occurs. Also the overall values of $|R|$ are increased as the surface density of the material of strip increases.

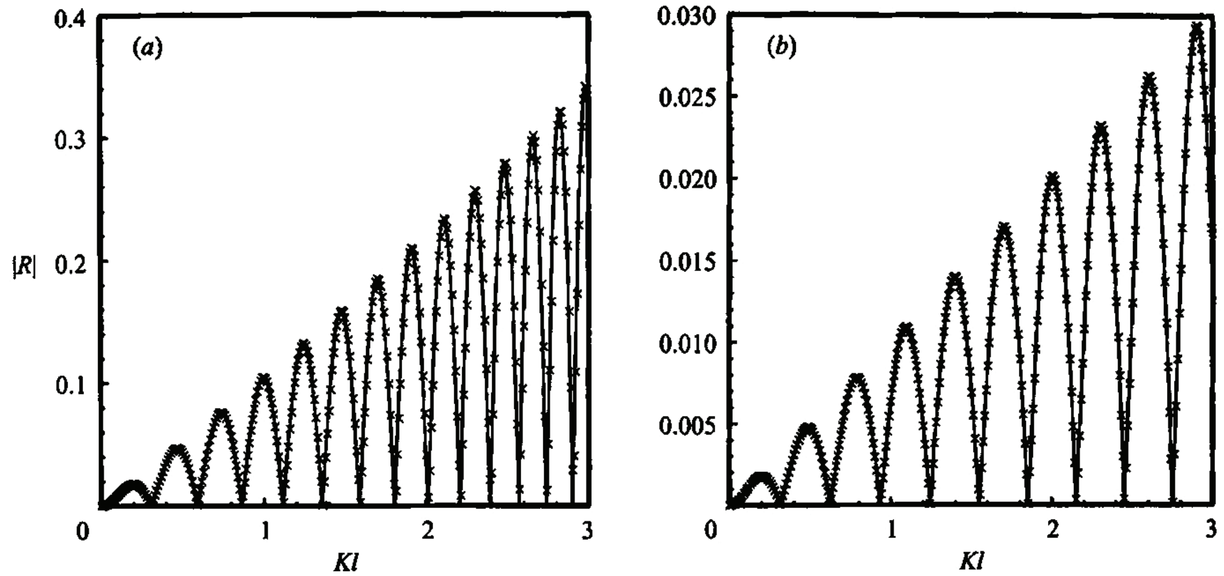

FIGURE 2. $|R|$ for $\epsilon_1/L = 0$, $l/L = 10$ and (*a*) $\epsilon_2/L = 0.1$ and (*b*) $\epsilon_2/L = 0.01$. Crosses denote data obtained by Kanoria *et al.* (1999). The line represents corresponding data from the present analysis.

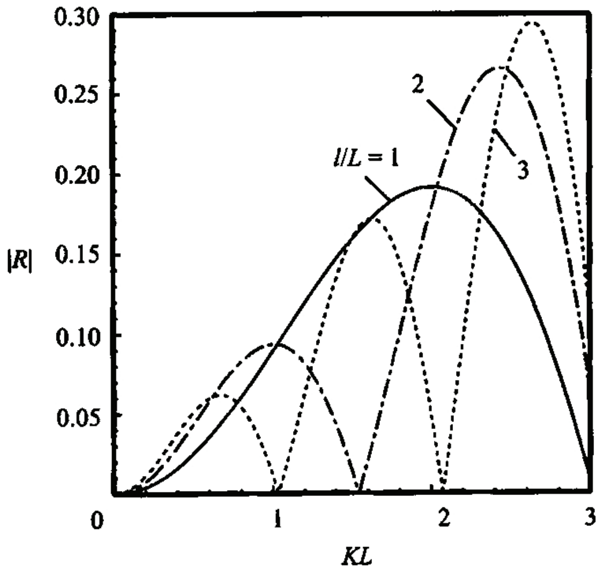

FIGURE 3. $|R|$ for $\epsilon_1/L = 0.1$, $\epsilon_2/L = 0$ and different values of the strip width: $l/L = 1, 2$ and 3.

In figures 2(*a*) and 2(*b*) the present results are compared with those of Kanoria *et al.* (1999) for a wide strip. The data for $|R|$ computed under the assumption of wide strip are indicated by crosses in figures 2(*a*) and 2(*b*). These are seen to lie exactly on the curves for the data obtained following the present method. This provides a good check on the validity of the results obtained here.

If the strip is termed as a scatterer, then the product $(|\epsilon_2 - \epsilon_1|K)(Kl)$ is defined as its strength. Since $0 < \epsilon_1 K$ and $\epsilon_2 K < 1$, $|\epsilon_2 - \epsilon_1|K$ is less than unity, so the strength can be increased by increasing Kl. Hence an increase in strip width (l/L) means an increase in the strength of the scatterer. A wide strip is thus a strong scatterer. Most of the figures show that as l/L increases, $|R|$ becomes more oscillatory with increasing amplitude, as expected.

Figure 3 shows $|R|$ for $\epsilon_1/L = 0.1$, $\epsilon_2/L = 0$ and $l/L = 1, 2, 3$, i.e. for the case when there is a gap of finite width between two semi-infinite inertial surfaces of the same

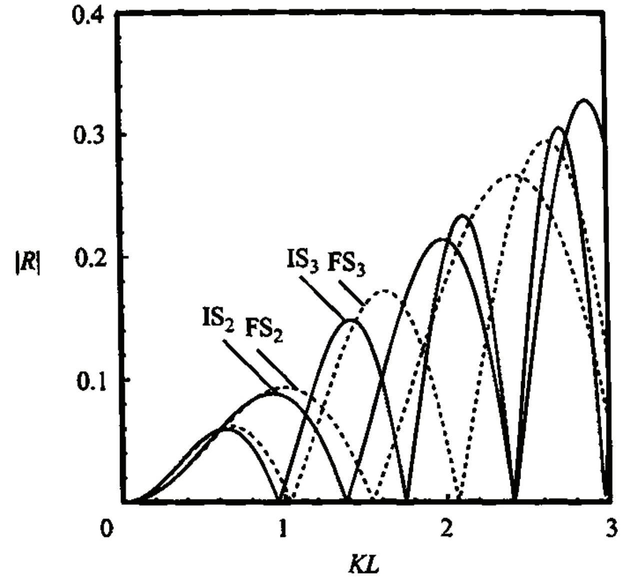

FIGURE 4. $|R|$ for two complementary cases. IS_2: $\epsilon_1/L = 0$, $\epsilon_2/L = 0.1$; FS_2: $\epsilon_1/L = 0.1$, $\epsilon_2/L = 0$ ($l/L = 2$). IS_3: $\epsilon_1/L = 0$, $\epsilon_2/L = 0.1$; FS_3: $\epsilon_1/L = 0.1$, $\epsilon_2/L = 0$ ($l/L = 3$).

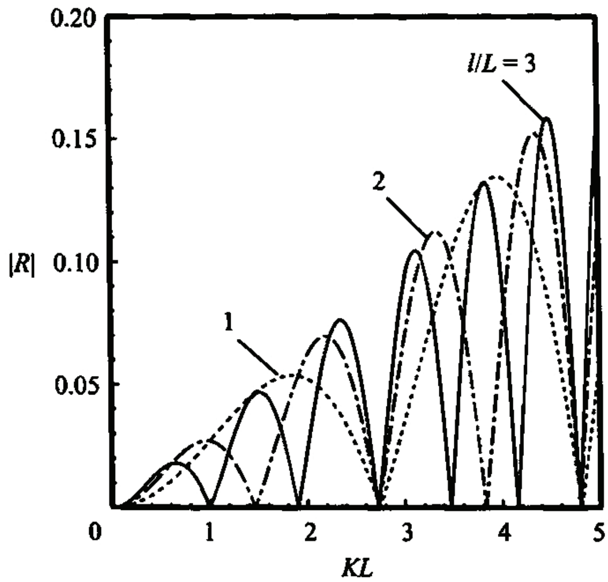

FIGURE 5. $|R|$ for non-zero surface densities. $\epsilon_1/L = 0.02$ $\epsilon_2/L = 0.05$, $l/L = 1, 2$ and 3.

surface density. In this case the number of zeros of $|R|$ increases with the increase in the strip width, as was observed in figure 1.

The two complementary cases, when the intermediate surface is composed of an inertial surface surrounded by a free surface or a free surface surrounded by inertial surface are compared in figure 4 for $l/L = 2, 3$. The continuous lines IS_2 and IS_3 represent $|R|$ for the case of scattering by a strip of inertial surface floating between free surfaces for $l/L = 2$ and 3 respectively while the dotted lines FS_2 and FS_3 show $|R|$ for the complementary case for l/L of the same order. From this figure it is observed that the zeros of $|R|$ for a strip of inertial surface are shifted towards the left of those for a strip of free surface.

For the general case of two non-zero inertial surfaces ($\epsilon_1/L = 0.02$, $\epsilon_2/L = 0.05$), the effect of strip width is depicted in figure 5 which shows similar features of the curves as in figures 1, 3, 4.

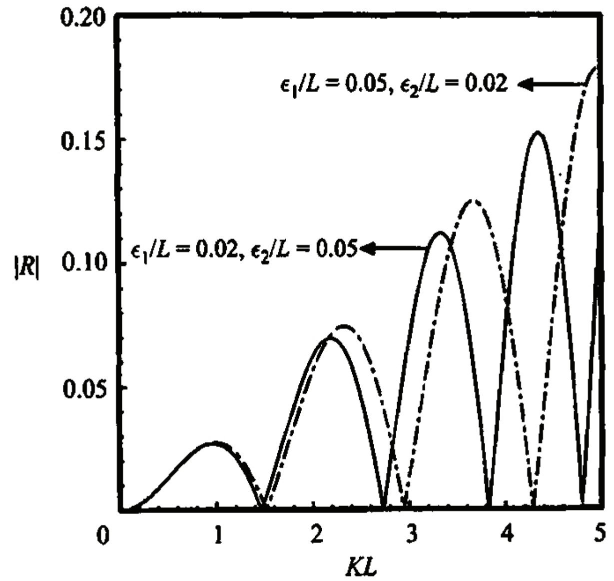

FIGURE 6. $|R|$ for non-zero surface densities with $l/L = 2$.

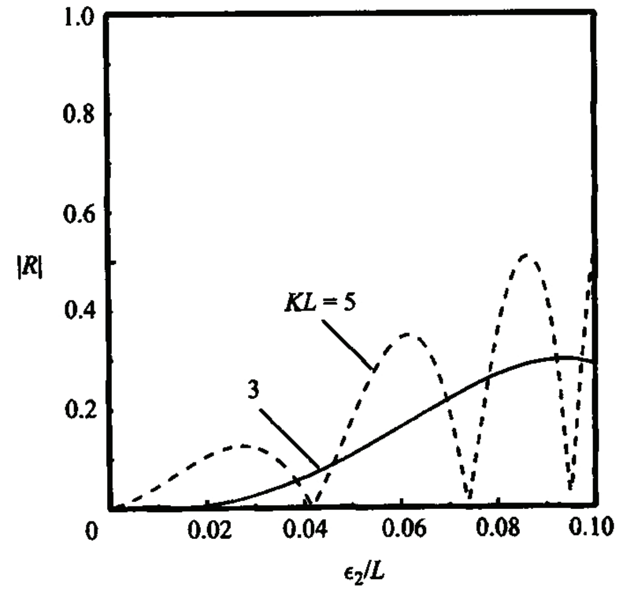

FIGURE 7. $|R|$ for a strip floating on the water surface vs. its surface density for two different wavelengths. $l/L = 2$, $\epsilon_1/L = 0$, $KL = 3$ and 5.

Figure 6 shows $|R|$ associated with a strip which is heavier or lighter compared to the surrounding surfaces. When the strip is heavier, more multiple reflection occurs than when the strip is lighter.

In figure 7, $|R|$ is depicted against the surface density parameter ϵ_2/L of a strip surrounded by free surface ($\epsilon_1/L = 0$) for two different wavenumbers ($KL = 3$, 5) and $l/L = 2$. It is observed that for smaller wavenumber ($KL = 3$) there is no reflection for $\epsilon_1/L < 0.02$. However, with the increase in the surface density parameter ϵ_2/L, the amount of reflection gradually increases. The is also true for $KL = 5$ but in this case the curve for $|R|$ is more oscillatory in nature versus ϵ_2/L.

6. Discussion

(*a*) Carleman singular integral equations occur in a natural way when handling the mixed bunudary value problem (BVP) described in §2, when a Fourier analysis is employed directly to the various conditions of the given BVP in conjunction with Havelock's inversion theorem. Note that the same problem could have been solved by the use of the Wiener–Hopf technique. However, to use the WH technique for such a BVP, some additional assumptions are needed on the boundary condition, the governing partial differential equation, the addition of a small imaginary part to a real parameter, etc. These are artificial. The present method of reduction of the problem to Carleman singular integral equations requires no such artificial assumptions.

(*b*) It has been mentioned in § 3 that the integral equations (3.18) and (3.19) can be decoupled on the assumption of large strip width. However, these can be decoupled simply by addition and subtraction in the following manner:
if we define

$$p(\xi) = B_1(\xi) + C_1(\xi), \quad q(\xi) = B_1(\xi) - C_1(\xi) \tag{6.1}$$

then addition and the subtraction of equations (3.18) and (3.19) produce

$$\lambda(\xi)p(\xi) + \frac{1}{\pi}\int_0^\infty \frac{p(u)}{u-\xi}\,du - \frac{1}{\pi}\int_0^\infty \frac{p(u)}{u+\xi}e^{-ul}\,du = F_1(\xi), \tag{6.2}$$

$$\lambda(\xi)q(\xi) + \frac{1}{\pi}\int_0^\infty \frac{q(u)}{u-\xi}\,du + \frac{1}{\pi}\int_0^\infty \frac{q(u)}{u+\xi}e^{-ul}\,du = F_2(\xi), \tag{6.3}$$

where

$$F_1(\xi), F_2(\xi) = F_B(\xi) \pm F_C(\xi). \tag{6.4}$$

The two equations (6.2) and (6.3) are not coupled. However, these cannot be solved directly unless l is assumed to be sufficiently large. To solve these integral equations a similar approach as used in § 4 may be employed. This is described here briefly.
By virtue of (4.3), (6.2) and (6.3) reduce to

$$\mathscr{S}p(\xi) + \mathscr{S}'p(\xi) = F_1(\xi), \tag{6.5}$$

$$\mathscr{S}q(\xi) - \mathscr{S}'q(\xi) = F_2(\xi). \tag{6.6}$$

Applying the operator $\mathscr{S}^{-1}$ to the above equations we find that

$$[I + \mathscr{L}]\,p(\xi) = \mathscr{S}^{-1}F_1(\xi), \tag{6.7}$$

$$[I - \mathscr{L}]\,q(\xi) = \mathscr{S}^{-1}F_2(\xi), \tag{6.8}$$

where the operator $\mathscr{L} = \mathscr{S}^{-1}\mathscr{S}'$ is defined in (4.13).
The right-hand sides of (6.7) and (6.8) are of the form

$$\mathscr{S}^{-1}F_j(\xi) = \alpha F_j^\alpha(\xi) + \beta F_j^\beta(\xi), \quad j = 1, 2, \tag{6.9}$$

so that

$$p(\xi) = [I + \mathscr{L}]^{-1}\left(\alpha F_1^\alpha(\xi) + \beta F_1^\beta(\xi)\right) = \alpha p_\alpha(\xi) + \beta p_\beta(\xi) \tag{6.10}$$

and

$$q(\xi) = [I - \mathscr{L}]^{-1}\left(\alpha F_2^\alpha(\xi) + \beta F_2^\beta(\xi)\right) = \alpha q_\alpha(\xi) + \beta q_\beta(\xi), \tag{6.11}$$

where $p_\alpha(\xi), p_\beta(\xi), q_\alpha(\xi), q_\beta(\xi)$ are to be found.
Comparing (6.7) and (6.8) (together with (6.9)) to (6.10) and (6.11) we see that equations (6.7) and (6.8) will be satisfied if the functions $p_\alpha(\xi), p_\beta(\xi), q_\alpha(\xi), q_\beta(\xi)$ satisfy the following Fredholm integral equations of the second kind:

$$\left.\begin{aligned} &[I + \mathscr{L}]p_\alpha(\xi) = F_1^\alpha(\xi), \quad [I + \mathscr{L}]p_\beta(\xi) = F_1^\beta(\xi), \\ &[I - \mathscr{L}]q_\alpha(\xi) = F_2^\alpha(\xi), \quad [I - \mathscr{L}]q_\beta(\xi) = F_2^\beta(\xi). \end{aligned}\right\} \tag{6.12}$$

Once the four equations (6.12) are solved numerically the functions $p(\xi)$ and $q(\xi)$ can be found in terms of α and β using (6.10) and (6.11) and then $B_1(\xi)$ and $C_1(\xi)$ will be determined from (6.1). In this alternative method of solving the Carleman equations one has also to solve numerically four Fredholm integral equations.

(*c*) Although by this method the reflection coefficient ($|R|$) is determined numerically, it is not possible to interpret analytically the positions of its zeros as a function of KL. On the other hand, if the same problem is solved by using the Wiener–Hopf technique for a wide strip, an approximate form for R can be obtained. From this the zeros of $|R|$ can be found analytically by solving a transcendental equation, in principle. However, this equation may not have analytical solutions, and will have to be solved numerically using standard methods.

7. Conclusion

The problem of wave diffraction by a strip of inertial surface floating sandwiched between other inertial surfaces is investigated here. The mathematical method employed essentially involves application of a mixed Fourier transform to solve the boundary value problem describing the physical problem, which eventually leads to solving two Carleman singular integral equations. These equations can be solved by an iterative process which is valid only for large width of the strip as in Gayen(Chowdhury) *et al.* (2005). A newly developed method valid for any width of the strip is employed in the present work leading to solving four Fredholm integral equations. Finally, the reflection and transmission coefficients are determined by solving a system of linear equations numerically and the numerical results for the reflection coefficient $|R|$ are presented in a number of figures. The results are compared with the known results for a wide strip obtained by Kanoria *et al.* (1999) and excellent agreement between these two results is seen. Other graphs of $|R|$ for strips of moderate widths are also presented.

Our approach provides a general technique to handle problems associated with wave diffraction by a strip or a slit. It can be generalized to study the effect of a floating elastic plate of finite width or a slit in a floating elastic plate. The problem can also be extended to the case of one or more identical strips separated by a finite distance.

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Appendix A. The operator $\mathscr{L}$

In this Appendix the form of the integral operator $\mathscr{L}$ as given by (4.13) is derived.

Using the definitions of the integral operators $\mathscr{S}$ and $\mathscr{S}'$ as given in (4.3), (4.7) it is easy to see that

$$\begin{aligned}\mathscr{L}m(\xi) &= (\mathscr{S}^{-1}\mathscr{S}')m(\xi)\\ &= \frac{\Lambda_0^+(\xi)}{\lambda(\xi)-\mathrm{i}}\left[\frac{\lambda(\xi)}{\Lambda_0^+(\xi)(\lambda(\xi)+\mathrm{i})}\left(-\frac{1}{\pi}\int_0^\infty \frac{m(u)\mathrm{e}^{-ul}}{u+\xi}\,\mathrm{d}u\right)\right.\\ &\quad\left.+\frac{1}{\pi^2}\int_0^\infty m(u)\mathrm{e}^{-ul}\,\mathrm{d}u\left(⨍_0^\infty \frac{\mathrm{d}t}{\Lambda_0^+(t)(\lambda(t)+\mathrm{i})(t+u)(t-\xi)}\right)\right].\end{aligned} \tag{A 1}$$

To evaluate the inner integral in the second term of (A1), we consider the integral

$$\int_{\Gamma} \frac{d\tau}{\Lambda_0(\tau)(\tau+u)(\tau-\zeta)}, \quad \zeta \notin \Gamma, \tag{A 2}$$

where $\Lambda_0(\zeta)$ satisfies the homogeneous RHP

$$[\lambda(\xi)+\mathrm{i}]\Lambda^+(\xi) - [\lambda(\xi)-\mathrm{i}]\Lambda^-(\xi) = 0, \quad \xi > 0, \tag{A 3}$$

in the complex ζ-plane cut along the positive real axis and Γ is a positively oriented contour consisting of a loop around the positive real axis having indentations above the point $\zeta = \xi + \mathrm{i}0$ and below the point $\zeta = \xi - \mathrm{i}0$ and a circle of large radius with centre at the origin in the complex τ-plane.

We observe that

$$\begin{aligned}\int_{\Gamma} \frac{d\tau}{\Lambda_0(\tau)(\tau+u)(\tau-\zeta)} &= \int_0^{\infty} \left\{\frac{1}{\Lambda_0^+(t)} - \frac{1}{\Lambda_0^-(t)}\right\} \frac{dt}{(t+u)(t+\zeta)} \\ &= 2\mathrm{i}\int_0^{\infty} \frac{dt}{\Lambda_0^+(t)(\lambda(t)+\mathrm{i})(t+u)(t-\zeta)}\end{aligned} \tag{A 4}$$

after using (A 3).

Also from the residue calculus theorem,

$$\int_{\Gamma} \frac{d\tau}{\Lambda_0(\tau)(\tau+u)(\tau-\zeta)} = \frac{2\pi\mathrm{i}}{u+\zeta}\left\{\frac{1}{\Lambda_0(\zeta)} - \frac{1}{\Lambda_0(-u)}\right\}. \tag{A 5}$$

Comparing (A 4) and (A 5) we find

$$\frac{1}{u+\zeta}\left\{\frac{1}{\Lambda_0(\zeta)} - \frac{1}{\Lambda_0(-u)}\right\} = \frac{1}{2\pi\mathrm{i}}\int_0^{\infty} \frac{2\mathrm{i}\,dt}{\Lambda_0^+(t)(\lambda(t)+\mathrm{i})(t+u)(t-\zeta)}.$$

Applying Plemelj's formulae to the above relation the inner integral in the second term on the right-hand side of (A 1) is evaluated as

$$⨍_0^{\infty} \frac{dt}{\Lambda_0^+(t)(\lambda(t)+\mathrm{i})(t+u)(t-\xi)} = \frac{\pi}{u+\xi}\left\{\frac{\lambda(\xi)}{(\lambda(t)+\mathrm{i})\Lambda_0^+(\xi)} - \frac{1}{\Lambda_0(-u)}\right\}$$

which when substituted into (A 1), produces

$$\mathscr{L}m(\xi) = -\frac{1}{\pi}\frac{\Lambda_0^+(\xi)}{\lambda(\xi)-\mathrm{i}}\int_0^{\infty} \frac{m(u)\mathrm{e}^{-ul}\,du}{(u+\xi)\Lambda_0(-u)}.$$

Appendix B. Simplification of $r_1(\xi)$ and $r_2(\xi)$

The basic step for the evaluation of the integral equations (4.23) and the functions $r_1(\xi)$ and $r_2(\xi)$ is to determine the functions $\Lambda_0(-\xi)$ and $\Lambda_0^+(\xi)$ for $\xi > 0$ in computable forms. In this Appendix we present an explicit derivation of these functions and also simplify $r_1(\xi)$ and $r_2(\xi)$.

The function $\Lambda_0^+(\xi)$ is given by

$$\Lambda_0^+(\xi) = \left(\frac{\xi-\mathrm{i}K_1}{\xi+\mathrm{i}K_1}\frac{\xi+\mathrm{i}K_2}{\xi-\mathrm{i}K_2}\right)^{1/2} \exp\left(\frac{1}{2\pi\mathrm{i}}⨍_0^{\infty} \frac{\ln\left(\dfrac{t-\mathrm{i}K_1}{t+\mathrm{i}K_1}\dfrac{t+\mathrm{i}K_2}{t-\mathrm{i}K_2}\right)}{t-\xi}\,dt\right), \quad \xi > 0. \tag{B 1}$$

If we define

$$\left.\begin{aligned} Y(\xi) &= \frac{1}{2\pi i}\int_0^\infty \frac{\ln\left(\dfrac{t-iK_1}{t+iK_1}\dfrac{t+iK_2}{t-iK_2}\right)}{t-\xi}\,dt, \quad \xi > 0, \\ Y_j(\xi) &= \frac{1}{2\pi i}\int_0^\infty \ln\frac{t-iK_j}{t+iK_j}\frac{dt}{t-\xi} \quad (j=1,2), \quad \xi > 0, \\ X(\xi) &= Y(-\xi) \text{ and } X_j(\xi) = Y_j(-\xi), \quad \xi > 0, \end{aligned}\right\} \tag{B 2}$$

then

$$\left.\begin{aligned} &Y(\xi) = Y_1(\xi) - Y_2(\xi),\ \Lambda_0(-\xi) = \exp(X(\xi)), \\ &\Lambda_0^+(\xi) = \left(\frac{\xi-iK_1}{\xi+iK_1}\frac{\xi+iK_2}{\xi-iK_2}\right)^{1/2}\exp(Y(\xi)). \end{aligned}\right\} \tag{B 3}$$

Following Varley & Walker (1989) the derivative of $Y_j(\xi)$ is found to be

$$Y_j'(\xi) = -\frac{K_j}{2\pi}\left[\frac{\ln(\xi/(-iK_j))}{\xi(\xi+iK_j)} + \frac{\ln(\xi/(iK_j))}{\xi(\xi-iK_j)}\right], \quad i = 1,2.$$

It may be observed that $Y_j(\infty) = 0$. Integration of $Y_j'(\xi)$ gives

$$\begin{aligned} Y_j(\xi) &= -\frac{K_j}{2\pi}\int_\infty^\xi \left[\frac{\ln(t/(-iK_j))}{t(t+iK_j)} + \frac{\ln(t/(iK_j))}{t(t-iK_j)}\right]dt \\ &= -\frac{1}{2\pi i}\int_{-iK_j/\xi}^{iK_j/\xi} \frac{\ln u}{u-1}\,du. \end{aligned} \tag{B 4}$$

After some manipulations $Y(\xi)$ reduces to

$$Y(\xi) = \frac{1}{4}\ln\frac{\xi-iK_1}{\xi-iK_2} - \frac{3}{4}\ln\frac{\xi+iK_1}{\xi+iK_2} - \frac{1}{\pi}\int_{K_2/\xi}^{K_1/\xi}\frac{\ln v}{v^2+1}\,dv. \tag{B 5}$$

Hence $\Lambda_0^+(\xi)$ has the alternative form

$$\begin{aligned} \Lambda_0^+(\xi) &= \left(\frac{\xi-iK_1}{\xi-iK_2}\right)^{1/2}\left(\frac{\xi+iK_1}{\xi+iK_2}\right)^{-1/2}\exp(Y(\xi)) \\ &= \left(\frac{\xi-iK_1}{\xi-iK_2}\right)^{3/4}\left(\frac{\xi+iK_1}{\xi+iK_2}\right)^{-5/4}E(\xi) \\ &= \left(\frac{\xi^2+K_1^2}{\xi^2+K_2^2}\right)^{-1/4} e^{-2i(\theta_1-\theta_2)}E(\xi) \end{aligned} \tag{B 6}$$

where $\theta_j = \tan^{-1}(K_j/\xi)$, $j = 1,2$, and

$$E(\xi) = \exp\left(-\frac{1}{\pi}\int_{K_2/\xi}^{K_1/\xi}\frac{\ln v}{v^2+1}dv\right). \tag{B 7}$$

$X(\xi)$ is simplified in a similar manner and we find that

$$X_j(\xi) = Y_j(-\xi) = -Y_j(\xi),\ j = 1,2.$$

Thus $X(\xi) = -Y(\xi)$, and

$$\begin{aligned} \Lambda_0(-\xi) &= \exp(X(\xi)) \\ &= \left(\frac{\xi - iK_1}{\xi - iK_2}\right)^{-1/4}\left(\frac{\xi + iK_1}{\xi + iK_2}\right)^{3/4}(E(\xi))^{-1} \\ &= \left(\frac{\xi^2 + K_1^2}{\xi^2 + K_2^2}\right)^{1/4} e^{i(\theta_1-\theta_2)}(E(\xi))^{-1}. \end{aligned} \tag{B 8}$$

The various complex-valued functions appearing in $r_1(\xi)$ and $r_2(\xi)$ are simplified as follows:

(*a*)
$$\frac{\Lambda_0^+(\xi)}{\lambda(\xi) - i} = (K_1 - K_2)\xi\left(\xi^2 + K_1^2\right)^{-3/4}\left(\xi^2 + K_2^2\right)^{-1/4} e^{-i(\theta_1-\theta_2)} E(\xi)$$

where we have used

$$\lambda(\xi) - i = \frac{(\xi - iK_1)(\xi + iK_2)}{\xi(K_1 - K_2)}. \tag{B 9}$$

(*b*)
$$\frac{\Lambda_0^+(\xi)}{\lambda(\xi) - i}\frac{1}{\Lambda_0(-\xi)} = \frac{(K_1 - K_2)\xi}{\xi^2 + K_1^2} e^{-2i(\theta_1-\theta_2)}(E(\xi))^2.$$

(*c*)
$$\frac{\Lambda_0^+(\xi)}{\lambda(\xi) - i}\frac{1}{\Lambda_0(-\xi)}\left(\frac{1}{\xi - iK_2}, \frac{1}{\xi + iK_2}\right) = \frac{(K_1 - K_2)\xi e^{-2i\theta_1}}{\left(\xi^2 + K_1^2\right)\left(\xi^2 + K_2^2\right)^{1/2}}(E(\xi))^2(e^{3i\theta_2}, e^{i\theta_2}).$$

(*d*)
$$\frac{\Lambda_0^+(\xi)}{\lambda(\xi) - i}\left(\frac{1}{\xi - iK_2}, \frac{1}{\xi + iK_2}\right) = \frac{(K_1 - K_2)\xi e^{-i\theta_1}}{\left(\left(\xi^2 + K_1^2\right)\left(\xi^2 + K_2^2\right)\right)^{3/4}} E(\xi)(e^{2i\theta_2}, 1).$$

Using (*a*) to (*d*), $r_1(\xi)$ and $r_2(\xi)$ are simplified as

$$r_1(\xi) = r_0(\xi)\left[\frac{1}{\left(\xi^2 + K_2^2\right)^{\frac{1}{2}}} + e^{iK_2 l}\int_0^\infty M(u, \xi)e^{3i\psi_2}\,du\right],$$

$$r_2(\xi) = r_0(\xi)\left[\frac{e^{i(K_2 l + 2\theta_2)}}{\left(\xi^2 + K_2^2\right)^{\frac{1}{2}}} + \int_0^\infty M(u, \xi)e^{i\psi_2}\,du\right],$$

where

$$r_0(\xi) = \frac{K_1 - K_2}{\pi}\frac{\xi E(\xi)e^{-i\theta_1}}{\left(\xi^2 + K_1^2\right)^{3/4}}\left(\xi^2 + K_2^2\right)^{1/4},$$

$$M(u, \xi) = \frac{K_1 - K_2}{2c}\frac{u e^{-ul}(E(\xi))^2 e^{i(\theta_2 - 2\psi_1)}}{\left(u^2 + K_1^2\right)\left(u^2 + K_2^2\right)^{1/2}(u + \xi)}$$

and $\psi_j = \tan^{-1}(K_j/u);\ j = 1, 2$.

Appendix C. Determination of α, β, R and T

We first replace $B(\xi)$ and $C(\xi)$ in (3.14) to (3.17) in terms of $B_1(\xi)$ and $C_1(\xi)$ using (3.20), then $B_1(\xi)$ and $C_1(\xi)$ in terms of (4.24) and (4.25) involving α and β. This gives rise to four equations for the determination of the four unknown constants α, β, R

and T. Eliminating R and T from the first two and the last two of these equations respectively, we obtain two equations for α and β which when solved produce

$$\alpha = -\mu_0 \frac{\mu_3\lambda_8 + \lambda_{10}}{\lambda_7\lambda_{10} - \lambda_8\lambda_9}, \quad \beta = \mu_0 \frac{\mu_3\lambda_7 + \lambda_9}{\lambda_7\lambda_{10} - \lambda_8\lambda_9}, \tag{C1}$$

where

$$\mu_0 = \left\{ \left(\frac{K_1 - K_2}{K_1 + K_2} \right)^2 e^{2iK_2 l} - 1 \right\}, \quad \mu_3 = \frac{K_1 - K_2}{K_1 + K_2} e^{iK_2 l},$$

$$\lambda_7 = 1 - \mu_0(\lambda_{3,l} - \lambda_4), \quad \lambda_8 = \mu_0(\lambda_{5,l} - \lambda_6),$$

$$\lambda_9 = \mu_0(\lambda_{4,l} - \lambda_3), \quad \lambda_{10} = 1 - \mu_0(\lambda_{6,l} - \lambda_5),$$

$$\lambda_j = \int_0^\infty \frac{f_j(\xi)}{\xi + iK_1} d\xi, \quad \lambda_{j,l} = \int_0^\infty \frac{f_j(\xi) e^{-\xi l}}{\xi - iK_1} d\xi, \quad j = 3, 4, 5, 6,$$

$$f_3(\xi) = \mu_1 B_1^\alpha(\xi) - \mu_2 C_1^\alpha(\xi), \quad f_4(\xi) = \mu_1 C_1^\alpha(\xi) - \mu_2 B_1^\alpha(\xi),$$

$$f_5(\xi) = \mu_1 B_1^\beta(\xi) - \mu_2 C_1^\beta(\xi), \quad f_6(\xi) = \mu_1 C_1^\beta(\xi) - \mu_2 B_1^\beta(\xi),$$

$$\mu_1 = \frac{2}{\pi i}(K_1 - K_2), \quad \mu_2 = \frac{2}{\pi i} \frac{(K_1 - K_2)^2}{K_1 + K_2} e^{iK_2 l}, \tag{C2}$$

and the functions $B_1^\alpha(\xi)$, $C_1^\alpha(\xi)$, $B_1^\beta(\xi)$ and $C_1^\beta(\xi)$ are given by (4.26) to (4.29). These can be reduced to numerically computable forms by using the calculations of Appendix B.

Once α and β are computed from (C 1), R can be determined directly from either (3.14) or (3.15) and T from either (3.16) or (3.17), after writing $B(\xi)$ and $C(\xi)$ in terms of $B_1(\xi)$ and $C_1(\xi)$ and the later functions in terms of α and β.

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