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# MEASURES OF POVERTY BASED ON THE REPRESENTATIVE INCOME GAP

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*SUMMARY*. This paper provides a generalisation of a poverty index of Clark, Hemming and Ulph. It is based on the notion of the representative income gap of the poor. Sen's index of poverty is revisited in this general framework.

## 1. Introduction

In a recent paper Clark, Hemming and Ulph (1981) have suggested an index of poverty based on the notion of 'equally distributed equivalent income gap'. Given a poverty line *a priori*, this index has several properties : (i) it is sensitive to the percentage of the population that is below the line (the 'head-count' ratio), (ii) it depends on the average income of the poor, and (iii) it depends on the 'relative deprivation measure'.

In this note we offer a generalisation of the index. Each index is implied by and implies at least one group deprivation function of the poor.

Essential to the construction of these indices is the notion of 'representative income gap of the poor'.

In Section 2 we show that there is a Clark *et al.*—like poverty index for every group deprivation function of the poor. Sen's index (Sen, 1976) is also interpreted in this section. In Section 3 we define 'absolute indices of poverty' and present several poverty indices.

## 2. Relative measures of poverty

With a population of size $n$, the distribution of incomes is represented by a vector $y = (y_1, y_2, \ldots, y_q, \ldots, y_n)$, where $y_i \geqslant 0$. Let us assume that the incomes are arranged in nondecreasing order, that is,

$$y_1 \leqslant y_2 \leqslant \ldots \leqslant y_q \leqslant \ldots \leqslant y_n.$$

$q(< n)$ is the number of the poor who have income below the poverty line $z$ (given exogenously). The income short-fall of the $i$-th poor from the poverty line is

$$g_i = (z - y_i),\ i = 1, 2, \ldots, q. \qquad \ldots \quad (1)$$

Throughout the paper, we will assume that the welfare of the poor is separable from that of the non-poor (for a discussion of the notion of separability see Blackorby, Primont and Russell, 1978, Ch. 3).

Clark, Hemming and Ulph (1981) assume that the identical deprivation functions take the form

$$d(g_i) = \frac{1}{\alpha} g_i^{\alpha},\ i = 1, 2, \ldots, q; \qquad \ldots \quad (2)$$

where $\alpha \geqslant 1$ for concavity in income. The group social welfare function is assumed to be additively separable and can therefore be written

$$-W(g,\ \alpha) = \sum_{i=1}^{q} d(g_i), \qquad \ldots \quad (3)$$

where $g = (g_1, g_2, \ldots, g_q)$.

To measure inequality in the distribution of income gaps they define the 'equally distributed equivalent income gap', which is that income gap, which if shared by all the poor, would be regarded as yielding the same level of welfare as the existing level and distribution of gaps. This is given by

$$g^* = \left[\frac{1}{q} \sum_{i=1}^{q} g_i^{\alpha}\right]^{1/\alpha}. \qquad \ldots \quad (4)$$

Poverty can then be measured using the following index

$$P^* = \frac{q \cdot g^*}{n \cdot z}$$

$$= \frac{q}{n} I \frac{g^*}{\bar{g}}, \qquad \ldots \quad (5)$$

where $I = \frac{1}{qz} \sum_{i=1}^{q} g_i$, and $\bar{g}$ is the mean income short-fall of the poor. $P^*$ is increasing in $\left(\frac{q}{n}\right)$, $I$ and $\frac{g^*}{\bar{g}}$, the relative deprivation measure.

Here we propose a generalisation of the approach adopted by Clark *et al.* For this, let the group deprivation function of the poor be given by

$$F = F(g_1, g_2, \ldots, g_q), \qquad \ldots \quad (6)$$

where, $-F$ is assumed to be continuous, non-decreasing in $y_i$'s $(i = 1, 2, \ldots, q)$ and also $S$-concave.[1]

We define the representative income gap ($g_e$) of the poor as that gap which, if shared by all the poor would make the distribution of income gaps socially indifferent (indifferent as measured by the group deprivation function) to the observed distribution and is given by

$$F(g_e \cdot \mathbf{1}) = F(g), \qquad \ldots \quad (7)$$

where $\mathbf{1}$ is an appropriate vector of ones. Solving (7) (uniquely) for $g_e$, we obtain

$$g_e = E(g), \qquad \ldots \quad (8)$$

where $E$ is a particular numerical representation of $F$.

As a general poverty index we introduce the measure

$$P(g) = \frac{q}{n} \cdot \frac{g_e}{z}. \qquad \ldots \quad (9)$$

The measure $P$ is the aggregate income gap of the poor which, if equally shared, would yield the same level of deprivation as the actual distribution of gaps generate, expressed as a proportion of the aggregate gap when each member of the population has a zero income.

The measure given by (9) is a relative poverty measure (that is, it remains unchanged when all the incomes and the poverty line itself are multiplied by a positive scalar) if and only if $F$ is homothetic[2]. This means

$$F = \phi(\overline{F}(g_1, \ldots, g_q)), \qquad \ldots \quad (10)$$

where $\phi$ is increasing in its argument and $\overline{F}$ is positively linearly homogeneous.

*Examples*: (i) Let $\overline{F}$, the image of the group deprivation function be given by

$$\overline{F}(g_1, \ldots, g_q) = \sum_{i=1}^{q} g_i a_i, \qquad \ldots \quad (11)$$

[1] $-F$ is said to be $S$-concave if $-F(\beta g) > -F(g)$ for all $g$ and for all bistochastic matrices $\beta$ of order $q$. $-F$ is strictly $S$-concave if the inequality is strict whenever the vector $\beta g$ is not a permutation of $g$. $S$-concavity is the requirement that $-F$ should agree with weak Lorenz quasiordering (see Dasgupta *et al*, 1973).

[2] The proof of this assertion follows from results in Blackorby and Donaldson (1978).

where the sequence $\{a_i\}$ is positive and non-increasing[3].

$$P = \frac{q}{nz} \cdot \frac{1}{\sum_{i=1}^{q} a_i} \sum_{i=1}^{q} g_i a_i. \qquad \dots (12)$$

Suppose that $a_i = q+1-i$, then

$$P = \frac{2}{(q+1)nz} \sum_{i=1}^{q} g_i(q+1-i) \qquad \dots (13)$$

$$= \text{Sen's measure.}$$

(ii) Let $g_e$ correspond to the symmetric mean income gap (of the poor) of order $\alpha$ ($\alpha > 1$), i.e.,

$$g_e = \left[\frac{1}{q} \sum_{i=1}^{q} g_i^{\alpha}\right]^{1/\alpha} \qquad \dots (14)$$

Then

$$P = \frac{q}{nz} \left[\frac{1}{q} \sum_{i=1}^{q} g_i^{\alpha}\right]^{1/\alpha} \qquad \dots (15)$$

$$= \text{The Clark, Hemming and Ulph measure.}$$

This index becomes more sensitive to transfer of income the larger is $\alpha$.

Therefore, the index $P$ is a generalisation of the Clark, Hemming and Ulph index given by (5). Given (9), we note that for every homothetic group deprivation function of the poor there exists a corresponding relative poverty index. These indices will differ only in the way in which the relative deprivations of the poor are taken into account. The measure is also sensitive to the head-count ratio.

## 3. Absolute measures of poverty

For many policy purposes it might be necessary to introduce absolute measures of poverty which are invariant with respect to translation of $z$ and $y^p$, where $y^p$ is the income vector of the poor.

For a given income profile $y$, we may define the absolute poverty index $Q$ as the product of the head-count ratio and the representative income gap of the poor corresponding to $y$, that is,

$$Q(g) = \frac{q}{n} \cdot g_e \qquad \dots (16)$$

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[3]The requirement that $(a_i)$ is non-increasing is necessary and sufficient for $\bar{F}$ to be quasi-convex or $S$-convex in $(y_1, \dots, y_q)$.

The value of the deprivation function remains unaltered when the same amount of income is added to or subtracted from the incomes of the poor and the poverty line itself. Hence the measure given by (16) remains invariant with respect to translation of $z$ and $y^p$.

*Examples*: (i) Consider the representative income gap of the poor corresponding to the symmetric mean income gap of order $\alpha$. Then

$$Q = \frac{q}{n}\left[\frac{1}{q}\sum_{i=1}^{q}(z-y_i)^{\alpha}\right]^{1/\alpha}, \qquad \dots \quad (17)$$

where $\alpha \geqslant 1$.

(ii) Let $g_e$ correspond to the Kolm-Pollak group deprivation function of the poor. Then

$$g_e = -\frac{1}{\theta}\log\left[\frac{1}{q}\sum_{i=1}^{q} e^{-\theta(z-y_i)}\right]. \qquad \dots \quad (18)$$

Therefore

$$Q = -\left(\frac{q}{n}\right)\frac{1}{\theta}\log\left[\frac{1}{q}\sum_{i=1}^{q} e^{-\theta(z-y_i)}\right], \qquad \dots \quad (19)$$

where $\theta > 0$.

Here $\theta$ is a free parameter which determines the curvature of the social indifference surfaces. As $\theta$ increases, the measure attaches more weight to transfers of income lower down the income scale.

Indeed, any group deprivation function $F = F(g_1, \dots, g_q)$ where $-F$ is continuous, non-decreasing in $y_i$'s $(i = 1, \dots, q)$ and also $S$-concave, will serve for constructing an absolute poverty index of the form (16). These indices will differ only in the way in which the deprivations of the poor are accounted for. The approach yields a rich class of measures to choose from.

## 4. Conclusions

We have generalised the Clark, Hemming and Ulph index [*vide* equation (5)] in this paper and have shown that (i) for every homothetic group deprivation function of the poor there is one relative poverty index of the type proposed; (ii) Clark, Hemming and Ulph's index is the relative poverty index when the group deprivation function of the poor is given by the symmetric mean income gap (of the poor) of order $\alpha$ $(\alpha \geqslant 1)$; (iii) Sen's index can be interpreted in this general framework; (iv) for every group deprivation function $F$ with minimal properties there is an absolute poverty index of the type proposed here.

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REFERENCES

BLACKORBY, C. and DONALDSON, D. (1978): Measures of relative equality and their meaning in terms of social welfare. *Journal of Economic Theory*, 18, 59-80.

BLACKORBY, C., PRIMONT, D. and RUSSELL, R. (1978): *Duality, Separability and Functional Structure: Theory and Economic Applications*, American Elsevier/North-Holland.

CLARK, S., HEMMING, R. and ULPH, D. (1981): On indices for the measurement of poverty. *Economic Journal*, 91, 515-526.

DASGUPTA, P., SEN, A. and STARRETT, D. (1973): Notes on the measurement of inequality. *Journal of Economic Theory*, 6, 180-187.

SEN, A. (1976): Poverty: An ordinal approach to measurement. *Econometrica*, 44, 219-231.

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