Journal of Combinatorial Theory, Series A 96, 388–395 (2001)

NOTE

Rigidity Theorems for Partial Linear Spaces

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Communicated by the Managing Editors

Let $(\mathscr{X}, \mathscr{L})$ be a partial linear space such that each of its points is contained in at least $n+1$ lines. Fix a prime p. Let $\mathscr{C}_{\mathscr{L}} \subseteq \mathbb{F}_p^{\mathscr{X}}$ denote the p-ary code generated by lines in $\mathscr{L}$ and $\mathscr{C}_{\mathscr{L}}^{\perp}$ denote its dual. In this article, we prove that the minimum weight of $\mathscr{C}_{\mathscr{L}}^{\perp}$ is at least $2n+2-\frac{2n}{p}$. (As a special case of our result we see that the minimum weight of the p-ary code dual to the code generated by lines in a projective plane of order n is at least $2n+2-\frac{2n}{p}$. This bound sharply improves upon the earlier known bounds as given in [1, Corollary 6.3.1 and Theorem 6.4.2]). We also prove that the induced structure on the support of a word of weight $2n+2-\frac{2n}{p}$ in $\mathscr{C}_{\mathscr{L}}^{\perp}$, if it exists, is isomorphic to the join of two Steiner 2-designs with $\frac{n}{p}$ lines through every point and p points on every line (Theorem 2).

For a projective plane Π of order n, let $\mathscr{C}$ denote the p-ary code generated by lines in Π and $\mathscr{C}^{\perp}$ its orthogonal. We denote by $Hull_p(\Pi)$ the intersection $\mathscr{C} \cap C^{\perp}$. The plane Π is said to be *tame at p* if the minimum weight of $Hull_p(\Pi)$ is $2n$ and the words of minimum weight are scalar multiples of vectors of the form $\psi_L - \psi_M$, where ψ_L and ψ_M denote characteristic functions of two lines L and M in Π. It is called *tame* if it is tame at all the primes that divide n. Thus, our result also implies that a projective plane of order p is tame. As only desarguesian planes are believed to be tame (see [1, Theorem 6.9.1] and the discussion thereon), the results of this article provide strong evidence towards the validity of the prime order case of the Hamada–Sachar conjecture.

2. RIGIDITY THEOREMS

In this section, in Theorem 1, we prove a generalisation of Bagchi's conjecture for linear spaces of prime order. Let us fix a prime p.

Let $\mathscr{X}$ be a non-empty set and $\mathscr{L} \subseteq P(\mathscr{X})$ be a collection of non-empty subsets of $\mathscr{X}$. The pair $(\mathscr{X}, \mathscr{L})$ is called a *partial linear space* if for any $A \neq B$ in $\mathscr{X}$, $\{A, B\}$ is a subset of at most one $L \in \mathscr{L}$. Elements of $\mathscr{X}$ are called points and elements of $\mathscr{L}$ are called lines. By a line through A, we mean a line containing the point A. The p-ary code generated by the lines in $\mathscr{L}$ is denoted by $\mathscr{C}_{\mathscr{L}}$ and its dual by $\mathscr{C}_{\mathscr{L}}^{\perp}$. We call a partial linear space $(\mathscr{X}, \mathscr{L})$ *non-trivial at p* if the p-ary code $\mathscr{C}_{\mathscr{L}}$ is not equal to $\mathbb{F}_p^{\mathscr{X}}$.

Let $(\mathscr{X}, \mathscr{L})$ be a partial linear space such that at least $n+1$ lines pass through every point in $\mathscr{X}$, and let $\mathscr{Y}$ be a subset of $\mathscr{X}$ such that no line in $\mathscr{L}$ intersects $\mathscr{Y}$ in one point. Let $\mathscr{L}'$ denote the set of lines in $\mathscr{Y}$ obtained by intersecting lines in $\mathscr{L}$ with $\mathscr{Y}$. Then $(\mathscr{Y}, \mathscr{L}')$ is also a partial linear space with $\geqslant n+1$ lines through every point. This is called *the induced structure* on $\mathscr{Y}$. Let $(\mathscr{X}, \mathscr{L}_1)$ and $(\mathscr{Y}, \mathscr{L}_2)$ be two partial linear spaces with disjoint point sets. Their *join* is the partial linear space $(\mathscr{X} \cup \mathscr{Y}, \mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3)$ where $\mathscr{L}_3$ is the collection of sets $\{x, y\}$ for all $x \in \mathscr{X}$ and $y \in \mathscr{Y}$. If $(\mathscr{X}_1, \mathscr{L}_1)$ and $(\mathscr{X}_2, \mathscr{L}_2)$ are two partial linear spaces such that p divides the cardinality of every line in $\mathscr{L}_1 \cup \mathscr{L}_2$, then their join is non-trivial at p since the non-zero word w defined by $w(x) = (-1)^i$ for all $x \in \mathscr{X}_i$ is in the dual of the code generated by its lines.

A partial linear space in which any pair of distinct points lie on a (unique) line is called a *linear space*. A *Steiner 2-design* is a linear space whose lines have same cardinality and the same number of lines pass through any point. The cardinality of a set S is denoted by $|S|$.

THEOREM 1. *If a partial linear space $(\mathscr{X}, \mathscr{L})$ is non-trivial at p and has at least $n+1$ lines through every point, then $|\mathscr{X}| \geqslant 2n+2-\frac{2n}{p}$. Moreover, the equality holds if and only if $(\mathscr{X}, \mathscr{L})$ is isomorphic to the join of two Steiner 2-designs with $\frac{n}{p}$ lines through every point and p points on every line.*

Proof. A Steiner 2-design with $\frac{n}{p}$ lines through every point and p points on each line has $n+1-\frac{n}{p}$ points. Thus the join of two such designs is a linear space with $2n+2-\frac{2n}{p}$ points and exactly $n+1$ lines through every point. Moreover, by the above remark, it is non-trivial at p.

To prove the converse, let $(\mathscr{Y}, \mathscr{L}')$ be a partial linear space with $\geqslant n+1$ lines through every point, with $|\mathscr{Y}| \leqslant 2n+2-\frac{2n}{p}$ and which is non-trivial at p. Let $\mathscr{X}$ denote the support of a word w of least weight in $\mathscr{C}_{\mathscr{L}'}^{\perp}$. The induced structure $(\mathscr{X}, \mathscr{L}_0)$ is a partial linear space with $\geqslant n+1$ lines through every point, with $|\mathscr{X}| \leqslant 2n+2-\frac{2n}{p}$ and such that $\mathscr{C}_{\mathscr{L}_0}^{\perp}$ is generated by the restriction of w to $\mathscr{X}$. To simplify the notation, we also denote the restriction of w to its support by w. Thus $(\mathscr{X}, \mathscr{L}_0)$ is non-trivial at p and the characteristic function of a subset L' of $\mathscr{X}$ is in $\mathscr{C}_{\mathscr{L}_0}$ if it is in the dual of $\langle w \rangle$. We now replace a line $L \in \mathscr{L}_0$ by two non-empty subsets L' and L'' whenever $L = L' \cup L''$ is a partition of L with the property that the characteristic function of L' (and hence of L'' also) is in $\mathscr{C}_{\mathscr{L}_0}$. When we reach a stage where no such replacements can be made, we get a set of lines on $\mathscr{X}$ such that the characteristic function of no proper subset of a line is in $\mathscr{C}_{\mathscr{L}_0}$ (we thank the referee for pointing out this construction). We denote the new set of lines by $\mathscr{L}$. Note that the construction of $\mathscr{L}$ from $\mathscr{L}_0$ is not canonical. However, for any such $\mathscr{L}$, the partial linear space $(\mathscr{X}, \mathscr{L})$ will have at least $n+1$ lines passing through every point and will have the following two properties:

1. $\mathscr{C}_{\mathscr{L}} = \mathscr{C}_{\mathscr{L}_0}$ and $\mathscr{C}_{\mathscr{L}}$ does not contain the characteristic function of a proper non-empty subset of a line in $\mathscr{L}$.
2. $\mathscr{C}_{\mathscr{L}}^{\perp}$ is one dimensional and $\mathscr{X}$ is the support of its generator w.

We deal with such a partial linear space in the remainder of the proof.

For any $A \in \mathscr{X}$, let x_A, y_A, z_A denote the number of lines through A of cardinality 2, 3, 4 respectively. Fix a point Q of $\mathscr{X}$ such that $x_Q \leqslant x_A$ for all $A \in \mathscr{X}$. We normalise w by assuming $w(Q) = -1$. We now colour $\mathscr{X}$ by elements of $\mathbb{F}_p$ using this w, wherein a point P gets the colour $w(P)$. As the characteristic function of a line is in the dual of $\langle w \rangle$, the sum of the colours occurring on any line is $0 (\mathrm{mod}\, p)$. Also, by the construction of $(\mathscr{X}, \mathscr{L})$, the colours occurring on any proper subset of a line do not add up to $0 (\mathrm{mod}\, p)$. Hence the lines containing a point each of colour α and $-\alpha$ have length 2 and the lines whose every point has same colour have length p. Let $\mathscr{S} = \{\alpha \in \mathbb{F}_p \mid w(P) = \alpha \text{ for some point } P\}$ denote the set of colours and let $X_\alpha \subseteq \mathscr{X}$ denote the set of points with colour α. Note that $0 \notin \mathscr{S}$ since $\mathscr{X}$ is the support of w.

Let S_A denote the union of all the lines through a point A. Then $1 + x_A + 2y_A + 3z_A + 4(n+1-x_A-y_A-z_A) \leqslant |S_A| \leqslant 2n+2-\frac{2n}{p}$ so that

$$3x_A + 2y_A + z_A \geqslant 2n+3+\frac{2n}{p}. \tag{1}$$

Similarly, $1 + x_A + 2y_A + 3(n+1-x_A-y_A) \leqslant 2n+2-\frac{2n}{p}$. Hence

$$2x_A + y_A \geqslant n+2+\frac{2n}{p}. \tag{2}$$

A line of cardinality s is called an s-line. If $L_1, ..., L_m$ denote all the s-lines through A with $s > 3$, then $|S_A| \geqslant 1 + x_A + 2(n+1-x_A-m) + \sum_{i=1}^{m} (|L_i|-1)$. The bound on $|\mathscr{X}|$ now implies that

$$x_A \geqslant 1 + \frac{2n}{p} + \sum_{i=1}^{m} (|L_i| - 3)\,. \tag{3}$$

The theorem is trivial for $p = 2$. Thus we assume that $p \geqslant 3$. The number of 2-lines through any point is at least x_Q. Hence for every $\alpha \in \mathscr{S}$, at least x_Q points of $\mathscr{X}$ are coloured $-\alpha$. As $x_Q > 0$, we have $\alpha \in \mathscr{S} \Rightarrow -\alpha \in \mathscr{S}$ and $|X_\alpha| \geqslant x_Q$ for all $\alpha \in \mathscr{S}$. Also, $|\mathscr{S}|$ is even as $0 \notin \mathscr{S}$. Let $|\mathscr{S}| = 2r$ for some r with $1 \leqslant r \leqslant \frac{p-1}{2}$. We then have

$$r x_Q \leqslant n + 1 - \frac{n}{p}\,. \tag{4}$$

First consider the case when $r = 1$. In this case $\mathscr{S} = \{1, -1\}$ and $\mathscr{L}$ contains only 2-lines and p-lines. Also, since $x_Q \leqslant n + 1 - \frac{n}{p}$, the number of p-lines through Q is at least $\frac{n}{p}$. Therefore, we must have

$$2n + 2 - \frac{2n}{p} = 1 + \frac{n}{p}(p-1) + \left(n + 1 - \frac{n}{p}\right) \leqslant |S_Q| \leqslant 2n + 2 - \frac{2n}{p}\,.$$

Thus $x_Q = n + 1 - \frac{n}{p}$ and the number of p-lines through Q must be $\frac{n}{p}$. Therefore, exactly $n+1$ lines pass through Q and $|\mathscr{X}| = |S_Q| = 2n + 2 - \frac{2n}{p}$. Moreover, $|X_1| = |X_{-1}| = n + 1 - \frac{n}{p}$. As the number of 2-lines through a point $A \in X_i$ is at most $|X_{-i}|$, we must have $x_A = x_Q$ for all $A \in \mathscr{X}$. Thus the above argument for Q also holds for any $A \in \mathscr{X}$ so that the number of p-lines through any point of $\mathscr{X}$ is $\frac{n}{p}$. Thus X_i with all the p-lines contained in it is isomorphic to a Steiner 2-design whose lines have length p and having $\frac{n}{p}$ lines through every point. As $\{x, y\} \in \mathscr{L}$ for all $x \in X_1$ and $y \in X_{-1}$, the partial linear space $(\mathscr{X}, \mathscr{L})$ is their join.

Thus $(\mathscr{X}, \mathscr{L})$ is a linear space. Since any substitution of a line in $\mathscr{L}_0$ by its partition would prevent $(\mathscr{X}, \mathscr{L})$ from being a linear space, we get $\mathscr{L} = \mathscr{L}_0$. Also, $2n + 2 - \frac{2n}{p} = |\mathscr{X}| \leqslant |\mathscr{Y}| \leqslant 2n + 2 - \frac{2n}{p}$. Hence $\mathscr{X} = \mathscr{Y}$ so that $\mathscr{L}_0 = \mathscr{L}'$. Thus $(\mathscr{X}, \mathscr{L})$ is the partial linear space $(\mathscr{Y}, \mathscr{L}')$ that we started off with. This proves the theorem for $r = 1$.

Thus we may assume that $r > 1$. To complete the proof of the theorem, we wish to prove that no such $(\mathscr{X}, \mathscr{L})$ exists.

As $1 < r \leqslant \frac{p-1}{2}$, we have $p > 3$ so that $(p-1)(1 + \frac{2n}{p}) > 2n + 2 - \frac{2n}{p}$. Hence, by (3) and (4), $r \neq \frac{p-1}{2}$. We thus have $1 < r < \frac{p-1}{2}$.

Let G_Q denote the graph whose vertex set is $\mathscr{S}$ and whose edges are given by the rule

$$\alpha \text{ and } \beta \text{ are adjacent if and only if } \alpha + \beta = 0 \text{ or } 1 \text{ in } \mathbb{F}_p.$$

As the sum of the colours occuring on any line is zero, a 3-line having non-empty intersections with X_{-1} and X_α also has non-empty intersection with $X_{1-\alpha}$. If α has degree one in G_Q, $X_{1-\alpha}$ is empty. Thus if α has degree one in G_Q, then $L \cap X_\alpha$ is either empty or is $\{Q\}$ for a 3-line L through Q. Since $0 \notin \mathscr{S}$, the degree of 1 is one in G_Q. The only possible loop of G_Q is at the vertex $\frac{p+1}{2}$ and this loop occurs if and only if $\frac{p+1}{2}$ belongs to $\mathscr{S}$.

If $\alpha_1 \cdots \alpha_m \alpha_1$ is a non-trivial cycle in G_Q, m must be even, as the edges of types $\{\alpha, -\alpha\}$ and $\{\alpha, 1-\alpha\}$ must alternate in the cycle. Also,

$$(\alpha_1+\alpha_2)+\cdots+(\alpha_{m-1}+\alpha_m)=(\alpha_2+\alpha_3)+\cdots+(\alpha_m+\alpha_1).$$

Hence m must be a multiple of $2p$, as one of these sums is zero and the other is $\frac{m}{2}$. However, $m \leqslant |\mathscr{S}| < p-1$, and hence G_Q does not contain any cycles. Therefore, each connected component of G_Q is a path. In case $\frac{p+1}{2} \in \mathscr{S}$, one of these paths has a loop at one end.

Case 1. The graph G_Q is connected.

In this case, G_Q is a path. If 1 is the only vertex of degree one, then the other end of this path must be a loop at $\frac{p+1}{2}$. Therefore, in this case G_Q is the path $1(-1)\,2(-2)\cdots(\frac{p-1}{2})(\frac{p+1}{2})(\frac{p+1}{2})$. However, this forces $|\mathscr{S}|=p-1$ which cannot happen. Therefore, G_Q must have two vertices of degree one and it is the path $1(-1)\,2(-2)\cdots r(-r)$ with $1<r<\frac{p-1}{2}$.

Since $r>1$, $Q \notin X_{-r}$ so that $|\{Q\} \cup X_{-r}| \geqslant 1+x_Q$. Let $T=\mathscr{X} \setminus (\{Q\} \cup X_{-r})$ and let l denote the number of s-lines through Q with $s>2$ and which contain at most one point from T. Note that every 2-line through Q contains exactly one point from T. Counting points of T which lie on lines through Q, we get

$$2(n+1-x_Q-l)+x_Q \leqslant |T| \leqslant 2n+2-\frac{2n}{p}-(1+x_Q).$$

This means $l \geqslant \frac{n}{p}+\frac{1}{2}>\frac{n}{p}$.

Let L be one of these l lines. If $L \subset X_{-r} \cup \{Q\}$, then L must contain at least $\frac{p-1}{r}$ points from X_{-r} as the colours on L add up to $0 (\operatorname{mod} p)$. As $Q \notin X_{-r}$, in this case $|L| \geqslant \frac{p+r-1}{r}$. If L contains one point from T, the colour of that point in L must be $(1+(|L|-2)\,r)(\operatorname{mod} p)$. As an integer, $1+(|L|-2)\,r>r$ as $|L|>2$. If this number is greater than p, then we must have $|L|-2>\frac{p-1}{r}$ so that $|L|>\frac{p+2r-1}{r}$. If it is less than p, then this number itself represents the colour of the remaining point (which is not $p-r$ by our assumption). Thus, as $\mathscr{S}=\{1, \ldots, r, p-r, \ldots, p-1\}$, we must have $p-r+1 \leqslant 1+(|L|-2)\,r$. This forces $|L|-2 \geqslant \frac{p-r}{r}$, so that $|L| \geqslant \frac{p+r}{r}$. Considering all the cases, we see that if a line L through Q contains at most one point from T, then $|L| \geqslant \frac{p+r-1}{r}$. As $r < \frac{p-1}{2}$, we have $|L| > 3$. We may then apply (3) and the fact that $l > \frac{n}{p}$ to get

$$x_Q > 1 + \frac{2n}{p} + \frac{n}{p}\left(\frac{p+r-1}{r} - 3\right) = 1 + \frac{n}{r} - \frac{n}{pr}.$$

However, this bound contradicts (4).

Case 2. G_Q is disconnected.

In this case, let $\mathscr{S}' \subseteq \mathscr{S}$ denote the set of colours which have degree one in G_Q. As G_Q is disconnected, we must have $|\mathscr{S}'| \geqslant 3$. Let $T = (\bigcup_{\alpha \in \mathscr{S}'} X_\alpha) \cup 3L_Q$ where $3L_Q$ is the set consisting of points $P \in \mathscr{X} \setminus \{Q\}$ which lie on 3-lines through Q. Since $3L_Q \cap X_\alpha$ is empty for all $\alpha \in \mathscr{S}'$, this union is disjoint. Hence $|T| \geqslant 2y_Q + |\mathscr{S}'| x_Q$. The bound on $|\mathscr{X}|$ together with (2) now implies that $|\mathscr{S}'| < 4$.

Thus $|\mathscr{S}'| = 3$ and we must have $2y_Q + 3x_Q \leqslant |\mathscr{X}| \leqslant 2n + 2 - \frac{2n}{p}$. By (2), we also have $2y_Q + 4x_Q \geqslant 2n + 4 + \frac{4n}{p}$. Hence $x_Q \geqslant 2 + \frac{6n}{p}$ so that (4) implies that $|\mathscr{S}| = 2r$ with $1 < r < \frac{p}{6}$. Also, G_Q must contain a loop as the number of vertices in G_Q of degree 1 is odd. Thus the graph G_Q consists of two components. One is

$$1(-1)\,2(-2)\cdots t(-t) \qquad \text{for some } t \text{ such that } 1 \leqslant t < r,$$

and the other is

$$\left(\frac{p+1}{2}\right)\left(\frac{p+1}{2}\right)\left(\frac{p-1}{2}\right)\left(\frac{3-p}{2}\right)\left(\frac{p-3}{2}\right)\cdots\left(\frac{p+1-(2r-2t)}{2}\right).$$

Thus $1, -t$ and $\alpha = \frac{p+1-(2r-2t)}{2} = \frac{p+1}{2} - (r-t)$ are the three vertices of degree one in G_Q.

Let $T = \{Q\} \cup 2L_Q \cup 3L_Q \cup X_{-t} \cup X_\alpha$, where iL_Q denotes the set of points in $\mathscr{X} \setminus \{Q\}$ which lie on i-lines through Q. Clearly, $|T| \geqslant 2y_Q + 3x_Q$. If every 4-line through Q contains a point of $\mathscr{X} \setminus T$, then we get

$$|\mathscr{X}| \geqslant z_Q + |T| \geqslant z_Q + 2y_Q + 3x_Q.$$

Since this, by (1), contradicts the bound on $|\mathscr{X}|$, we must have a 4-line L through Q contained in T. As $2L_Q$ and $3L_Q$ consist of the points on 2 and 3-lines through Q, respectively, $L \subset \{Q\} \cup X_{-t} \cup X_\alpha$.

Let L contain i points from X_α, where $0 \leqslant i \leqslant 3$. Then the sum of the colours occuring on L is $-1 + i\alpha - (3-i)\,t$ which must be $0 \pmod{p}$. Substituting the value of α, for i varying from 0 to 3, we infer that one of the following four integers is a multiple of p:

$$3t+1, \qquad 2(r+t)+1, \qquad 2r-t, \qquad 6(r-t)-1.$$

However, as $1 \leqslant t < r < \frac{p}{6}$, none of the above integers can be a multiple of p. This completes the proof of the theorem.

THEOREM 2. *Let $(\mathscr{X}, \mathscr{L})$ be a partial linear space whose every point is contained in at least $n+1$ lines. If it is non-trivial at p, then the minimum weight of $\mathscr{C}_{\mathscr{L}}^{\perp}$ is at least $2n+2-\frac{2n}{p}$. If this weight is attained, then the induced structure on the support of a word of weight $2n+2-\frac{2n}{p}$ in $\mathscr{C}_{\mathscr{L}}^{\perp}$ is isomorphic to the join of two Steiner 2-designs with $\frac{n}{p}$ lines through every point and p points on every line.*

Proof. The induced structure on the support of a non-zero word w of $\mathscr{C}_{\mathscr{L}}^{\perp}$ is a partial linear space with $\geqslant n+1$ lines through every point that is non-trivial at p. Theorem 1 now implies the theorem.

THEOREM 3. *A projective plane of order p is tame.*

Proof. This is the special case of the above theorem when $n = p$ and $(\mathscr{X}, \mathscr{L})$ is a projective plane.

A *partial linear space of order n* is a partial linear space with exactly $n+1$ lines through every point.

Remark 1. If a Steiner 2-design with order $\frac{n}{p}-1$ and line length p is to exist, then the standard divisibility properties of its parameters imply that $n \equiv 0$ or $p(\text{mod } p^2)$. Therefore, in case n does not satisfy this requirement, the minimum weight of $\mathscr{C}_{\mathscr{L}}^{\perp}$ is more than $2n+2-\frac{2n}{p}$. The join of a projective plane of order 2 with a set S of cardinality 4 (with no lines!) is a partial linear space of order $n = 6$. With $p = 3$, its cardinality is $2n+3-\frac{2n}{p} = 11$ and it is non-trivial at p.

Remark 2. By arguments similar to those in the proof of Theorem 1, we can prove that the minimum weight of $\mathscr{C}_{\mathscr{L}}^{\perp}$ corresponding to a *linear space* of order $2p$ is at least $4p$ and can obtain an explicit description of words of weight $4p$ in $\mathscr{C}_{\mathscr{L}}^{\perp}$ if they exist. One consequence of our result is that a projective plane of order $2p$, if it exists, must be tame at p. (The reader interested in knowing the detailed proof may contact the author.)

Remark 3. The join of an affine plane of order p and a set S of cardinality $p(p-1)$ with one line $L = S$ is an example of a linear space of order p^2 which is non-trivial at p and whose cardinality is strictly less than $2p^2$.

The results that we have obtained so far prompt us to believe in:

Conjecture 1. The minimum weight of the p-ary code $\mathscr{C}_{\mathscr{L}}^{\perp}$ corresponding to a linear space of order $n < p^2$ is at least $2n$.

ACKNOWLEDGMENTS

Theorem 1 was conjectured by B. Bagchi when $(\mathscr{X}, \mathscr{L})$ is a linear space of prime order. The author is grateful to B. Bagchi for numerous stimulating discussions and for providing ideas towards improving the presentation. We also gratefully thank the referee whose incisive comments led us to many simplifications of our earlier approach. The results of this paper could not have been obtained without their constructive criticism.

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